

A BRIEF
HISTORY OF MATHEMATICS

AN AUTHORIZED TRANSLATION OF

DR. KARL FINK'S GESCHICHTE DER
ELEMENTAR-MATHEMATIK

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are considered in order number-systems and number-symbols, arithmetic, algebra, geometry and trigonometry, allowing, as far as possible, within the narrow confines of a single branch of the elements, a rapid and sure orientation. Against such a procedure the objection may be raised that in this way the general survey of the culture history of a certain epoch will suffer. On the other hand, in a history of elementary mathematics, especially one confined within such modest bounds, an exhaustive description of whole periods with all their correlations of past and future cannot well be presented.

It is not the purpose of this work to set forth the interesting historical development of mechanics and astronomy. Although it cannot be denied that by this separation of related branches there is wanting a certain definitiveness to the work, yet the hope may be expressed that this lack will not be felt too keenly. The elementary parts of mathematics have only few points of contact with these branches, and our endeavor is to present in brief compass only that which is most essential.

Further, in the interest of a presentation as condensed as possible, the biographical notices which often lend great attraction to a more extended treatment of a subject must be relegated to the appendix and there treated but briefly.

The work had its inception in certain suggestions which the author received at the semi-monthly meetings of a mathematical club in Tübingen, founded and conducted by Prof. Dr. A. Brill, for which suitable thanks ought here to be expressed. Acknowledgment is especially due to the president of the club whose interpretations have been decisive for certain parts of the present work. These meetings furnished the author the desired opportunity, through the lectures connected with the most diverse branches of the science and through the discussions which often followed, with references to recent literature, to penetrate into those circles of thought which to-day dominate the higher branches of mathematics. The writer was thus led to complete his studies

by going into the recent history of the science. The results of such investigations are here presented with perhaps greater fullness than seems necessary for the main purpose of the book or justified by its title. But in default of such a digest, a first experiment may lay claim to a friendly judgment, in spite of the continually increasing subdivisions of the science; nor will such an attempt be thought inappropriate, inasmuch as it does not seem possible to draw a sharp line of demarcation between the elementary and higher mathematics. For on the one hand certain problems of elementary mathematics have from time to time furnished the occasion for the development of higher branches, and on the other from the acquisitions of these new branches a clear light has fallen upon the elementary parts. Accordingly it may be gratifying to many a student and teacher to find here at least that which is fundamental.

The exceedingly rich literature, especially in German, at the disposal of the author is referred to in the footnotes. He has made free use of the excellent *Jahrbuch über die Fortschritte der Mathematik*, which with clear and systematic arrangement enumerates and discusses the most recent mathematical literature.

K. FINK.

TÜBINGEN, June, 1890.

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GENERAL SURVEY.

THE beginnings of the development of mathematical truths date back to the earliest civilizations of which any literary remains have come down to us, namely the Egyptian and the Babylonian. On the one hand, brought about by the demands of practical life, on the other springing from the real scientific spirit of separate groups of men, especially of the priestly caste, arithmetic and geometric notions came into being. Rarely, however, was this knowledge transmitted through writing, so that of the Babylonian civilization we possess only a few traces. From the ancient Egyptian, however, we have at least one manual, that of Ahmes, which in all probability appeared nearly two thousand years before Christ. !

The real development of mathematical knowledge, obviously stimulated by Egyptian and Babylonian influences, begins in Greece. This development shows itself predominantly in the realm of geometry, and enters upon its first classic period, a period of no great duration, during the era of Euclid, Archimedes, Eratosthenes, and Apollonius. Subsequently it inclines more toward the arithmetic side; but it soon becomes so completely engulfed by the heavy waves

of stormy periods that only after long centuries and in a foreign soil, out of Greek works which had escaped the general destruction, could a seed, new and full of promise, take root.

One would naturally expect to find the Romans entering with eagerness upon the rich intellectual inheritance which came to them from the conquered Greeks, and to find their sons, who so willingly resorted to Hellenic masters, showing an enthusiasm for Greek mathematics. Of this, however, we have scarcely any evidence. The Romans understood very well the practical value to the statesman of Greek geometry and surveying—a thing which shows itself also in the later Greek schools—but no real mathematical advance is to be found anywhere in Roman history. } Indeed, the Romans often had so mistaken an idea of Greek learning that not infrequently they handed it down to later generations in a form entirely distorted.

More important for the further development of mathematics are the relations of the Greek teachings to the investigations of the Hindus and the Arabs. The Hindus distinguished themselves by a pronounced talent for numerical calculation. What especially distinguishes them is their susceptibility to the influence of Western science, the Babylonian and especially the Greek, so that they incorporated into their own system what they received from outside sources and then worked out independent results.

The Arabs, however, in general do not show this same independence of apprehension and of judgment. Their chief merit, none the less a real one however, lies in the untiring industry which they showed in translating into their own language the literary treasures of the Hindus, Persians and Greeks. The courts of the Mohammedan princes from the ninth to the thirteenth centuries were the seats of a remarkable scientific activity, and to this circumstance alone do we owe it that after a period of long and dense darkness Western Europe was in a comparatively short time opened up to the mathematical sciences.

The learning of the cloisters in the earlier part of the Middle Ages was not by nature adapted to enter seriously into matters mathematical or to search for trustworthy sources of such knowledge. It was the Italian merchants whose practical turn and easy adaptability first found, in their commercial relations with Mohammedan West Africa and Southern Spain, abundant use for the common calculations of arithmetic. Nor was it long after that there developed among them a real spirit of discovery, and the first great triumph of the newly revived science was the solution of the cubic equation by Tartaglia. It should be said, however, that the later cloister cult labored zealously to extend the Western Arab learning by means of translations into the Latin.

In the fifteenth century, in the persons of Peurbach and Regiomontanus, Germany first took position.

1400

in the great rivalry for the advancement of mathematics. From that time until the middle of the seventeenth century the German mathematicians were chiefly calculators, that is teachers in the reckoning schools (*Rechenschulen*). Others, however, were algebraists, and the fact is deserving of emphasis that there were intellects striving to reach still loftier heights. Among them Kepler stands forth pre eminent, but with him are associated Stifel, Rudolff, and Bürgi. Certain is it that at this time and on German soil elementary arithmetic and common algebra, vitally influenced by the Italian school, attained a standing very conducive to subsequent progress.

The modern period in the history of mathematics begins about the middle of the seventeenth century. Descartes projects the foundation theory of the analytic geometry. Leibnitz and Newton appear as the discoverers of the differential calculus. The time has now come when geometry, a science only rarely, and even then but imperfectly, appreciated after its banishment from Greece, enters along with analysis upon a period of prosperous advance, and takes full advantage of this latter sister science in attaining its results. Thus there were periods in which geometry was able through its brilliant discoveries to cast analysis, temporarily at least, into the shade.

The unprecedented activity of the great Gauss divides the modern period into two parts: before Gauss—the establishment of the methods of the dif

1650

B.G.

ferential and integral calculus and of analytic geometry as well as more restricted preparations for later advance; with Gauss and after him—the magnificent development of modern mathematics with its special regions of grandeur and depth previously undreamed of. The mathematicians of the nineteenth century are devoting themselves to the theory of numbers, modern algebra, the theory of functions and projective geometry, and in obedience to the impulse of human knowledge are endeavoring (to carry their light into remote realms which till now have remained in darkness.)

A.G.

1800's

I. NUMBER-SYSTEMS AND NUMBER-SYMBOLS.

AN inexhaustible profusion of external influences upon the human mind has found its legitimate expression in the formation of speech and writing in numbers and number-symbols. It is true that a counting of a certain kind is found among peoples of a low grade of civilization and even among the lower animals. "Even ducks can count their young."* But where the nature and the condition of the objects have been of no consequence in the formation of the number itself, there human counting has first begun.

The oldest counting was even in its origin a process of reckoning, an adjoining, possibly also in special elementary cases a multiplication, performed upon the objects counted or upon other objects easily employed, such as pebbles, shells, fingers. Hence arose number-names. The most common of these undoubtedly belong to the primitive domain of language; with the advancing development of language their aggregate was gradually enlarged, the legitimate combina-

*Hankel, *Zur Geschichte der Mathematik im Altertum und Mittelalter*, 1874, p. 7. Hereafter referred to as Hankel. Tylor's *Primitive Culture* also has a valuable chapter upon counting.

tion of single terms permitting and favoring the creation of new numbers. Hence arose number-systems.

The explanation of the fact that 10 is almost everywhere found as the base of the system of counting is seen in the common use of the fingers in elementary calculations. In all ancient civilizations finger-reckoning was known and even to-day it is carried on to a remarkable extent among many savage peoples. Certain South African races use three persons for numbers which run above 100, the first counting the units on his fingers, the second the tens, and the third the hundreds. (They always begin with the little finger of the left hand and count to the little finger of the right.) The first counts continuously, the others raising a finger every time a ten or a hundred is reached.*

Some languages contain words belonging fundamentally to the scale of 5 or 20 without these systems having been completely elaborated; only in certain places do they burst the bounds of the decimal system. In other cases, answering to special needs, 12 and 60 appear as bases. The New Zealanders have a scale of 11, their language possessing words for the first few powers of 11, and consequently 12 is represented as 11 and 1, 13 as 11 and 2, 22 as two 11's, and so on.†

*Cantor, M., *Vorlesungen über Geschichte der Mathematik*. Vol. I, 1880; 2nd ed., 1894, p. 6. Hereafter referred to as Cantor. Conant, L. L., *The Num-*

In the verbal formation of a number-system addition and multiplication stand out prominently as definitive operations for the composition of numbers; very rarely does subtraction come into use and still more rarely division. For example, 18 is called in Latin $10 + 8$ (*decem et octo*), in Greek $8 + 10$ (*ὀκτώ-καί-δέκα*), in French $10\ 8$ (*dix-huit*), in German $8\ 10$ (*acht-zehn*), in Latin also $20 - 2$ (*duo-de-viginti*), in Lower Breton $3 \cdot 6$ (*tri-omc'h*), in Welsh $2 \cdot 9$ (*dew-naw*), in Aztec $15 + 3$ (*caxtulli-om-cy*), while 50 is called in the Basque half-hundred, in Danish two-and-a-half times twenty.* In spite of the greatest diversity of forms, the written representation of numbers, when not confined to the mere rudiments, shows a general law according to which the higher order precedes the lower in the direction of the writing.† Thus in a four-figure number the thousands are written by the Phœnicians at the right, by the Chinese above, the former writing from right to left, the latter from above downward. A striking exception to this law is seen in the subtractive principle of the Romans in IV, IX, XL, etc., where the smaller number is written before the larger.

Among the Egyptians we have numbers running from right to left in the hieratic writing, with varying direction in the hieroglyphics. In the latter the numbers were either written out in words or represented by symbols for each unit, repeated as often as neces-

*Hankel, p. 22.

†Hankel, p. 32.

sary. In one of the tombs near the pyramids of Gizeh have been found hieroglyphic numerals in which 1 is represented by a vertical line, 10 by a kind of horse-shoe, 100 by a short spiral, 10 000 by a pointing finger, 100 000 by a frog, 1 000 000 by a man in the attitude of astonishment. In the hieratic symbols the figure for the unit of higher order stands to the right of the one of lower order in accordance with the law of sequence already mentioned. The repetition of symbols for a unit of any particular order does not obtain, because there are special characters for all nine units, all the tens, all the hundreds, and all the thousands.* We give below a few characteristic specimens of the hieratic symbols:

			—	𐤀	Λ	·Λ	𐤁	÷
1	2	3	4	5	10	20	30	40

The Babylonian cuneiform inscriptions† proceed from left to right, which must be looked upon as exceptional in a Semitic language. In accordance with the law of sequence the units of higher order stand on the left of those of lower order. The symbols used in writing are chiefly the horizontal wedge ➤, the vertical wedge ▼, and the combination of the two at an angle ◀. The symbols were written beside one another, or, for ease of reading and to save space, over one another. The symbols for 1, 4, 10, 100, 14, 400, respectively, are as follows :

*Cantor, I., pp. 43, 44.

†Cantor, I., pp. 77, 78.



For numbers exceeding 100 there was also, besides the mere juxtaposition, a multiplicative principle; the symbol representing the number of hundreds was placed at the left of the symbol for hundreds as in the case of 400 already shown. The Babylonians probably had no symbol for zero.* The sexagesimal system (i. e., with the base 60), which played such a part in the writings of the Babylonian scholars (astronomers and mathematicians), will be mentioned later.

The Phoenicians, whose twenty-two letters were derived from the hieratic characters of the Egyptians, either wrote the numbers out in words or used special numerical symbols—for the units vertical marks, for the tens horizontal.† Somewhat later the Syrians used the twenty-two letters of their alphabet to represent the numbers 1, 2, . . . 9, 10, 20, . . . 90, 100, . . . 400; 500 was 400 + 100, etc. The thousands were represented by the symbols for units with a subscript comma at the right.‡ The Hebrew notation follows the same plan.

The oldest Greek numerals (aside from the written words) were, in general, the initial letters of the fundamental numbers. I for 1, II for 5 (πέντε), Δ for 10 (δέκα), § and these were repeated as often as necessary.

* Cantor, I., p. 84.

† Cantor, I., p. 113.

‡ Cantor, I., pp. 113-114.

§ Cantor, I., p. 110.

These numerals are described by the Byzantine grammarian Herodianus (A. D. 200) and hence are spoken of as Herodianic numbers. Shortly after 500 B. C. two new systems appeared. One used the 24 letters of the Ionic alphabet in their natural order for the numbers from 1 to 24. The other arranged these letters apparently at random but actually in an order fixed arbitrarily; thus, $\alpha=1$, $\beta=2$, , $\iota=10$, $\kappa=20$, , $\rho=100$, $\sigma=200$, etc. Here too there is no special symbol for the zero.

The Roman numerals* were probably inherited from the Etruscans. The noteworthy peculiarities are the lack of the zero, the subtractive principle whereby the value of a symbol was diminished by placing before it one of lower order ($IV=4$, $IX=9$, $XL=40$, $XC=90$), even in cases where the language itself did not signify such a subtraction; and finally the multiplicative effect of a bar over the numerals ($\overline{XXX}=30\ 000$, $\overline{C}=100\ 000$). Also for certain fractions there were special symbols and names. According to Mommsen the Roman number-symbols I, V, X represent the finger, the hand, and the double hand. Zangemeister proceeds from the standpoint that *decem* is related to decussare which means a perpendicular or oblique crossing, and argues that every straight or curved line drawn across the symbol of a number in the decimal system multiplies that number by ten. In fact, there are on monuments

*Cantor, I., p. 486.

representations of 1, 10, and 1000, as well as of 5 and 500, to prove his assertion.*

Of especial interest in elementary arithmetic is the number-system of the Hindus, because it is to these Aryans that we undoubtedly owe the valuable position-system now in use. Their oldest symbols for 1 to 9 were merely abridged number-words, and the use of letters as figures is said to have been prevalent from the second century A. D.† The zero is of later origin; its introduction is not proven with certainty till after 400 A. D. The writing of numbers was carried on, chiefly according to the position-system, in various ways. One plan, which Aryabhatta records, represented the numbers from 1 to 25 by the twenty-five consonants of the Sanskrit alphabet, and the succeeding tens (30, 40 100) by the semi-vowels and sibilants. A series of vowels and diphthongs formed multipliers consisting of powers of ten, *ga* meaning 3, *gi* 300, *gu* 30 000, *gau* $3 \cdot 10^{16}$.‡ In this there is no application of the position-system, although it appears in two other methods of writing numbers in use among the arithmeticians of Southern India. Both of these plans are distinguished by the fact that

**Sitzungsberichte der Berliner Akademie vom 10. November 1887*. Wordsworth, in his *Fragments and Specimens of Early Latin*, 1874, derives C for *centum*, M for *mille*, and L for *quingenta* from three letters of the Chalcidian alphabet, corresponding to θ , ϕ , and χ . He says: "The origin of this notation is, I believe, quite uncertain, or rather purely arbitrary, though, of course, we observe that the initials of *mille* and *centum* determined the final shape taken by the signs, which at first were very different in form."

† See *Encyclopædia Britannica*, under "Numerals"

‡ Cantor, I., p. 566.

the same number can be made up in various ways. Rules of calculation were clothed in simple verse easy to hold in mind and to recall. For the Hindu mathematicians this was all the more important since they sought to avoid written calculation as far as possible. One method of representation consisted in allowing the alphabet, in groups of 9 symbols, to denote the numbers from 1 to 9 repeatedly, while certain vowels represented the zeros. If in the English alphabet according to this method we were to denote the numbers from 1 to 9 by the consonants *b, c, . . . z* so that after two countings one finally has $z=2$, and were to denote zero by every vowel or combination of vowels, the number 60502 might be indicated by *siren* or *heron*, and might be introduced by some other words in the text. A second method employed type-words and combined them according to the law of position. Thus *abdhi* (one of the 4 seas) $=4$, *surya* (the sun with its 12 houses) $=12$, *açvin* (the two sons of the sun) $=2$. The combination *abdhisuryaçvinas* denoted the number 2124.*

Peculiar to the Sanskrit number-language are special words for the multiplication of very large numbers. *Arbuda* signifies 100 millions, *padma* 10 000 millions; from these are derived *maharbuda* $=1000$ millions, *mahapadma* $=100\,000$ millions. Specially-formed words for large numbers run up to 10^{17} and even further. This extraordinary extension of the

*Cantor, I., p. 567.

decimal system in Sanskrit resembles a number-mania to grasp the infinitely great. Of this endeavour to bring the infinite into the realm of number-presentation and representation, traces are found also in the Babylonians and Greeks. This appearance finds its explanation in mystic-religious concepts and philosophic speculations.

The ancient Chinese number-symbols are composed of a comparatively few fundamental elements arranged in a perfectly developed decimal system. The combination takes place sometimes by multiplication, sometimes by addition. Thus *san* = 3, *che* = 10, *che san* denotes 13, but *san che* 30.* Later, as a result of foreign influence, there arose two new kinds of notation whose figures show some resemblance to the ancient Chinese symbols. Numbers formed by these figures were not written from above downwards but after the Hindu fashion from left to right beginning with the highest order. The one kind comprising the merchants' figures is never printed but is found in writings of a business character. Ordinal and cardinal numbers are arranged in lines one above another, with zeros when necessary in the form of small circles. In this notation

$|| = 2, X = 4, \perp = 6, + = 10, \bar{h} = 10\,000,$

" X

and hence $\bar{h} \circ \circ + \perp = 20\,046.$

* Cantor, I., p. 630.

Among the Arabs, those skilful transmitters of Oriental and Greek arithmetic to the nations of the West, the custom of writing out number-words continued till the beginning of the eleventh century. Yet at a comparatively early period they had already formed abbreviations of the number-words, the *Divanî* figures. In the eighth century the Arabs became acquainted with the Hindu number-system and its figures, including zero. From these figures there arose among the Western Arabs, who in their whole literature presented a decided contrast to their Eastern relatives, the Gubar numerals (dust-numerals) as variants. These Gubar numerals, almost entirely forgotten to-day among the Arabs themselves, are the ancestors of our modern numerals,* which are immediately derived from the apices of the early Middle Ages. These primitive Western forms used in the abacus-calculations are found in the West European MSS. of the eleventh and twelfth centuries and owe much of their prominence to Gerbert, afterwards Pope Sylvester II. (consecrated 999 A. D.).

The arithmetic of the Western nations, cultivated to a considerable extent in the cloister-schools from the ninth century on, employed besides the abacus the Roman numerals, and consequently made no use of a symbol for zero. In Germany up to the year 1500 the Roman symbols were called German numerals in distinction from the symbols—then seldom employed—

* Hankel, p. 255.

of Arab-Hindu origin, which included a zero (Arabic *as-sifr*, Sanskrit *sunya*, the void). The latter were called ciphers (*Ziffern*). From the fifteenth century on these Arab-Hindu numerals appear more frequently in Germany on monuments and in churches, but at that time they had not become common property.* The oldest monument with Arabic figures (in Katharein near Troppau) is said to date from 1007. Monuments of this kind are found in Pforzheim (1371), and in Ulm (1388). A frequent and free use of the zero in the thirteenth century is shown in tables for the calculation of the tides at London and of the duration of moonlight.† In the year 1471 there appeared in Cologne a work of Petrarch with page-numbers in Hindu figures at the top. In 1482 the first German arithmetic with similar page-numbering was published in Bamberg. Besides the ordinary forms of numerals everywhere used to-day, which appeared exclusively in an arithmetic of 1489, the following forms for 4, 5, 7 were used in Germany at the time of the struggle between the Roman and Hindu notations:

8 · 9 · Λ ·

The derivation of the modern numerals is illustrated by the examples below which are taken in succession from the Sanskrit, the apices, the Eastern Arab, the

* Unger, *Die Methodik der praktischen Arithmetik*, 1888, p. 70. Hereafter referred to as Unger.

† Günther, *Geschichte des mathematischen Unterrichts im deutschen Mittelalter bis zum Jahr 1525*, 1887, p. 175. Hereafter referred to as Günther.

early apices like
counters
↓
digits

Western Arab Gubar numerals, the numerals of the eleventh, thirteenth, and sixteenth centuries.*

٤	٥	٦	٧
٨	٩	١٠	١١
١٢	١٣	١٤	١٥
١٦	١٧	١٨	١٩
٢٠	٢١	٢٢	٢٣
٢٤	٢٥	٢٦	٢٧
٢٨	٢٩	٣٠	٣١
٣٢	٣٣	٣٤	٣٥

In the sixteenth century the Hindu position-arithmetic and its notation first found complete introduction among all the civilized peoples of the West. By this means was fulfilled one of the indispensable conditions for the development of common arithmetic in the schools and in the service of trade and commerce.

*Cantor, table appended to Vol. I, and Hankel, p. 325.

II. ARITHMETIC.

A. GENERAL SURVEY.

THE simplest number-words and elementary counting have always been the common property of the people. Quite otherwise is it, however, with the different methods of calculation which are derived from simple counting, and with their application to complicated problems. As the centuries passed, that part of ordinary arithmetic which to-day every child knows, descended from the closed circle of particular castes or smaller communities to the common people, so as to form an important part of general culture. Among the ancients the education of the youth had to do almost wholly with bodily exercises. Only a ripe age sought a higher cultivation through intercourse with priests and philosophers, and this consisted in part in the common knowledge of to-day: people learned to read, to write, to cipher.

At the beginning of the first period in the historic development of common arithmetic stand the Egyptians. To them the Greek writers ascribe the invention of surveying, of astronomy, and of arithmetic. To their literature belongs also the most ancient book on

arithmetic, that of Ahmes, which teaches operations with whole numbers and fractions. The Babylonians employed a sexagesimal system in their position-arithmetic, which latter must also have served the purposes of a religious number-symbolism. The common arithmetic of the Greeks, particularly in most ancient times, was moderate in extent until by the activity of the scholars of philosophy there was developed a real mathematical science of predominantly geometric character. In spite of this, skill in calculation was not esteemed lightly. Of this we have evidence when Plato demands for his ideal state that the youth should be instructed in reading, writing, and arithmetic.

The arithmetic of the Romans had a purely practical turn; to it belonged a mass of quite complicated problems arising from controversies regarding questions of inheritance, of private property and of reimbursement of interest. The Romans used duodecimal fractions. Concerning the most ancient arithmetic of the Hindus only conjectures can be made; on the contrary, the Hindu elementary arithmetic after the introduction of the position-system is known with tolerable accuracy from the works of native authors. The Hindu mathematicians laid the foundations for the ordinary arithmetic processes of to-day. The influence of their learning is perceptible in the Chinese arithmetic which likewise depends on the decimal system; in still greater measure, however, among the

Arabs who besides the Hindu numeral-reckoning also employed a calculation by columns.

The time from the eighth to the beginning of the fifteenth century forms the second period. This is a noteworthy period of transition, an epoch of the transplanting of old methods into new and fruitful soil, but also one of combat between the well-tried Hindu methods and the clumsy and detailed arithmetic operations handed down from the Middle Ages. At first only in cloisters and cloister-schools could any arithmetic knowledge be found, and that derived from Roman sources. But finally there came new suggestions from the Arabs, so that from the eleventh to the thirteenth centuries there was opposed to the group of abacists, with their singular complementary methods, a school of algorists as partisans of the Hindu arithmetic.

Not until the fifteenth century, the period of investigation of the original Greek writings, of the rapid development of astronomy, of the rise of the arts and of commercial relations, does the third period in the history of arithmetic begin. As early as the thirteenth century besides the cathedral and cloister-schools which provided for their own religious and ecclesiastical wants, there were, properly speaking, schools for arithmetic. Their foundation is to be ascribed to the needs of the brisk trade of German towns with Italian merchants who were likewise skilled computers. In the fifteenth and sixteenth centuries

2nd
8700-1400

1400
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school affairs were essentially advanced by the humanistic tendency and by the reformation. Latin schools, writing schools, German schools (in Germany) for boys and even for girls, were established. In the Latin schools only the upper classes received instruction in arithmetic, in a weekly exercise: they studied the four fundamental rules, the theory of fractions, and at most the rule of three, which may not seem so very little when we consider that frequently in the universities of that time arithmetic was not carried much further. In the writing schools and German boys' schools the pupils learned something of calculation, numeration, and notation, especially the difference between the German numerals (in Roman writing) and the ciphers (after the Hindu fashion). In the girls' schools, which were intended only for the higher classes of people, no arithmetic was taught. Considerable attainments in computation could be secured only in the schools for arithmetic. The most celebrated of these institutions was located at Nuremberg. In the commercial towns there were accountants' guilds which provided for the extension of arithmetic knowledge. But real mathematicians and astronomers also labored together in developing the methods of arithmetic. In spite of this assistance from men of prominence, no theory of arithmetic instruction had been established even as late as in the sixteenth century. What had been done before had to be copied. In the books on arithmetic

a long history.

were found only rules and examples, almost never proofs or deductions.

The seventeenth century brought no essential change in these conditions. Schools existed as before where they had not been swallowed up by the horrors of the Thirty-Years' War. The arithmeticians wrote their books on arithmetic, perhaps contrived calculating machines to make the work easier for their pupils, or composed arithmetic conversations and poems. A specimen of this is given in the following extracts from Tobias Beutel's *Arithmetica*, the seventh edition of which appeared in 1693.*

"Numerieren lehrt im Rechen
Zahlen schreiben und aussprechen."

"In Summen bringen heisst addieren
Dies muss das Wörtlein Und vollführen."

"Wie eine Hand an uns die andre wäscht rein
Kann eine Species der andern Probe seyn."

"We are taught in numeration
Number writing and expression,"
etc., etc.

Commercial arithmetic was improved by the cultivation of the study of exchange and discount, and the abbreviated method of multiplication. The form of instruction remained the same, i. e., the pupil reckoned according to rules without any attempt being made to explain their nature.

The eighteenth century brought as its first and

*Unger, p. 124.