

Phil Lucht's

ELECTRICITY & MAGNETISM BINDER

History of Math

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A Ridiculously Brief History of Electricity and Magnetism

Note: I found this at maxwell.byu.edu/~spencerr/phys442/node4.html , brought it up in Word and highlighted various things in bold and underline. I later found a PDF at the same site. I then did some cut and pastes from the PDF into this Word doc for equations and various other symbols. If something is unreadable, see the PDF.

Mostly from E. T. Whittaker's *A History of the Theories of Aether and Electricity...*

Prehistory

900 BC - **Magnus**, a Greek shepherd, walks across a field of black stones which pull the iron nails out of his sandals and the iron tip from his shepherd's staff (authenticity not guaranteed). This region becomes known as **Magnesia**.

600 BC - **Thales** of Miletos rubs amber (**elektron** in Greek) with cat fur and picks up bits of feathers.

1269 - Petrus **Peregrinus** of Picardy, Italy, discovers that natural spherical magnets (lodestones) align needles with lines of longitude pointing between two pole positions on the stone.

The 17th Century

1600 - William **Gilbert**, court physician to Queen Elizabeth, discovers that the earth is a giant magnet just like one of the stones of Peregrinus, explaining how compasses work. He also discusses static electricity and invents an electric fluid which is liberated by rubbing.

ca. 1620 - Niccolo **Cabeo** discovers that electricity can be repulsive as well as attractive.

1630 - Vincenzo Cascariolo, a Bolognese shoemaker, discovers **fluorescence**.

1638 - Rene **Descartes** theorizes that light is a pressure wave through the second of his three types of matter of which the universe is made. He invents properties of this fluid that make it possible to calculate the reflection and refraction of light. The "modern" notion of the aether is born.

1638 - **Galileo** attempts to measure the speed of light by a lantern relay between distant hilltops. He gets a very large answer.

1644 - Rene **Descartes** theorizes that the magnetic poles are on the central axis of a spinning vortex of one of his fluids. This vortex theory remains popular for a long time, enabling Leonhard Euler and two of the Bernoullis to share a prize of the French Academy as late as 1743.

1657 - Pierre de **Fermat** shows that the principle of least time is capable of explaining refraction and reflection of light. Fighting with the Cartesians begins. (This principle for reflected light had been anticipated anciently by Hero of Alexandria.)

1665 - Francesco Maria **Grimaldi**, in a posthumous report, discovers and gives the name of diffraction to the bending of light around opaque bodies.

1667 - Robert **Hooke** reports in his *Micrographia* the discovery of the rings of light formed by a layer of air between two glass plates. These were actually first observed by Robert Boyle, which explains why they are now called Newton's rings.

In the same work he gives the matching-wave-front derivation of reflection and refraction that is still found in most introductory physics texts. These waves travel through the aether. He also develops a theory of color in which white light is a simple disturbance and colors are complex distortions of the basic simple white form.

1671 - Isaac **Newton** destroys Hooke's theory of color by experimenting with prisms to show that white light is a mixture of all the colors and that once a pure color is obtained it can never be changed into another color. Newton argues against light being a vibration of the ether, preferring that it be something else that is capable of traveling through the aether. He doesn't insist that this something else consist of particles, but allows that it may be some other kind of emanation or impulse. In Newton's own words, "...let every man here take his fancy."

1675 - Olaf **Roemer** repeats Galileo's experiment using the moons of Jupiter as the distant hilltop. He measures $c = 2.3 \times 10^8$ m/s.

1678 - Christiaan **Huygens** introduces his famous construction and principle, thinks about translating his manuscript into Latin, then publishes it in the original French in 1690. He uses his theory to discuss the double refraction of Iceland Spar. His is a theory of pulses, however, not of periodic waves.

The 18th Century

1717 - **Newton** shows that the "two-ness" of double refraction clearly rules out light being aether waves. (All aether wave theories were sound-like, so Newton was right; longitudinal waves can't be polarized.) [The pressure-sound wave theory through a fluid can only support longitudinal compression and does not support transverse polarization. Since light was showing two polarization states in this spar, that ruled out simple pressure wave in aether idea.]

1728 - James **Bradley** shows that the orbital motion of the earth changes the apparent motions of the stars in a way that is consistent with light having a finite speed of travel. (speed of light)

1729 - Stephen **Gray** shows that electricity doesn't have to be made in place by rubbing but can also be transferred from place to place with conducting wires. He also shows that the charge on electrified objects resides on their surfaces. [Still in the static world only.]

1733 - Charles Francois **du Fay** discovers that electricity comes in two kinds which he called **resinous(-)** and **vitreous(+)**.

1742 - Thomas **Le Seur** and Francis **Jacquier**, in a note to the edition of Newton's Principia that they publish, show that the force law between two magnets is inverse cube.

1749 - Abbe Jean-Antoine **Nollet** invents the two-fluid theory electricity.

1745 - Pieter van **Musschenbroek** invents the Leyden jar, or capacitor, and nearly kills his friend Cunaeus.

1747 - Benjamin **Franklin** invents the theory of one-fluid electricity in which one of Nollet's fluids exists and the other is just the absence of the first. He proposes the principle of conservation of charge and calls the fluid that exists and flows "positive". This educated guess ensures that undergraduates will always be confused about the direction of current flow. He also discovers that electricity can act at a distance in situations where fluid flow makes no sense.

1748 - Sir William **Watson** uses an electrostatic machine and a vacuum pump to make the first glow discharge. His glass vessel is three feet long and three inches in diameter: the first fluorescent light bulb.

1750 - John Michell discovers that the two poles of a magnet are equal in strength and that the force law for individual poles is inverse square.

1752 - Johann Sulzer puts lead and silver together in his mouth, performing the first recorded "tongue test" of a battery.

1759 - Francis Ulrich Theodore Aepinus shows that electrical effects are a combination of fluid flow confined to matter and action at a distance. He also discovers charging by induction.

1762 - Canton reports that a red hot poker placed close to a small electrified body destroys its electrification.

1764 - Joseph Louis **Lagrange** discovers **the divergence theorem** in connection with the study of gravitation. It later becomes known as Gauss's law. (See 1813).

1766 - Joseph **Priestly**, acting on a suggestion in a letter from Benjamin Franklin, shows that hollow charged vessels contain no charge on the inside and based on his knowledge that hollow shells of mass have no gravity inside correctly deduces that the electric force law is inverse square.

ca 1775 - Henry **Cavendish** invents the idea of capacitance and resistance (the latter without any way of measuring current other than the level of personal discomfort). But being indifferent to fame he is content to wait for his work to be published by Lord Kelvin in 1879.

1777 - Joseph Louis **Lagrange** invents the concept of the scalar potential for gravitational fields.

1780 - Luigi **Galvani** causes dead frog legs to twitch with static electricity, then also discovers that the same twitching can be caused by contact with dissimilar metals. His followers invent another invisible fluid, that of "animal electricity", to describe this effect. [He really thinks animals are creating the electricity somehow, not just responding to it.]

1782 - Pierre Simon **Laplace** shows that Lagrange's potential satisfies $\nabla^2 V = 0$.

1785 - Charles Augustin **Coulomb** uses a torsion balance to verify that the electric force law is inverse square. He also proposes a combined fluid/action-at-a-distance theory like that of Aepinus but with two conducting fluids instead of one. Fighting breaks out between single and double fluid partisans. He also discovers that the electric force near a conductor is proportional to its surface charge density and makes contributions to the two-fluid theory of magnetism.

1793 - Alessandro **Volta** makes the first batteries and argues that animal electricity is just ordinary electricity flowing through the frog legs under the impetus of the force produced by the contact of dissimilar metals. He discovers the importance of "completing the circuit."

The period 1800-1825 -- things heat up after Volta invents the battery so currents may be produced and studied

1800 - In 1800 **Volta** discovers the Voltaic pile (dissimilar metals separated by wet cardboard) which greatly increases the magnitude of the effect.

1800 - William Nicholson and Anthony Carlisle discover that water may be separated into hydrogen and oxygen by the action of Volta's pile.

1801 - Thomas **Young** gives a theory of Newton's rings based on constructive and destructive interference of waves. He explains the dark spot in the middle by proposing that there is a phase shift on reflection between a less dense and more

dense medium, then uses essence of sassafras (whose index of refraction is intermediate between those of crown and flint glass) to get a light spot at the center. [diffraction]

1803 - Thomas **Young** explains the fringes at the edges of shadows by means of the wave theory of light. The wave theory begins its ascendancy, but has one important difficulty: light is thought of as a longitudinal wave, which makes it difficult to explain double refraction effects in certain crystals. [scalar theory bypasses polarization question]

1807 - Humphrey Davy shows that the essential element of Volta's pile is chemical action since pure water gives no effect. He argues that chemical effects are electrical in nature.

1808 - Laplace gives an explanation of double refraction using the particle theory, which Young attacks as improbable.

1808 - Etienne Louis **Malus**, a military engineer, enters a prize competition sponsored by the French Academy "To furnish a mathematical theory of double refraction, and to confirm it by experiment." He discovers that light reflected at certain angles from transparent substances as well as the separate rays from a double-refracting crystal have the same property of polarization. In 1810 he receives the prize and emboldens the proponents of the particle theory of light because no one sees how a wave theory can make waves of different polarizations.

1811 - **Arago** shows that some crystals alter the polarization of light passing through them. [rotation!]

1812 - **Biot** shows that Arago's crystals rotate the plane of polarization about the propagation direction.

1812 - Simeon Denis **Poisson** further develops the two-fluid theory of electricity, showing that the charge on conductors must reside on their surfaces and be so distributed that the electric force within the conductor vanishes. This surface charge density calculation is carried out in detail for ellipsoids. He also shows that the potential within a distribution of electricity satisfies the equation $\nabla^2 \phi = -\rho/\epsilon_0$.

1812 - Michael **Faraday**, a bookbinders apprentice, writes to Sir Humphrey Davy asking for a job as a scientific assistant. Davy interviews Faraday and finds that he has educated himself by reading the books he was supposed to be binding. He gets the job.

ca. 1813 - **Laplace** shows that at the surface of a conductor the electric force is perpendicular to the surface and that $E = \sigma/\epsilon_0$.

1813 - Karl Friedrich **Gauss** rediscovers the divergence theorem of Lagrange. It will later become known as Gauss's law.

1815 - David **Brewster** establishes his law of complete polarization upon reflection at a special angle now known as Brewster's angle. He also discovers that in addition to uniaxial crystals there are also biaxial ones. For uniaxial crystals there is the faint possibility of a wave theory of longitudinal-type, but this appears to be impossible for biaxial ones.

1816 - David Brewster invents the kaleidoscope.

1816 - Francois **Arago**, an associate of Augustin Fresnel, visits Thomas Young and describes to him a series of experiments performed by Fresnel and himself which shows that light of differing polarizations cannot interfere. Reflecting later on this curious effect Young sees that it can be explained if light is transverse instead of longitudinal. This idea is communicated to Fresnel in 1818 and he immediately sees how it clears up many of the remaining difficulties of the wave theory. Six years later the particle theory is dead.

1817 - Augustin **Fresnel** annoys the French Academy. The Academy, hoping to destroy the wave theory once and for all, proposes diffraction as the prize subject for 1818. To the chagrin of the particle-theory partisans in the Academy the winning memoir in 1818 is that of Augustin Fresnel who explains diffraction as the mutual interference of the secondary waves emitted by the unblocked portions of the incident wave, in the style of Huygens. One of the judges from the particle camp of the Academy is **Poisson**, who points out that if Fresnel's theory were to be indeed correct, then there should be a bright spot at the center of the shadow of a circular disc. This, he suggests to Fresnel, must be tested experimentally. The experiment doesn't go as Poisson hopes, however, and the spot becomes known as "Poisson's spot."

1820 - Hans Christian **Oersted** discovers that electric current in a wire causes a compass needle to orient itself perpendicular to the wire.

1820 - Andre Marie **Ampere**, one week after hearing of Oersted's discovery, shows that parallel currents attract each other and that opposite currents attract.

1820 - Jean-Baptiste **Biot** and Felix **Savart** show that the magnetic force exerted on a magnetic pole by a wire falls off like $1/r$ and is oriented perpendicular to the wire. Whittaker then says that "This result was soon further analyzed," to obtain

$$d\mathbf{B} \propto \frac{I d\mathbf{s} \times \mathbf{r}}{r^3}$$

1820 - John **Herschel** shows that quartz samples that rotate the plane of polarization of light in opposite directions have different crystalline forms. This difference is helical in nature.

1821 - **Faraday** begins electrical work by repeating Oersted's experiments.

1821 - Humphrey **Davy** shows that direct current is carried throughout the volume of a conductor and establishes that

$$R \propto \frac{\ell}{A}$$

for long wires. He also discovers that resistance is increased as the temperature rises.

1822 - Thomas Johann **Seebeck** discovers the thermoelectric effect by showing that a current will flow in a circuit made of dissimilar metals if there is a temperature difference between the metals.

1824 - **Poisson** invents the concept of the magnetic scalar potential and of surface and volume pole densities described by the formulas

$$V = - \int \mathbf{M} \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dV' = \int \frac{\nabla' \cdot \mathbf{M}}{|\mathbf{r} - \mathbf{r}'|} dV' - \int \frac{\mathbf{M} \cdot \mathbf{n}'}{|\mathbf{r} - \mathbf{r}'|} dA'$$

. He also finds the magnetic field inside a spherical cavity within magnetized material.

1825 - **Ampere** publishes his collected results on magnetism. His expression for the magnetic field produced by a small segment of current is different from that which follows naturally from the Biot-Savart law by an additive term which integrates to zero around closed circuit [was this the time term in the curl B equation?] It is unfortunate that electrodynamics and relativity decide in favor of Biot and Savart rather than for the much more sophisticated Ampere,

whose memoir contains both mathematical analysis and experimentation, artfully blended together. In this memoir are given some special instances of the result we now call Stokes theorem.

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I.$$

Maxwell describes this work as "one of the most brilliant achievements in science. The whole, theory and experiment, seems as if it had leaped, full-grown and full-armed, from the brain of the 'Newton of electricity'. It is perfect in form and unassailable in accuracy; and it is summed up in a formula from which all the phenomena may be deduced, and which must always remain the cardinal formula of electrodynamics."

The period 1825-1850

1825 - **Fresnel** shows that combinations of waves of opposite circular polarization traveling at different speeds can account for the rotation of the plane of polarization.

1826 - Georg Simon **Ohm** establishes the result now known as Ohm's law. $V=IR$ seems a pretty simple law to name after someone, but the importance of Ohm's work does not lie in this simple proportionality. What Ohm did was develop the idea of voltage as the driver of electric current. He reasoned by making an analogy between Fourier's theory of heat flow and electricity. In his scheme temperature and voltage correspond as do heat flow and electrical current. It was not until some years later that Ohm's electroscopic force (V in his law) and Poisson's electrostatic potential were shown to be identical. [again, the new world of DC currents from batteries is not quite linked to the static electricity world]

1827 - Augustin **Fresnel** publishes a decade of research in the wave theory of light. Included in these collected papers are explanations of diffraction effects, polarization effects, double refraction, and Fresnel's sine and tangent laws for reflection at the interface between two transparent media.

1827 - Claude Louis Marie Henri **Navier** publishes the correct equations for vibratory motions in one type of elastic solid. This begins the quest for a detailed mathematical theory of the aether based on the equations of continuum mechanics.

1827 - F. Savary, after noticing that the current from a Leyden jar magnetizes needles in alternating layers, conjectures that the electric motion during the discharge consists of a series of oscillations.

1828 - George **Green** generalizes and extends the work of Lagrange, Laplace, and Poisson and attaches the name potential to their scalar function. **Green's theorems** are given, as well as the **divergence theorem** (Gauss's law), but Green doesn't know of the work of Lagrange and Gauss and only references Priestly's deduction of the inverse square law from Franklin's experimental work on the charging of hollow vessels.

1828 - Augustine Louis **Cauchy** presents a theory similar to Navier's, but based on a direct study of elastic properties rather than using a molecular hypothesis. These equations are more general than Navier's. In Cauchy's theory, and in much of what follows, the aether is supposed to have the same inertia in each medium, but different elastic properties.

1828 - Poisson shows that the equations of Navier and Cauchy have wave solutions of two types: transverse and longitudinal. Mathematical physicists spend the next 50 years trying to invent an elasticsaether for which the longitudinal waves are absent. [By this time, it was known that solids can support transverse polarizations, so it probably seemed only a matter of time until the nature of the aether as some kind of solid was elucidated. But why then did the ether not support longitudinal polarization, as all solids do?]

1831 - **Faraday** shows that [the transformer] changing currents in one circuit induce currents in a neighboring circuit. Over the next several years he performs hundreds of experiments and shows that they can all be explained by the idea of changing magnetic flux. No mathematics is involved, just picture thinking using his field-lines. [

1831 - **Ostrogradsky** rediscovers the divergence theorem of Lagrange, Gauss, and Green.

1832 - Joseph **Henry** independently discovers induced currents.

1833 - **Faraday** begins work on the relation of electricity to chemistry. In one of his notebooks he concludes after a series of experiments, "...there is a certain absolute quantity of the electric power associated with each atom of matter."

1834 - **Faraday** discovers self inductance.

1834 - Jean Charles **Peltier** discovers the flip side of Seebeck's thermoelectric effect. He finds that current driven in a circuit made of dissimilar metals causes the different metals to be at different temperatures.

1834 - Emil **Lenz** formulates his rule for determining the direction of Faraday's induced currents. In its original form it was a force law rather than an induced emf law: "Induced currents flow in such a direction as to produce magnetic forces that try to keep the magnetic flux the same." So Lenz would predict that if you try to push a conductor into a strong magnetic field, it will be repelled. He would also predict that if you try to pull a conductor out of a strong magnetic field that the magnetic forces on the induced currents will oppose the pull.

1835 - James MacCullagh and Franz Neumann extend Cauchy's theory to crystalline media

1837 - **Faraday** discovers the idea of the dielectric constant.

1837 - George **Green** attacks the elastic aether problem from a new angle. Instead of deriving boundary conditions between different media by finding which ones give agreement with the experimental laws of optics, he derives the correct boundary conditions from general dynamical principles. This advance makes the elastic theories not quite fit with light.

1838 - **Faraday** shows that the effects of induced electricity in insulators are analogous to induced magnetism in magnetic materials. Those more mathematically inclined immediately appropriate Poisson's theory of induced magnetism, inventing P, D, and ϵ .

1838 - Faraday discovers Faraday's dark space, a dark region in a glow discharge near the negative electrode.

1839 - James **MacCullagh** invents an elastic aether in which there are no longitudinal waves. In this aether the potential energy of deformation depends only on the rotation of the volume elements and not on their compression or general distortion. This theory gives the same wave equation as that satisfied by E and B in Maxwell's theory. [interesting...]

1839 - **William Thomson (Lord Kelvin)** removes some of the objections to MacCullagh's rotation theory by inventing a mechanical model which satisfies MacCullagh's energy of rotation hypothesis. It has spheres, rigid bars, sliding contacts, and flywheels. [It derives its name from the title, Baron Kelvin of Largs, that Thomson received from the British government in 1892, and named after Thomson because of his proposal in this 1848 paper. Sir William was later named Lord Kelvin when he was made a Baron and a member of the House of Lords. He adopted the name **Kelvin** from the river Kelvin in Scotland with due regard to the association he had with Glasgow University and the area.]

1839 - Cauchy and Green present more refined elastic aether theories, Cauchy's removing the longitudinal waves by postulating a negative compressibility, and Green's using an involved description of crystalline solids.

1841 - Michael **Faraday** is completely exhausted by his efforts of the previous 2 decades, so he rests for 4 years.

1841 - James Prescott **Joule** shows that energy is conserved in electrical circuits involving current flow, thermal heating, and chemical transformations.

1842 - F. Neumann and Matthew O'Brien suggest that optical properties in materials arise from differences in the amount of force that the particles of matter exert on the aether as it flows around and between them.

1842 - Julius Robert **Mayer** asserts that heat and work are equivalent. His paper is rejected by *Annalen der Physik*.

1842 - Joseph **Henry** rediscovers the result of F. Savery about the oscillation of the electric current in a capacitive discharge and states, "The phenomena require us to admit the existence of a principal discharge in one direction, and then several reflex actions backward and forward, each more feeble than the preceding, until equilibrium is restored."

1842 - Christian **Doppler** gives the theory of the Doppler effect.

1845 - **Faraday** quits resting and discovers that the plane of polarization of light is rotated when it travels in glass along the direction of the magnetic lines of force produced by an electromagnet (Faraday rotation).

1845 - Franz **Neumann** uses (i) Lenz's law, (ii) the assumption that the induced emf is proportional to the magnetic force on a current element, and (iii) Ampere's analysis to deduce Faraday's law. In the process he finds a potential function from which the induced electric field can be obtained, namely the vector potential A (in the Coulomb gauge), thus discovering the result which Maxwell wrote as

$$\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t.$$

1846 - **George Airy** modifies MacCullagh's elastic aether theory to account for Faraday rotation.

1846 - **Faraday**, inspired by his discovery of the magnetic rotation of light, writes a short paper speculating that light might be electro-magnetic in nature. He thinks it might be transverse vibrations of his beloved field lines. [as opposed to a wave traveling in an ether?]

1846 - **Faraday** discovers diamagnetism. He sees the effect in heavy glass, bismuth, and other materials.

1846 - Wilhelm **Weber** combines Ampere's analysis, Faraday's experiments, and the assumption of Fechner that currents consist of equal amounts of positive and negative electricity moving opposite to each other at the same speed to derive an electromagnetic theory based on forces between moving charged particles. This theory has a velocity-dependent potential energy and is wrong, but it stimulates much work on electromagnetic theory which eventually leads to the work of Maxwell and Lorenz. It also inspires a new look at gravitation by William Thomson to see if a velocity-dependent correction to the gravitational energy could account for the precession of Mercury's perihelion.

1846 - William **Thomson** shows that Neumann's electromagnetic potential is in fact the vector potential from which B may be obtained via $\mathbf{B} = \nabla \times \mathbf{A}$.

1847 - **Weber** proposes that diamagnetism is just Faraday's law acting on molecular circuits. In answering the objection that this would mean that everything should be diamagnetic he correctly guesses that diamagnetism is masked in

paramagnetic and ferromagnetic materials because they have relatively strong permanent molecular currents. This work rids the world of magnetic fluids.

1847 - Hermann von **Helmholtz** writes a memoir "On the Conservation of Force" which emphatically states the principle of conservation of energy: "Conservation of energy is a universal principle of nature. Kinetic and potential energy of dynamical systems may be converted into heat according to definite quantitative laws as taught by Rumford, Mayer, and Joule. Any of these forms of energy may be converted into chemical, electrostatic, voltaic, and magnetic forms." He reads it before the Physical Society of Berlin whose older members regard it as too speculative and reject it for publication in *Annalen der Physik*.

1848-9 - Gustav **Kirchoff** [misspelled] extends Ohm's work to conduction in three dimensions, gives his laws for circuit networks, and finally shows that Ohm's "electroscopic force" which drives current through resistors and the old electrostatic potential of Lagrange, Laplace, and Poisson are the same. He also shows that in steady state electrical currents distribute themselves so as to minimize the amount of Joule heating.

1849 - A. **Fizeau** repeats Galileo's hilltop experiment (9 km separation distance) with a rapidly rotating toothed wheel. [speed of light] and measures $c = 3.15 \times 10^8$ m/s.

1849 - George Gabriel **Stokes** studies diffraction around opaque bodies both theoretically and experimentally and shows that the vibration of aether particles are executed at right angles to the plane of polarization. Three years later he comes to the same conclusion by applying aether theory to light scattered from the sky. This result is, however, inconsistent with optics in crystals.

The period 1850-1875

ca. 1850 - Stokes overcomes some of the difficulties with crystals by turning Cauchy's hypothesis around and letting the elastic properties of the aether be the same in all materials, but allowing the inertia to differ. This gives rise to the conceptual difficulty of having the inertia be different in different directions (in anisotropic crystals).

ca. 1850 - Jean **Foucault** improves on Fizeau's measurement and uses his apparatus to show that the speed of light is less in water than in air.

1850 - **Stokes law** is stated without proof by Lord Kelvin (William Thomson). Later Stokes assigns the proof of this theorem as part of the examination for the Smith's Prize. Presumably, he knows how to do the problem. Maxwell, who was a candidate for this prize, later remembers this problem, traces it back to Stokes and calls it Stokes theorem.

1850 - William **Thomson** (Lord Kelvin) invents the idea of magnetic permeability and susceptibility along with the separate concepts of B, M, and H.

1851 - **Thomson** gives a general theory of thermoelectric phenomena, describing the effects seen by Seebeck and Peltier.

1853 - **Thomson** uses Poisson's magnetic theory to derive the correct formula for magnetic energy:

$$U = \int \mu H^2 dV / 2.$$

He also gives the formula $U = LI^2/2$ and gives the world the powerful, but confusing, analysis where the forces on circuits are obtained by taking either the positive or negative gradient of the magnetic energy. Knowing which sign to use is, of course, the confusing part.

1853 - **Thomson** gives the theory of the RLC circuit providing a mathematical description for the observations of Henry and Savary.

1854 - **Faraday** clears up the problem of disagreements in the measured speeds of signals along transmission lines by showing that it is crucial to include the effect of capacitance.

1854 - **Thomson**, in a letter to Stokes, gives the equation of telegraphy ignoring the inductance: , where R is the cable resistance and where C is the capacitance per unit length. Since this is the diffusion equation, the signal does not travel at a definite speed. [transmission line theory]

1855 - **Faraday** retires, living quietly in a house provided by the Queen until his death in 1867. [good work!]

1855 - James Clerk **Maxwell** writes a memoir in which he attempts to marry Faraday's intuitive field line ideas with Thomson's mathematical analogies. In this memoir the physical importance of the divergence and curl operators for electromagnetism first become evident.

The equations $\nabla \cdot (\epsilon \mathbf{E}) = 4\pi\rho$, $\nabla \times \mathbf{A} = \mathbf{B}$, and $\nabla \times \mathbf{H} = 4\pi\mathbf{J}$ appear in this memoir.

1857 - Gustav **Kirchoff** derives the equation of telegraphy for an aerial coaxial cable where the inductance is important and derives the full telegraphy equation.

$$\partial^2 V / \partial x^2 = LC \partial^2 V / \partial t^2 + RC \partial V / \partial t.$$

He recognizes that when the resistance is small, this is the wave equation with propagation speed $v = \sqrt{LC}$, which for a coaxial cable turns out to be very close to the speed of light. Kirchoff notices the coincidence, and is thus the first to discover that electromagnetic signals can travel at the speed of light.

1861 - Bernhard **Riemann** develops a variant of Weber's electromagnetic theory which is also wrong.

1861, - **Maxwell** publishes a mechanical model of the electromagnetic field. Magnetic fields correspond to rotating vortices with idle wheels between them and electric fields correspond to elastic displacements, hence displacement currents. The equation for $\mathbf{H}$ now becomes $\nabla \times \mathbf{H} = 4\pi\mathbf{J}_{\text{tot}}$, where $\mathbf{J}_{\text{tot}}$ is the total current, conduction plus displacement, and is conserved $\nabla \cdot \mathbf{J} = 0$. This addition of a displacement current completes Maxwell's equations and it is now easy for him to derive the wave equation exactly as done in our textbooks on electromagnetism and to note that the speed of wave propagation was close to the measured speed of light. Maxwell writes, "We can scarcely avoid the inference that light in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena." **Thomson**, on the other hand, says of the displacement current, "(it is a) curious and ingenious, but not wholly tenable hypothesis."

1864 - **Maxwell** reads a memoir before the Royal Society in which the mechanical model is stripped away and just the equations remain. He also discusses the vector and scalar potentials, using the Coulomb gauge. He attributes physical significance to both of these potentials. He wants to present the predictions of his theory on the subjects of reflection and refraction, but the requirements of his mechanical model keep him from finding the correct boundary conditions, so he never does this calculation.

1867 - **Stokes** performs experiments that kill his own anisotropic inertia theory.

1867 - Joseph Boussinesq suggests that instead of aether being different in different media, perhaps the aether is the same everywhere, but it interacts differently with different materials, similar to the modern electromagnetic wave theory.

1867 - **Riemann** proposes a simple electric theory of light in which Poisson's equation is replaced by

$$\nabla^2 V - (1/c^2) \partial^2 V / \partial t^2 = -4\pi\rho.$$

1867 - **Ludwig Lorenz** develops an electromagnetic theory of light in which the scalar and vector potentials, in retarded form, are the starting point. He shows that these retarded potentials each satisfy the wave equation and Maxwell's equations for the fields and can be derived from his potentials. His vector potential does not obey the Coulomb gauge, however, but another relation now known as the Lorenz gauge. Although he is able to derive Maxwell's equations from his retarded potentials, he does not subscribe to Maxwell's view that light involves electromagnetic waves in the aether. He feels, rather, that the fundamental basis of all luminous vibrations is electric currents, arguing that space has enough matter in it to support the necessary currents.

1868 - **Maxwell** decides that giving physical significance to the scalar and vector potentials is a bad idea and bases his further work on light on E and H. [$A\mu$]

1869 - **Maxwell** presents the first calculation in which a dispersive medium is made up of atoms with natural frequencies. This makes possible detailed modeling of dispersion with refractive indices having resonant denominators.

1869 - **Hittorf** finds that cathode rays can cast a shadow.

1870 - **Helmholtz** derives the correct laws of reflection and refraction from Maxwell's equations by using the following boundary conditions: D_n , E_t , and B are continuous. Once these boundary conditions are taken Maxwell's theory is just a repeat of MacCullagh's theory. The details were not given by Helmholtz himself, but appear rather in the inaugural dissertation of H. A. Lorentz.

1870-1900 - The hunt is on for physical models of the aether which are natural and from which Maxwell's equations can be derived. The physicists who work on this problem include Maxwell, Thomson, Kirchhoff, Bjerknes, Leahy, Fitz Gerald, Helmholtz, and Hicks.

1872 - E. Mascart looks for the motion of the earth through the aether by measuring the rotation of the plane of polarization of light propagated along the axis of a quartz crystal. No motion is found with a sensitivity of $v/c = 10^{-5}$.

1873 - **Maxwell** publishes his Treatise on Electricity and Magnetism, which discusses everything known at the time about electromagnetism from the viewpoint of Faraday. His own theory is not very thoroughly discussed, but he does introduce his electromagnetic stress tensor in this work, including the accompanying idea of electromagnetic momentum.

The period 1875-1900

1875 - **John Kerr** shows that ordinary dielectrics subjected to strong electric fields become double refracting, showing directly that electric fields and light are closely related. [Kerr effect, an electric Faraday rotation effect]

1876 - **Henry Rowland** performs an experiment inspired by Helmholtz which shows for the first time that moving electric charge is the same thing as an electric current.

1876 - A. Bartoli infers the necessity of light pressure from thermal arguments, thus beginning the exploration of the connection between electromagnetism and thermodynamics.

1879 - J. **Stefan** discovers the Stefan-Boltzmann law, i.e., that radiant emission is proportional to T^4 .

1879 - Edwin **Hall** performs an experiment that had been suggested by Henry Rowland and discovers the Hall effect including its theoretical description by means of the Hall term in Ohm's law.

1879 - Sir William **Crookes** invents the radiometer and studies the interaction of beams of cathode ray particles in vacuum tubes.

1879 - Ludwig **Boltzmann** uses Hall's result to estimate the speed of charge carriers (assuming that charge carriers are only of one sign.)

1880 - Rowland shows that Faraday rotation can be obtained by combining Maxwell's equations and the Hall term in Ohm's law, assuming that displacement currents are affected in the same way as conduction currents.

1881 - **J. J. Thomson** attempts to verify the existence of the displacement current by looking for magnetic effects produced by the changing electric field made by a moving charged sphere.

1881 - George Fitz Gerald points out that J. J. Thomson's analysis is incorrect because he left out the effects of the conduction current of the moving sphere. Including both currents makes the separate effect of the displacement current disappear.

1881 - **Helmholtz**, in a lecture in London, points out that the idea of charged particles in atoms can be consistent with Maxwell's and Faraday's ideas, helping to pave the way for our modern picture of particles and fields interacting instead of thinking about everything as a disturbance of the aether, as was popular after Maxwell.

1881 - Albert **Michelson** and Edwin **Morley** attempt to measure the motion of the earth through the aether by using interferometry. They find no relative velocity. Michelson interprets this result as supporting Stokes hypothesis in which the aether in the neighborhood of the earth moves at the earth's velocity.

1883 - **Fitz Gerald** proposes testing Maxwell's theory by using oscillating currents in what we would now call a magnetic dipole antenna (loop of wire). He performs the analysis and discovers that very high frequencies are required to make the test. Later that year he proposes obtaining the required high frequencies by discharging a capacitor into a circuit.

1883-5 - Horace **Lamb** and Oliver **Heaviside** analyze the interaction of oscillating electromagnetic fields with conductors and discover the effect of skin depth.

1884 - John **Poynting** shows that Maxwell's equations predict that energy flows through empty space with the energy flux given by $\mathbf{E} \times \mathbf{B} / 4\pi$. He also investigates energy flow in Faraday fashion by assigning energy to moving tubes of electric and magnetic flux.

1884 - Heinrich **Hertz** asserts that E made by charges and made by a changing magnetic field are identical. Working from dynamical ideas based on this assumption and some of Maxwell's equations, Hertz is able to derive the rest of them.

1887 - Svante **Arrhenius** deduces that in dilute solutions electrolytes are completely dissociated into positive and negative ions.

1887 - **Hertz** finds that ultraviolet light falling on the negative electrode in a spark gap facilitates conduction by the gas in the gap.

- 1888 - R. T. Glazebrook revives one of Cauchy's wave theories and combines it with Stokes anisotropic aether inertia theory to get agreement with the experiments of Stokes in 1867.
- 1888 - **Hertz** discovers that oscillating sparks can be produced in an open secondary circuit if the frequency of the primary is resonant with the secondary. He uses this radiator to show that electrical signals are propagated along wires and through the air at about the same speed, both about the speed of light. He also shows that his electric radiations, when passed through a slit in a screen, exhibit diffraction effects. Polarization effects using a grating of parallel metal wires are also observed. [in other words, radio waves do the same things that light waves do]
- 1888 - **Roentgen** shows that when an uncharged dielectric is moved at right angles to a magnetic field is produced. ??
- 1889 - **Hertz** gives the theory of radiation from his oscillating spark gap.
- 1889 - Oliver **Heaviside** finds the correct form for the electric and magnetic fields of a moving charged particle, valid for all speeds $v < c$.
- 1889 - J. J. Thomson shows that Canton's effect (1762) in which a red hot poker can neutralize the electrification of a small charged body is due to electron emission causing the air between the poker and the body to become conducting.
- 1890 - **Fitz Gerald** uses the retarded potentials of L. Lorentz to calculate electric dipole radiation from Hertz's radiator.
- 1892 - Oliver Lodge performs experiments on the propagation of light near rapidly moving steel disks to test Stokes hypothesis that moving matter drags the aether with it. No such effect is observed.
- 1892 - Hendrik Anton **Lorentz** presents his electron theory of electrified matter and the aether. This theory combines Maxwell's equations, with the source terms ρ and J , with the Lorentz force law for the acceleration of charged particles: $ma = qE + q \mathbf{v} \times \mathbf{B}$. Lorentz's aether is simply space endowed with certain dynamical properties. Lorentz gives the modern theory of dielectrics involving D and P , and also includes the effect of magnetized matter. He also gives what we now call the Drude-Lorentz harmonic oscillator model of the index of refraction. But Lorentz's theory has a "stationary aether", which conflicts with the negative Michelson-Morley result.
- 1892 - George **Fitz Gerald** proposes length contraction as a way to reconcile Lorentz's theory and the null results on the motion of the earth through the aether. At the end of this year Lorentz endorses this idea.
- 1894 - **J. J. Thomson** measures the speed of cathode rays and shows that they travel much more slowly than the speed of light. The aether model of cathode rays begins to die.
- 1894 - Philip **Lenard** studies the penetration of cathode rays through matter.
- 1895 - Pierre **Curie** experimentally discovers Curie's law for paramagnetism and also shows that there is no temperature effect for diamagnetism.
- 1895 - **Lorentz**, in his "Search for a theory of electrical and optical effects in moving bodies" gives the Lorentz transformation to first order in v/c . The transformed time variable he calls "local time".
- 1895 - Wilhelm **Roentgen** discovers X-rays produced by bremsstrahlung in cathode ray tubes.
- 1896 - Arthur Shuster, Emil Wiechert, and George Stokes propose that X-rays are aether waves of exceedingly small wavelength. [so now we have light waves, radio waves, and X-rays all being the same thing]

1896 - **J. J. Thomson** discovers that materials through which X-rays pass are rendered conducting.

1896 - **Henri Becquerel** discovers that some sort of natural radiation from uranium salts can expose a photographic plate wrapped in thick black paper.

1896 - **P. Zeeman** discovers the splitting of atomic line spectra by a magnetic field.

1896 - Lorentz gives an electron theory of the Zeeman effect.

1897 - J. J. Thomson argues that cathode rays must be charged particles smaller in size than atoms (Emil Wiechert made the same suggestion independently in this same year). In response Fitz Gerald suggests that "we are dealing with free electrons in these cathode rays."

1897 - **W. Wien** discovers that positively-charged moving particles can also be made (the so-called canal rays of E. Goldstein) and that they have a much smaller q/m ratio than cathode rays.

1897 - **J. J. Thomson** deflects cathode rays by crossed electric and magnetic fields and measures e/m .

1898 - Marie and Pierre Curie separate from pitchblende two highly radioactive elements which they name polonium and radium.

1899 - Ernest Rutherford discovers that the rays from uranium come in two types, which he calls alpha and beta radiation.

1900 - Marie and Pierre Curie show that beta rays and cathode rays are identical.

The period 1900-1925

1900 - Emil Wiechert shows that simply replacing the distributed charge from Lorentz's theory with the charge of a moving point particle gives incorrect results. Instead the Lienard-Wiechert retarded potentials must be used.

1900 - Joseph Larmor obtains the second order corrections to the Lorentz Transformation.

1901 - R. Blondlot performs experiments that show that Lorentz's theory in which there is no moving aether gives the correct result in cases where the hypothesis of a moving aether gives the wrong result.

1902 - **Lord Rayleigh** performs experiments to test whether the Fitz Gerald contraction is capable of causing double refraction in moving transparent substances. No such effect is found.

1903 - The Hagen-Rubens connections between the conductivity of metals and their optical properties are experimentally established.

1903 - Lorentz gives the famous square root formulas for the Lorentz transformation giving the effect to all orders in v/c .

1904 - Lorentz gives his electron-collision theory of electrical conduction

1905 - H. A. Wilson performs experiments similar to those of Blondlot; again, Lorentz's theory is found to give the correct result.

1905 - Albert **Einstein** completes Lorentz's work on space-time transformations and relativity is born. [there is no ether, so there is no special frame in which the ether is at rest, so all frames are equivalent and relative]

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Ross Spencer
Tue Apr 13 10:47:17 MDT 1999

JACKSON

- John David Jackson - Classical Electrodynamics
3rd Ed.

Chapter 1: Introduction to Electrostatics

This is a pretty heavy-duty introduction! It sets up the entire mechanism for computing potential ϕ with the Dirichlet or Neumann boundary conditions using Green's Functions.

1.1 Coulomb's Law.

1.2 Electric Field. Of a point charge, then many point charges, then integral over ρ . Mention of the "statcoulomb" with mks put in appendix. Review of delta function technology.

1.3 Gauss's Law. Proves this for a point charge in the integral sense, result (1.11)

1.4 Differential Form of Gauss's Law. Jackson just quotes "the divergence theorem" without comment. At least Portis proves it. I remember being hit flat in the face by this strange new theorem. So $\nabla \cdot \mathbf{E} = 4\pi\rho$.

1.5 The Potential. Defines it for general ρ by saying $\mathbf{E} = -\nabla\phi$ and then ϕ is what it is. Since $\mathbf{E}$ has this form, we are guaranteed by vector identity to have $\nabla \times \mathbf{E} = 0$. Relation to work. ϕ as line integral of $\mathbf{E}$.

1.6 Surface charge σ and the Dipole Layer. Shows how to compute it using pillbox and get $4\pi\sigma = \Delta E_{\text{normal}}$. The **dipole layer** is defined as two σ layers separated by d such that $\sigma d = D$ as $d \rightarrow 0$, all functions of $\mathbf{x}$. So you deal only with the $D(\mathbf{x})$ result. Now $\phi = -\int D d\Omega$ and $\Delta\phi = 4\pi D$ across the layer.

1.7. Poisson and Laplace Equations. Poisson has ρ on the right, Laplace does not. Finds that Green's is $1/R$ see (1.31).

1.8 Green's Theorem Applied to ϕ . In the divergence theorem set $A = \phi\nabla\psi$ and you get the "first identity". Swap the functions, subtract, and you get the "second identity" we usually just call "Green's Theorem" (1.35). So we have just derived this from the divergence theorem. Now set $\psi = 1/R$ and $\phi = \phi$. Note that $\nabla^2\psi = 4\pi\delta$ while $\nabla^2\phi = 4\pi\rho$, and voila!, out pops the underpinning equation of electrostatics (1.36). It tells you how to find the potential given some ρ and some boundary conditions on a closed surface.

Interpretation is very important and very tricky here!

If point $\mathbf{x}$ is *outside* the closed surface, the derivation shows that the LHS of (1.36) is zero and so the two terms on the right must exactly cancel. In this case, you are getting no statement at all about what ϕ is inside or outside the surface. If the surface is just a mathematical one around some charge distribution ρ , all you can say is that the right term surface term will cancel the term on the RHS. Now the first term is the infinite-surface solution for the potential ϕ that we are familiar with, due to charges ρ . The second term can be *interpreted* as arising from a mysterious distribution of charge σ and dipole layer D on the surface that makes a potential which exactly cancels the first term. Suppose we really could put this distribution of real charge and D on such a surface, gluing it into free space somehow. Then if we went outside (now with an infinite surface beyond us), we would add the potentials from the original ρ term and then from the *real* surface σ and D terms, these would cancel, and we would *conclude* that $\phi = 0$

everywhere outside the surface. We can then reverse this situation by saying the following. Suppose we have a cavity in grounded metal which matches our surface. We *know* that $\phi = 0$ everywhere outside the surface! Then we can conclude that the σ and D in this case *really exist* on the inside surface of the cavity and these real things are what cancels the ρ -generated potential! { I am not quite sure what the D term means, although I know what D means; the σ term means real surface charge. }

If point x is *inside* the closed surface, we get an equation for ϕ which is the usual ρ term plus the surface terms. If we have a mathematical surface only, which supports no charge or D , then the surface term will give exactly 0 (assuming there are no metal surfaces anywhere). But if we are in a metal cavity and we have real σ and D on the metal surface arranged as described above so they cancel our ρ potential on the outside, then this equation tells us how to find the complete ϕ inside the cavity, and in this case the second term will NOT be zero.

We have a seeming paradox here. We want the surface terms to be 0 if we have x on the inside of a *purely mathematical* surface, but we want the surface terms to be non-zero and exactly cancel the ρ term if we are on the outside of the surface, because in this case the theorem says the LHS = 0. So how can the same surface terms be 0 in one case and non-zero in the other case! The answer is that the surface term integral really is different depending on whether x is inside or outside. I did not understand this until I did the following example case, and then it became clear.

Example: A point charge q in the center of a sphere of radius a . In this case, $\mathbf{r}'$ is the integration variable that wanders over the surface and we can replace da' with $a^2 d\Omega'$. We choose the z' axis along the direction of vector $\mathbf{r}$. We then replace da' with $2\pi a^2 dx'$ where $x' = \cos\theta'$. Here are some intermediate steps in computing the surface integrals:

$$\partial/\partial n' = \partial/\partial r' \quad R^2 = a^2 + r^2 - 2ar \cos\theta' = A + Bx' \quad A = a^2 + r^2 \quad B = -2ar$$

$$\text{and } \partial/\partial n'(1/R) = -(a - rx')/R^3 \quad \text{at the surface}$$

We know with a purely math surface that $\phi = q/r'$ at the surface, so

$$\partial/\partial n'(\phi) = -q/a^2 \quad \text{at the surface}$$

So the complete surface business is this:

$$(1/4\pi) 2\pi a^2 \int dx' \{ (-q/a^2)/R - (q/a) [-(a - rx')/R^3] \}$$

where of course $R = R(x')$, and integrate -1 to 1. There are three terms to integrate here. If we plug this integral symbolically into Maple, we get a pretty big mess which involves $\sqrt{A+B}$ and $\sqrt{A-B}$. We know that $A > B$, so that is not the problem. This is where the situation $a > r$ or $a < r$ makes a huge difference. We have:

$$\begin{aligned} \sqrt{A+B} &= \sqrt{(a-r)^2} = \text{sign}(a-r) * (a-r) = |a-r| & // \text{critical thing is sign}(a-r) ! \\ \sqrt{A-B} &= \sqrt{(a+r)^2} = (a+r) & // \text{no confusion here} \end{aligned}$$

When $r < a$ (meaning our point x is inside the surface), the entire surface integration gives 0, showing us that our answer is just $\phi = q/r$ from the ρ term. On the other hand, when $r > a$ (meaning we are on the outside), then the big mess comes out being $-q/r$ and exactly cancels the ρ term!

The main point is to understand that, although the surface integral "looks the same" in both the x inside and x outside cases, it is not the same!

Note: Regarding the integrand $\sqrt{A+Bx'}$, it has a branch cut to the left of -1 which lies outside our integration range. We are working on the real analytic sheet where we take the $+$ square root, the other sheet has the $-$ square root, we never go there. When we have to deal with the endpoint value $\sqrt{(a-r)^2}$, we must therefore take the positive square root which is $|a-r|$. There is no more to it than that!

This same situation occurs in a more trivial situation. Imagine computing the potential inside a sphere of uniform surface charge σ . We do exactly as above and we get $\int dx' 1/R$ and we get an answer that looks like $\{|a-r| - (a+r)\}/2ar$. If $r > a$, we get a $1/r$ answer, and if $r < a$ we get a constant $1/a$ answer -- the potential is constant inside the sphere. At exactly $r=0$ both answers are the same.

Now we come to another main point. In general, for x inside, we should just regard (1.36) as a complicated "integral equation" that we have to solve for ϕ . But we look at the equation and wonder: is it possible that we can take some forced values for ϕ and ϕ' on the surface, and then use the equation to compute ϕ everywhere inside (also knowing ρ of course)? If so, that would be a powerful tool, and this is the subject of the next section.

1.9. Comments on Solutions in the Dirichlet, Neumann and Cauchy cases. It turns out that if you try to force both ϕ and ϕ' on a closed surface (Cauchy), you have "overspecified" the problem and there is no solution generally speaking. However, if you specify *either* ϕ (Dirichlet, the usual case) *or* ϕ' (Neumann) on the closed surface, then in fact this *does* completely determine ϕ inside, but you have to do some work to find the solution, which we will do below.

A table on page 17 summarizes things, and I put now some older notes right here.

In statics, you have the wave-equation with $k=0$ ($\lambda = \infty$) with ρ as the driving source [known as Poisson], as in (1.28), and the general solution is (1.17) where $1/R$ is the Green's function or propagator (think $\exp(ikR)/R$ with $k=0$). This is the "particular solution", and you can add solutions to $\nabla^2\phi = 0$ [known as Laplace], as in (1.36), in order to find a complete solution that matches some boundary conditions. The two boundary surface terms can be interpreted as from an effective surface charge and "dipole layer", page 15. [see discussion above!]

In the boundary conditions, you are either specifying the normal E field at the surface (Neumann), or you specify ϕ itself (Dirichlet). If you specify either one on a closed boundary, you get a unique and correct solution, but if you try to specify both (Cauchy), it is "too much" and the only solution is 0. Specifying any of these three on an open (partial) surface is "not enough" to get a solution. This is shown in the first column of the table on page 17.

Notice the second column which applies to "hyperbolic" ODE's like the wave equation with $k \neq 0$. In this case, specifying *anything* on a closed surface is too much, and the only chance you have is doing Cauchy (ie, specifying ϕ and ϕ') on an open partial surface. In Kirchhoff scalar diffraction theory (field ψ) applied to an aperture, you are in effect trying to specify ψ and $\partial\psi/\partial n$ on a *closed* surface (in the hole, you are assuming you have the incident field unaltered, on the screen you assume 0, on the great sphere you assume 0). Thus you are trying to do Cauchy on the wave equation, and that is known to only give a 0 solution. The fix is that in the aperture, things are not quite unaltered!

The third column applies to ODE's like the heat equation which I have never really studied! Jackson does not prove these things, but refers us to famous historic sources like Morse & Feshbach, and Sommerfeld himself.

1.10. How to solve for ϕ with boundary conditions using the G method. The basic problem is that you would like to eliminate one or the other of the two surface terms mentioned above. Earlier we used Green's Theorem with $\psi = 1/R$ and $\phi = \phi$. If we instead use $\psi = G(x,x')$, we get (1.42). If we could somehow solve this for $G(x,x')$, obviously we can find ϕ . If we can somehow arrange for G to both solve the Poisson equation AND to be 0 on a surface, then we will have eliminated the first surface term, and we can then get our answer by knowing only ϕ on the surface (that is, we don't have to preknow ϕ' , nor do we want to know it if we know ϕ there). So this is method to solve a Dirichlet problem!

On the other hand, suppose we could find a G with $G'=0$ on the surface. That would appear to eliminate the second term in (1.42). HOWEVER, this leads to an inconsistency! It is easy to show that the surface integral of G' must be -4π , so you can never have $G'=0$. The fix is to set $G' = -4\pi/\text{Area} = \text{a constant}$. If you do this, then the second surface term in (1.42) does not quite vanish and you end up with an extra term $1/\text{Area} * \int \phi \, dA$ which is the average of ϕ on the surface. This is the "extra" first term you

see in (1.46). If we agree to only do Neumann problems working in the space between a finite surface and an infinite one, which together make up our surface S , then this average ϕ term vanishes. This is tricky, though, because you have to remember that your true "interior" region is now "exterior" to the inner surface.

Finally, Jackson in (1.40) writes a general form for the Green's function which solves (1.39). the usual $1/R$ is the particular solution, but F can be in principle any solution of Laplace! Our game is going to be finding $F(x,x')$ that gives us stuff like $G = 0$ on some surface. The term F can be directly associated with the induced surface charges in a problem with conductors.

1.11 Electrostatic potential energy and energy of the electric field. Jackson starts by doing the simple not-equal sum between a set of charges to add up the potential energy as in (1.50). For continuous this appears as 1.52 and he notes that now we have included the diagonal or self-energy stuff. But 1.52 gives 1.53 where you integrate $\rho\phi$, and then you fiddle easily to show 1.54 which we know well, that the density is $|E|^2 / 8\pi$. He shows with an example what must always be true, that the self energy terms will make sure the energy density can never be negative, though the non-self-energy component can in fact be negative. He mentions that you can use the virtual displacement idea to see how energy changes and thus do a force computation.

We then have 13 problems. I seem to have done 7 of them during the course!

I notice that Jackson is in 3rd edition 1998 at \$94 Amazon. It seems to me he has added a lot more applications, in Portis style, and has gotten rid of the entire multipole expansion chapter 16 and replaced it with some stuff about radiation damping. Maybe he decided the multipole was just too hard! Otherwise, the 3rd edition has the same chapters and basic contents as my 1st edition.

Chapter 2: Green's Function solutions for Metal Sphere, and other things

Images is really the Green's thing in disguise. Why charges can't leave a charged sphere. Inversion. Then orthogonal functions and separation of variables.

2.1 Method of Images. The idea here is that you consider a point charge near some metal conductor, and you know that in this case, $\phi(x) = qG(x, x')$ where x' is the location of the charge. Think of image charges as creating the $F(x, x')$ discussed earlier. You then carefully select the image or images to make $G = \phi = 0$ on your conductor surface. If you can find a set of image charges that do the trick, then you have found A solution for ϕ in a volume bounded by a surface on which ϕ is known. Since we know that in this case (ϕ known on a closed surface, Dirichlet) the solution is unique, then the point charge + image charge configuration must be THE solution to the problem. A less fancy way to think of image charges is that you select them to make field lines perpendicular to the metal surface so there is no tangential E field. The usual situation is a point charge above an infinite plane, then the image has opposite charge and is an equal distance behind the surface. As usual, the field implied in the image problem "on the other side of the boundary" is fictitious and has no bearing on the physical problem.

2.2 Point Charge near Grounded Metal Sphere. I never realized until just now the significance of this problem and how it relates to something I recently was wondering about. Here is the logic flow. First, Jackson puts a point charge q distance y from the center of a grounded sphere of radius a , and puts it outside so $y > a$. He shows that you can get $\phi = G = 0$ on the sphere by assuming one image charge inside the sphere whose size is $-q(a/y)$ and whose distance from sphere center is a^2/y . The potential solution is then given by the first 2 terms of (2.8), and you can then go and compute whatever you want. In this solution, the grounded sphere brings in from ground a total charge of $q' = -q(a/y)$, as Gauss's law tells you. Again, we see that the image charge is simulating what the surface charge distribution really does.

2.3 Point Charge near charged, insulated metal sphere. Suppose you add to the above solution a sphere with total charge $Q_1 = Q - [-q(a/y)]$. The total solution then has Q on the sphere and solution (2.8) with the third term. This is then the solution you get by putting a point charge outside a charged insulated sphere.

But why is this interesting? The potential is the sum of three terms, and if you do $\nabla\phi$, you get the electric field ($F(x)$) at all points in space, and it is a function of a , y , and Q . However, if you want to know the force on the point charge q , you only consider the last two terms in (2.8) and set $x=y$, and this then gives (2.9). The point charge feels a force from the Q at sphere center, and from the image charge which lies under the sphere surface below the point charge. The distance between the point charge and the image charge is given by $y - a^2/y$. As the point charge approaches the surface, the image charge approaches from the other side and it's force dominates. The image charge always has a sign opposite q . If Q and q have opposite signs, then q is attracted by the central Q and by the image charge. The more interesting case is when Q and q have the same sign. Then q is repelled by the central Q , but attracted by

the image charge. The two forces are equal when q lies about $(a/2)\sqrt{q/Q}$ above the sphere surface in the case $Q \gg q$. For an electron over a macroscopic sphere of some reasonable charge, this is a very small distance. As q approaches the surface, the image charge wins out (the reforming surface charge, that is to say) and the force becomes infinite right at the surface. So here is the point: an individual electron on a charged sphere is held on by an infinite restoring force if it tries to leave. No doubt if you work with a non-idealized metal surface, you will find the force is finite, and a finite energy can get an electron over the equilibrium hump and this is called the work function. Not too long ago I made a bogus argument about why electrons did not leave a charged metal object in which I referred to the 1 eV chemical idea. I thought the electron was held by its chemical binding stability energy. That might be true, but this image charge force seems much more significant.

This whole discussion could have been tried with regard to a charge over a flat metal surface, but there it does not work so easily. You could imagine an insulated large round plate that is grounded and then you get the image charge solution assuming very large radius. But then you think of that plate as having charge Q in addition and you compute ϕ of this, things have log divergence, not very nice. You have to put in a cutoff radius R , but I think the whole thing can be made to work. For radius R of the plate, I think this is the potential the point charge sees from the image and the plate:

$$\phi(y) = -q/(2y) + (Q+q)/(\pi R^2) [\sqrt{R^2 + y^2} - y]$$

If I set $\phi'(y) = 0$ with large Q , I find that $y = R\sqrt[3]{\pi q/2Q}$ is the equilibrium point. Yes, it is quite ugly. The problem here is that distances do matter, you cannot just ignore far away things and say that locally the surface is flat and we will use the "flat solution". Jackson solution diverges for $a \rightarrow \infty$ as well. So there sphere is a very instructive case indeed that shows clearly the principle involved. In other cases, the same general idea will probably be true, but much harder to compute.

2.4 Point charge near a metal sphere of fixed potential. This is identical to the previous problem except you replace $Q+aq/y$ with Va in the third term. We are just superposing another sphere on the original problem. We have now changed the y dependence of everything a little by doing this.

2.5 Metal Sphere in a uniform E field. Uses image charges at $\pm R \rightarrow \pm \infty$ to simulate the E field and finds that the sphere acts as a dipole object. So this example uses 2 image charges and no point charge to start with. Resulting σ on the sphere is a very simple $\cos\theta$ form suggesting dipole.

2.6. Method of Inversion. Equation (2.17) with (2.18) makes the basic symmetry theorem. You can take the known solution of one problem and "invert it" to apply to another problem. An example is that you can solve the sphere above a plane in this way as in Figure 2.10. There are tricky points here, I have not studied it. I think this is a special case of the more general conformal mapping ideas that Jackson gave us notes on and probably has added to his later editions?

2.7 Green's function for a Sphere. We outlined the idea in the last chapter: find G that vanishes on a sphere, then use that in (1.42) to find ϕ given some value of ϕ on the sphere. We already know the right G from our sphere image solution, it is (2.22) which becomes (2.23) in polar. The derivative is (2.24). Since there is no ρ in the closed region (which, by the way, is exterior here to the sphere), only one term of the three in (1.42) survive and it is shown in (2.25).

2.8 Application to two hemispheres at +V and -V. Jam everything in, get (2.27) which, I am happy to see, "cannot be integrated in closed form". He can get the result on-axis, and he can power-series expand things and he gets something that looks like multipole, but that idea has not been set up yet.

2.9. Orthogonal Functions and Expansions. Covers the basic idea of expanding $f(x)$ in terms of some orthogonal functions $\phi_n(x)$, the orthonormality condition, the completeness condition. Fourier Series and Integral are stated. I am pretty solid on this stuff.

2.10 Separation of Variables. The problem is a cube with 5 faces at $\phi=0$ and some $V(x,y)$ on the top surface. The solution is to use cartesians, separate variables, find some basis functions that satisfy the 5 zero conditions, then force the last one by summing with coefficients as in (2.63) with coeffs in the next equation. The comment made is that you often end up "making your own" custom set of ortho-functions for a particular class of problems.

11 problems, I have done 4 it would appear.

Chapter 3: Solving spherical and cylindrical electrostatic problems.

This chapter talks about the technology needed for working in the (r, θ, ϕ) and (ρ, z, ϕ) worlds. In each world, the radial equation implies certain special functions (Legendre and Bessel). The final section treats the flat metal charged disk as an example of Cauchy mixed boundary conditions.

3.1 Laplace Equation in (r, θ, ϕ) . Assume (3.2) as separated form, find the three separated ODE equations. ϕ is easy and causes quantized integral parameter m . The r/θ separation constant is written $l(l+1)$ and it is later shown why l must be an integer. So now we have the three equations, and we have solved one of them. The radial equation is trivial and the solution is r^l or $1/r^{l+1}$ as in (3.8). So only the θ equation requires work, and this is Legendre.

3.2 Legendre Land. The θ equation is written with $x = \cos\theta$, but here only in the case $m=0$. Assume $+x$ power series, get recursion for coefficients, discover that to get convergence at $x = \pm 1$ you need truncation, and that in turn makes l be integral. The solutions for given l are the famous $P_l(x)$. The second kind solutions don't fall out here because the power series assumes convergence at $x=0$. The $P_l(x)$ form an ortho set, all the properties are given and some are derived in detail. Page 59 shows the expansion and coefficient formulas, the expansion and projection as I like to call them.

3.3 Azimuthal Symmetry problems in (r, θ, ϕ) . This makes $m=0$ so we then have the $P_l(x)$ and the absolutely most general solution is (3.33), I agree. This makes it easy to solve near a sphere with an arbitrary $V(\theta)$ potential on it, specialize to 3.34 and you are done as in 3.35, at least as a series. We now do the $\pm V$ hemispheres for a second time and get (3.37), a more organized answer than last time.

A nice trick is this. Suppose you know $\phi(z)$ only on the symmetry axis where expansion reduces to 3.38 since $P_l(1) = 1$, the forward direction. Then use that data to get the coefficients, and add back the P_l and you have your full answer! We found the axis solution exactly for the $\pm V$ problem, so we can do that here as a third way to solve this same problem!

Next, $1/R$ is expanded in $P_l(\cos\theta)$ as shown in (3.41) and a proof is given. Recall the $r_<$ notation trick! His proof is just to look on the z axis and get a series there, then add P_l back in.

Another sample problem: find ϕ near a ring of charge, page 63. Again, we know it easily on the axis. Expand and answer is (3.46) on the axis still, then add $P_l(\cos\theta)$ and 3.48 is all yours. I liked it last time, and I still like it.

3.4 The $P_m^l(x)$ and $Y_{lm}(\theta, \phi)$ Worlds. You need this for no azimuthal symmetry. All just the basic nuts and bolts. Our same most general expansion is now written as (3.61). Jackson is trying to throttle the rate at which complexity builds up, not hitting the poor students with m right off the bat.

3.5. The Y_{lm} Addition Theorem. The thing is shown in (3.62). The important point is that this lets you *separate* the variables of the two vectors which are making the $\cos\theta$ angle. When you then stick this expansion into our $1/R$ Legendre expansion, you get the glorious (3.70), which provides separation in all three variables, at the cost of a messy double sum. Still, it lets you do something when you might otherwise be blocked. You can bet this result will play a role in the multipole field expansion!

3.6 Cylindrical Symmetry and Bessel World. We start over, separate in z, ρ, ϕ . We get two constants as before k^2 and ν^2 . The ϕ and z equations are both trivial giving exponents as in 3.76 implying $\nu = m = \text{integer}$. The ρ equation then ends up as Bessel's Equation where the k constant is absorbed into the argument $J_\nu(k\rho)$, so $x = k\rho$ in this world. We get various J properties. For *integral* ν we need the second kinders $N_\nu(k\rho)$ which are the Neumann's. The Hankels are also defined (third kind). Various properties of things are given, very useful and here all in one place. Now when you try to make orthonormal functions over some range $0, a$ for ρ , you end up using $J_\nu(k\rho)$ with $k = \text{zeros}/a$. This is needed to make parts go away in the proof of orthogonality. This is called Fourier-Bessel series, I have never used it in any of my doings. Jackson finally mentions some other versions of Bessel series with names Kapteyn, Schlomilch, and Watson's Bessel book is where to look for this stuff. We assumed non-phase exponent for z , and this led to soft Bessels in the other direction. If we assume soft exponent phasor form in z , then we get the hard other kind of Bessels, the I and K functions, so-called "modified Bessel functions". Hard means they blow up in some direction, soft means sine like.

3.7 Cylinder/Bessel problems. Cylinder with $\phi=0$ everywhere but prescribed to a function on one end. Then an application of the Fourier-Bessel series. Skip this stuff, use when you need it! I don't think we will be doing our scattering problem in cylindricals, but if so, come back here.

3.8 Doing a general Green's analysis in r, θ, ϕ . We did the Green's deal in Dirichlet for a sphere (find G such that $G=0$ on a sphere) and the image charge trick made short work of it. But suppose you have some other spherically symmetrical situation, such as concentric spheres. The general form of G is (3.118) with (3.119), that is, a double harmonic expansion with some radial green's function $g(r, r')$ that we have to figure out for our geometry. For the single sphere that function is shown in 3.114, which we can think of as the $1/R$ factor (in its fancy expansion) plus F as the expansion of the image charge in a similar fashion. For the concentric spheres case, the resulting g is the much messier (3.122), and the full Green's is shown in 3.125. For this case, you would need an infinite set of image charges to get the same result, Jackson claims. Remember, you are trying to make $G=0$ on both spheres at the same time! This section also gives some useful delta function stuff in the spherical coordinates, page 79. This is a pretty tough row to hoe! Each sample problem is a whole paper you could write.

3.9 Spherical Problems. We did Spherical setup work above, then we did cylindrical setup work, and then cylinder problems, but we never did spherical problems in the general case, so here we are now. This is getting closer to my main interest theme right now which is applying multipole to plane wave scattering, but at the moment we are back in Electrostatic Land and we are about to do something here along these lines.

Here, we are first reminded of the general Dirichlet 2-term result (3.126). The first term is the p term if you have any, the second term is integrating a prescribed ϕ over your surface with dG/dn as a factor. In 3.127 we compute this dG/dn for sphere(b), then 3.128 gives the non- p term of the 3.126. So in the following two examples, we use a grounded sphere and practice doing the p term. In the general case, you can superpose this solution with solution to a sphere with some general ϕ on its surface. For example, you could do a ring-charge inside the now-infamous $\pm V$ sphere.

Example 1: put a ring of charge inside a grounded metal sphere. In this case of course the surface integral is 0, and we are going to practice doing the pG integral term. This p is given with delta functions which in turn kill off the dV' integration in the p term integral (ϕ is just 2π) giving 3.130. Note that one delta is forcing $\cos\theta' = 0$ since charge ring is in that plane, hence then original $Y_{lm}(\theta', \phi')$ within G gets

pinned to $Y_{lm}(0, -)$ and $m=0$ due to symmetry and that is why you see $P_l(0)$ sitting in 3.130. The other P_l is of course from the other Y factor in G . So 3.130 is the solution to our problem, a sum over ℓ .

Example 2: put a diameter line charge inside a grounded metal sphere. We have a new ρ with its deltas, and we take note of the $1/r^2$ in these forms. The result is 3.133. Since ρ in this case only has delta in θ' , we still have the r' -integration to do and the final result is 3.136. Jackson notes that this thing actually diverges on the z axis (it should, that is where the charge is), BUT manages to be 0 on the sphere, meeting the $\phi=0$ requirement there (grounded sphere).

3.10 Green's for (ρ, z, ϕ) . Write the usual equating defining G . Expand both sides. The RHS is obvious, the LHS expansion of G shown in 3.140 is justified because we say that the cylindrical solutions had $\exp(ikx) \exp(im\phi)$, where we have now used the I, K sign of k^2 . We have thus "separated variables" in G and have $g(\rho, \rho')$ to figure out. In a typical tour-de-force, Jackson shows that the solution to the radial ρ equation is $g(\rho, \rho') = 4\pi I_m(k\rho_<) K_m(k\rho_>)$. Recall that I is the one that is best-behaved at $\rho=0$ and no surprise that it is connected with $\rho_<$. So we can jam this into 3.140 to get the cylinder Green's G . But we know that this is $1/R$ as well, so we get the 3.148 expansion of $1/R$. This result is pretty messy! Jackson then shows that if you compute $\ln(1/R)$ from this messy thing, you get the relatively simple expansion shown as 3.152. Very nice. No examples however.

3.11 The Eigenfunction Method. This is a quantum-mechanics Schroedinger-equation approach. Take your ODE as in 3.153 and find some eigensolutions as in 3.154 which have some desired boundary conditions, such as vanishing at the walls of a box that you want to study. The greens must at once be of the general form 3.157 expanding onto the eigenfunction basis. Goof around and end up with 3.160. One point not quite made clearly is that you might have $\lambda=0$ in 3.153 (as in the Laplace equation), but you will still of course have non-zero eigenvalues in 3.154. In particular, for that mentioned box in cartesians, your eigenfunctions are 3.166 and λ_n as in 3.165, although there is no $\lambda = k^2$ in the Laplace. That is why we see just λ_n^2 in the denominator of the final G for this situation in 3.167, call this example 1. For example 2, we take the same $\lambda=0$ Laplace equation, but instead of a box we do infinite space. In this case we get that $\lambda_n = k^2$ which is now going to be a continuous eigenvalue, and 3.160 becomes 3.164. This is an interesting way to interpret this nice form for $1/R$. Example 3 is to do the thing incylindricals, and we then get $G = 1/R$ expressed as 3.168.

Comments on this method? The idea is to decide ahead of time on some boundary conditions, then solve for eigenfunctions by insisting on these boundary conditions for all the eigenfunctions, and by inserting some λ_n into your original equation, which really comes from separating the variables, even if $\lambda = 0$ in the full equation for G . That is, we get a separate "wave equation" for each variable-separated function. The full original equation still has $\lambda=0$. We know what each equation looks like from our earlier work, and only one of the three variables will be tough. Notice from page 48 that the sum of the effective k^2 values has to be zero, so if two of them are phasor type, the third must be hard expo type. You cannot have all three be phasor type because they would then add up to a negative value, not 0.

3.12 An Example of mixed (Cauchy) boundary conditions. The problem is a charged, insulated metal disk, sounds simple enough! We take as our closed surface an infinite hemisphere on one side of the disk. We know that $\phi = \text{constant}$ on the disk and $\phi=0$ on the infinite half-sphere, but we don't know ϕ in the disk plane outside the disk! However, we do know $\partial\phi/\partial z$ outside the disk, the along-axis E field, and that must be zero by symmetry. So here we have a Cauchy situation that really was not mentioned in

Jackson's table. We really do have a closed surface, and we have ϕ prescribed on some of it, and ϕ' on the rest of it.

OK, next since we know there's no ϕ dependence we specialize the cylindrical expansion to get 3.170, but we don't yet know what $f(k)$ is. We quickly get the two "integral equations" for $f(k)$ shown in 3.173 from the mixed boundary conditions. Jackson is forced to just tell us that the solution is 3.176 without a derivation, but of course we can verify that it works. This gives 3.177 for the full solution. The integral there can be done to give the final form 3.178, a nice compact closed-form solution to this problem with no special functions required! Jackson computes σ on each side of the disk (same on both sides of course) and finds a spike at the edge as you would expect. It is as if there were some kind of centrifugal force pushing the charge out to the edge to maintain no radial E field. Jackson finally notes that the problem can be exactly solved in elliptical coordinates, something he treats does in this book.

Fourteen problems, of which I have done 5. Stupidly, I did not keep my problem sets, is that possible?

Chapter 4: The Scalar Multipole Expansion, and Dielectrics

It is a little odd that these two subjects are mixed in one chapter. It is true, however, that dielectric is really based on one of the multipole moments (dipole), so it is good for the reader to at least know what these moments are! Jackson treats the usual set of dielectric problems, but also delves into models for χ and for the microscopic χ which he calls γ . This latter is treated for both polar and non-polar substances. This is a very good and useful chapter.

4.1 The Multipole Expansion. Finally we are getting where we want to be. We already know from 3.61 what the most general form of a potential is in spherical coordinates. You can pick whatever coefficients you want here, and the result will formally satisfy the Laplace Equation. Notice in 3.61 that we have the two characteristic radial functions r^ℓ and $r^{-(\ell+1)}$.

Now, we imagine some ρ distribution that is all contained within some mathematical bounding sphere. Then we know that *outside* that sphere, we cannot have r^ℓ terms in the potential because they will diverge at ∞ , so we only have the $r^{-(\ell+1)}$ terms, so we write 4.1 as shown. This equation is THE MULTIPOLE EXPANSION of the (scalar) potential. That is, you are expanding it onto eigenfunctions of the Laplace equation in spherical coordinates. We know this is a complete expansion, nothing is left out, and it is just a question of finding the coefficients which are here called $q_{\ell m}$. No surprise, these are given in 4.3 and you see that you are just integrating your friendly $Y_{\ell m}$ with r^ℓ against ρ over the sphere that contains all the action. For $\ell = 0$ we get the charge. For $\ell = 1$ we get the electric dipole as shown, although the p things appear as raising and lowering operator type things due to the way sphericals work. As for the $\ell = 2$, the Q_{ij} are what you get from a certain cartesian Taylor expansion of ϕ shown in 4.10. To get this result, you first expand $1/R$ as $(1/r) [1 - \mathbf{r} \cdot \mathbf{r}' / r^2 + \text{etc}]$ where $R = |\mathbf{r} - \mathbf{r}'|$ and we treat r' as small and just do a regular Taylor series. Then insert this into $\phi = \int \rho / R \, dV$ and you have it.

So that is really all there is to it! The idea is then to think of the specific ℓ, m terms in the expansion by themselves, call these things $\phi_{\ell m}$. Each one implies its own electric field $\mathbf{E}_{\ell m}$ as in 4.11. For $\ell = 1$ we get the usual dipole result shown in 4.13, not quoted yet in this text.

4.2. Multipole Expansion of Interaction Energy. The potential energy of a charge q in an external potential Φ is just $q\Phi$. This would be the work needed to bring in q from infinity to some location $\mathbf{r}$. The work to bring in two charges would be $(q_1 + q_2)\Phi$. Note that this does not include the "self energy", the work needed to assemble q_1 and q_2 near each other without Φ present, which is certainly non-zero. I think then that $\int \rho \Phi \, dV$ is the energy of some charges just due to the external field, and if you wanted the total energy you would have to compute $\int \rho(\Phi + \phi) \, dV$ where ϕ was the potential just due to ρ with no external Φ . Jackson is thinking of this $\int \rho \Phi \, dV$ as the "interaction energy" between an existing ρ and some externally generated ϕ , a sort of interaction Hamiltonian or Lagrangian type thing. If this integral is broken into multipole terms, you get 4.17 which shows how each moment of ρ interacts with the Φ field. As the moments go up, the derivatives of ϕ go up. Thus, you could rule out a quadrupole interaction if

you knew that your external E field had no gradient. Note that this is not the case in a plane wave in the z direction, but we really haven't done waves yet, so let's hold off on this idea.

Page 102 then gives a nuclear physics application. A nucleus has a quantum state and is expected to have a q_{20} moment. A nucleus in a crystal is expected to see a macroscopic ϕ which has first and second order gradients, so there is expected to be an interaction between the crystal field and the nuclear Q moment. In QM we know that this interaction will split the otherwise degenerate m levels, akin to the Zeeman effect for electrons as I dimly recall. So RF techniques can then be used to measure the Q moments in this way, given a knowledge of the local E field.

Jackson's last gasp in this section is to comment on the interaction between an external dipole field and an acted-upon dipole field, we get 4.19. I am not sure why we care about this but OK, maybe he will need it later for something. He might have said something about the dipoles of water molecules here, but he did not.

4.3 Dielectric Materials. This is certainly a change in topic within this chapter, but it is also something I want to know about, since I am thinking of floaters as dielectric spheres.

This discussion is VERY carefully constructed. We start with some microscopic Maxwell equations 4.20 where ϵ and ρ' are the detailed microfield and charge distributions, a total mess. We know that we have to get into average fields to work at a macroscopic scale. We are going to average the micro field and the micro ρ as shown in 4.22. The starting point is 4.23, standard E from ρ , a superposition of point charges if you like. We then do our multipole expansion of the ϵ field. This just means do an expansion of $1/R$ and put it in 4.23. We did this in 4.10, the cartesian Taylor expansion through quadrupole. Jackson leaves ∇ on $1/R$, there is not reason to evaluate it right now. Here, he drops quadrupole and above, with the claim that since field variations are larger than atomic size (5000 Å larger than 1 Å), you won't be doing much quadrupole action. This claim is not supported very well, but it must be true, because you don't see anyone talking about some kind of Q polarization tensor on a footing with P as being important in a dielectric. So things are simplified a lot, we only have two terms as in 4.26. Here Jackson is summing over distinct charges e_j . He replaces those with a fake integration in 4.28 by using delta function ρ and π , where now π is a microscopic level sea of little dipoles as in 4.27, just as I would do this. The subscript "mol" is just to remind us that these are not the macroscopic ρ and P . Jackson likes to think of "molecules" instead of "atoms" which is fine. We now take 4.28 and apply the averaging operation over a volume as shown in 4.22. He treats the ρ term, and says the p term works exactly the same way. In 4.30 the averaging is applied and we move it to the right, where of course it just gives the "average charge" in the volume due to the delta. But then you can write this as the molecule density N times the average charge per molecule in 4.31, all just words, nothing tricky is happening here. The net result is that in 4.28 we get to replace ρ_{mol} with $\langle N \rangle \langle e \rangle$ as shown in 4.33. Similarly, we replace π_{mol} with $\langle N \rangle \langle p \rangle$, so the average polarization in a volume is N times the average polarization per molecule. Then the big step is coming. After all, we want to know what $\nabla \cdot \epsilon$ is going to be [don't confuse Jackson's choice of ϵ as microfield with ϵ as dielectric constant soon to come!] Applying ∇ gives us a ∇^2 in each term applied to $1/R$ and we know that gives delta -- THIS is why Jackson just left the ∇ operators sitting. The final result is the very simple 4.34. Jackson interprets the "extra" term on the right as an effective polarization charge density that is getting crunched around as in figure 4.2, affecting the E field divergence. But the next step is to rewrite as 4.35 and identify the pieces as I have marked. And so we end up with the final facts that $P = Np$ and ρ he writes as $N\epsilon + \rho_{\text{ext}}$. All throughout, this $\langle e \rangle$ thing has been present to account for the average charge, but normally this is going to be exactly 0, and then you only want to worry about the "external to atoms" charge, the free charge that can move around and that is ρ_{ext} .

So this I think is a very clear explanation of what P is, and what D is, and why there is no Q or higher term to worry about! I will be interested to see if we really ignore Q in Mie Scattering theory.

Finally, we get the traditional restatement of the two Maxwell equations for E , and the divergence one shows that D sees only the free charge, so that it will be D_{normal} that will see the free surface charge σ . So this is going to be the one big difference at a boundary!

A very good section.

4.4 Dielectric Boundary Conditions. Here the fact that $P = \chi E$ is quoted as an experimental fact that has very little error in most substances, ie, there is no tensor aspect to it, see graph of some evidence on this on page 109. The formal boundary conditions are as in 4.45. I remember being confused about this on a final exam in some simple problem they gave, ouch, it really is stupidly simple.

4.5 Dielectric boundary problems. Our first problem is the traditional point charge near a plane interface. Earlier we might have done this with a conductor, but Jackson skipped that traditional problem and went right to the sphere. Looking from the right, the solution is to have a point charge at the image point on the left with a special magic charge q' . Looking from the left, we see a single charge at the q location but with magic charge q'' . The boundary conditions are as in 4.48, and these are then used to find the sizes of q' and q'' . You have to differentiate the potentials on both sides in the right directions and apply your BC's. The answer is in 4.51, so the complete problem is solved. There is a σ_{pol} on the interface as in 4.53 that you compute with a pill box, knowing P_1 and P_2 on the two sides.

Our second problem is another traditional one that has a very simple answer. The dielectric sphere in the constant z direction E field. I might imagine that if this E -field were in a plane wave, it might somehow interact with the overall dipole appearance that the sphere has looking from the outside, as we shall soon see and cause some dipole radiation. To solve this problem, we assume appropriate expansions for ϕ on the two sides in our spherical coords, as in 4.54. We apply the dielectric BC's at the surface (well suited certainly to spherical coordinates!) and do some shuffle to conclude that only the A_1 , B_1 and C_1 coefficients are non-zero. The final answer is in 4.60 and is very interesting. Inside, we have a perfectly parallel E field that is along the original field but reduced by $3/(\epsilon+2)$. Outside, we have our original field plus the field of what appears to be a point-dipole located at sphere center. This dipole points along the original field and has the strength shown in 4.62. , so P is as in 4.63 inside the sphere. Finally, we compute the σ_{pol} .

The third problem is a spherical cavity in a dielectric. The trick here is to look at the second equation in 4.56 and move the ϵ to the other side, which is like doing $\epsilon \rightarrow 1/\epsilon$. The answer is then the same except the dipole direction is now reversed against the original field and has a different magnitude as shown in 4.66.

So I think I get the basic idea here and I think I can solve any problem of this type.

Here is a fourth problem we will need right away. Imagine a cavity in an infinite dielectric. What is the E field at the center of the hole? We know how P sits in the dielectric, and we know the σ_{pol} charge sits on the boundary, and the E field sees this charge as well as free charge. The charge is $P \cos \theta$ and the z component of the field it makes at the center makes another $\cos \theta$, so we get 4.68 where the r^2 factors of the dA and the field $1/r^2$ cancel and we get $1/3x^3$ at 1 and -1 which is $2/3$, then 2π from azimuth and we get our answer $4\pi P/3$.

4.6. The Clausius-Mossotti Model for χ . Another excellent section. First, we ask: in a dielectric with uniform macroscopic (average) field E , what field does a particular molecule actually see? In section 4.3,

we did not ask this question. There we just computed E as an average value of the micro E and computed the divergence of this average E . We did not ask the question that is asked here. The answer is quite amazing I think. The answer is that a particular molecules *sees* $E' = E + (4\pi/3) P$, where E is the macroscopic field we always use. If from some other model you can compute that $p = \gamma E_i$ on the microscale for your molecules, then $P = Np = N\gamma E_i$ and you combine to get $P/N\gamma = E + (4\pi/3) P$, and you can then solve for $\chi = P/E$ and you get 4.74. The idea is that this relates the macroscopic χ parameter to the microscopic γ parameter. The result is Clausius-Mossotti and is OK for gases, roughly true for liquids and gases. I think for water the result is off by 50% or so. Comment that the static ϵ for water is 80, obviously not true at optical frequencies!

So, how do we arrive at the fact that a molecule sees $E' = E + (4\pi/3) P$? The E part is because we assume it is present from some source, so it is the extra correction term that is interesting. If you carve out a spherical cavity around your test molecule, the distance P bulk creates the $(4\pi/3) P$ field at the center. For this part, we are not surprised to find that this extra field is in the same direction as E since E is making it. This result is consistent with our solution of the cavity problem in the last section, where the field in a cavity is stronger than the field in the enclosing dielectric.

The real mystery is this: obviously you should now add the effect of a carved out sphere. Jackson shows that for a cubic lattice, the effect is exactly 0, just because you are in the midst of a symmetric environment. (We have already added the external field E , so don't want to add it here again) Well, you must ask, what happens if we take our little cavity to be huge, it will always give 0. Suppose our entire dielectric were a larger sphere. I guess the answer is that eventually you get some dielectric surface charge to deal with. Jackson then argues that this local contribution is roughly zero in any normal material.

4.7 Models for γ . We skipped this section in the course, I read it now.

First, we get a model for γ for non-polar objects (molecules) just from a toy electron model of a harmonic oscillator and we get $\gamma = e^2/m\omega_0^2$ where ω_0 is our resonant HO frequency which we would set to $h\nu = \text{optical levels}$. For a mixture of molecule types, we get 4.78. Jackson argues that this effect can only make dielectric constants on the order of 1.00XX and he gives examples of air, helium, etc. That is, his examples are non-polar gases.

Jackson wonders whether thermal agitation affects this result. Intuition says no since the molecules ought to self-polarize regardless of jiggling around, the E field is not much different in non-relativistic action. Jackson proves this is in fact true using a stat mech trick that I unfortunately have forgotten but know at one time. You weight things with the Boltzmann factor in computing a thermal average. So the answer here really is it makes no difference.

The more significant situation is when you have fixed dipole moments as in water or CO gas. Here the Hamiltonian term is $-p \cdot E$ and when we Boltzmanize this one we get that $\gamma = p^2/(3kT)$. Obviously thermal fights the line up, and larger kT relative to $p \cdot E$ energy is going to cost you and reduce γ . Sadly, Jackson does not quote ϵ values for polar substances, but I think water at DC frequencies has $\epsilon = 80$, so this completely swamps the non-polar effect.

So, a very clear and to the point section!

4.8 Electrostatic Energy in a Dielectric. I have only skimmed this section today, it is not of direct interest to me right now. The question is this: is 4.86 still true in terms of macroscopic quantities if you assemble charges in a dielectric environment? Jackson concludes that yes it is, provided the dielectric is

"linear" so 4.91 is OK. The total energy density is $\mathbf{E} \cdot \mathbf{D} / 8\pi$ as shown in 4.92. He concludes that the dielectric holds an extra energy density due to $\mathbf{P}$ shown in 4.97, and comments on the 1/2.

The next subject here is what happens when you "move in" a dielectric object into some fixed fields, or between some battery-fed electrodes held at constant potential. Since there are energy changes, we can expect forces on the dielectric object and these are discussed. Maybe you can levitate something in this way. That is, an $\mathbf{E}$ field with gradient ought to have a force on a neutral chunk of dielectric. We know this is true for a fixed dipole moment, etc etc. Have to keep it from rotating.

Nine problems, I did only 3 in the course it would appear.

Chapter 6: Faraday, Maxwell, Potentials, Gauges, Green's Solutions, Conservation Laws

This is a grab-bag chapter that constructs the known important results for "time-varying fields". Earlier in the book we did things in electrostatics and magnetostatics, here those things are updated.

6.1 Faraday's Law. In this section, Jackson reviews the 1831 discovery of Faraday that if you change the magnetic flux surface integral on a current loop, a voltage is induced around that loop which causes a current to flow in that loop according to Ohm's Law. This voltage is usually written $\mathcal{E}$ and is historically called the "electromotive force". It is, however, the same kind of potential (voltage) we had in electrostatics, and it is as if a battery were in series with the loop, although the battery cannot be localized anywhere in the loop. We now know this voltage is due to the Maxwell equation $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}/c$. Jackson notes that the term Lenz's Law is used to describe the sign of the induced voltage -- it opposes the change in flux.

In this little section, Jackson first assumes an unknown constant k in Faraday's law, as if $\nabla \times \mathbf{E} = -k \dot{\mathbf{B}}$. He then considers a loop in motion and uses Galilean invariance to show that in the moving loop, charges are seeing a field $\mathbf{E}' = \mathbf{E} + k(\mathbf{v} \times \mathbf{B})$. But from Biot-Savart (force on a current in a $\mathbf{B}$ field) we conclude that $k = 1/c$. Recall that Biot-Savart was developed in Chapter 5, as the force on a test current similar to Coulomb's law of the force on a test charge. [I have a separate note written up on this section.] By the way, this force on a particle is valid all the way up to the speed of light for v .

So, the end result here is $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}/c$. Jackson does not claim to "derive" this equation. He just found what the constant must be to be consistent with Biot-Savart.

6.2 Energy in a $\mathbf{B}$ field. I have spent about 3 hours trying to clean up Jackson's logic in this section, but to no avail. I tried to draw a reasonable tiling picture and so on. My conclusion is that this is a bogus presentation, something Jackson normally does not do! He starts with the idea that if you have a wire with a current flowing in it, and you increase the flux through the wire by making a change δB in the magnetic field in the vicinity, then some work $\delta W = I \delta V = (I/c)d(\text{flux})$ must be done by the current in the wire to make this new increased $\mathbf{B}$ field. Jackson then tries to tile his wire circuit somehow with a mesh of loops, and then he tries to apply this δW idea to one of the little tiling loops. It is very unclear whether the tiling loops are supposed to replace the wire, or whether there is supposed to be current density $\mathbf{J}$ in all space inside the starting wire, or what! He assumes that the tiling wires have a cross sectional area $\Delta\sigma$ and that $\mathbf{J} dV = I d\ell$ where $dV = \Delta\sigma d\ell$. So the fact that he has $\mathbf{J}$ being along the tiling loop suggests that he is trying to *replace* the original wire with a mesh of wires. But does this mesh then form a thin open surface with some small thickness that spans the original wire loop? If so, then the integral in 6.12 would be just over the volume of this mesh, not over the volume of all space. Then the conclusion would be that the total work done is 6.15 integrated over the volume of this mesh. But this seems wrong, because we know it should be over the volume of all space where we have fields from our wire. The whole presentation is very, very ugly and I am sure that Jackson regretted it and has replaced it with some better argument in his 2nd and 3rd editions.

Therefore, I am abandoning this entire section of Jackson's book, including the trailing comments on work done in a magnetization etc. The result is of course well known.

6.3 Maxwell's Contribution. Jackson claims that in 1865, Maxwell was staring at the equations 6.22 that seem to explain all electrical phenomena to that date. But Maxwell saw something wrong. Since $\text{div curl} = 0$, the second equation implied that $\text{div } \mathbf{J} = 0$, in violation of continuity. He knew this should be $d\rho/dt$. He got this fixed up by adding $1/c \partial \mathbf{D} / \partial t$ to the RHS, and since $\text{div } \mathbf{D} = 4\pi\rho$, this was exactly what was needed. Besides making the equations consistent with continuity, this little addition allowed one to show that either field satisfies the wave equation, and this allowed waves as solutions to the equations, and this led to Maxwell's idea that probably light was these waves, traveling through the EM ether. Maxwell noticed that the velocity implied by his equations was pretty close to the measured speed of light. This was a huge breakthrough and that is why he gets his name on these equations, although he only really added the displacement current term.

6.4 Vector and Scalar Potentials.

(1) Conjecture that any $\mathbf{B}$ can be written as $\mathbf{B} = \nabla \times \mathbf{A}$ where $\mathbf{A}$ is some "vector potential". At least we know that this form guarantees that $\nabla \cdot \mathbf{B} = 0$ since in general $\text{div curl} = 0$ for anything.

(2) In statics we knew that $\nabla \times \mathbf{E} = 0$ so we conjectured that $\mathbf{E} = -\nabla\phi$ since $\text{curl grad} = 0$ for anything. We were able to also satisfy $\nabla \cdot \mathbf{E} = 0$ if we insisted that $\nabla^2 \phi = 0$.

(3) With time dependence, we have $\nabla \times \mathbf{E} = -(1/c) \dot{\mathbf{B}}$ instead of 0. With $\mathbf{B} = \nabla \times \mathbf{A}$ this means we have that $\nabla \times (\mathbf{E} + \dot{\mathbf{A}}/c) = 0$ so now we should choose $\mathbf{E} + \dot{\mathbf{A}}/c = -\nabla\phi$ since that form satisfies this equation since $\text{curl grad} = 0$. So we end up with:

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} & \text{and} \\ \mathbf{E} &= -\dot{\mathbf{A}}/c - \nabla\phi \end{aligned}$$

We have now replaced 6 quantities with 4 quantities, perhaps an improvement. At this point, we have satisfied two of Maxwell's equations: $\nabla \cdot \mathbf{B} = 0$ and Faraday. The other two Maxwell's equations are shown top of page 180. The above equations can both be written as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, so the $\mathbf{E}$ and $\mathbf{B}$ fields are really part of a tensor. See page 379.

6.5A The Lorentz Gauge

(4) If we assume the condition $\partial_\mu A^\mu = 0$ (known as the Lorentz Condition) on the potential (where $\partial_0 = \partial/\partial t$, $\partial^0 = -\partial/\partial t$ and for proper scaling, and $x^0 = ct$), then the two equations decouple into the two shown on page bottom, which we can combine as $\square^2 A^\mu = 4\pi J^\mu$. So at this point we have "three" equations, those two on the bottom, and the Lorentz condition.

(5) Consider the gauge transformation $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$ along with $\phi' = \phi - \dot{\Lambda}/c$, or $A'^\mu = A^\mu + \partial^\mu \Lambda$. Notice that such a transformation *does not alter* the $\mathbf{E}$ and $\mathbf{B}$ fields shown in 6.29 and 6.31. Thus the potentials A'^μ and A^μ are equivalent solutions to a problem that give the same observable field results. There appears to be a whole "family" of solutions A^μ that are equivalent. In the tensor notation from above we

had that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ so when you do $A^\mu = A^\mu + \partial^\mu \Lambda$, you get $F'^{\mu\nu} = F^{\mu\nu}$ because the two second derivative terms obviously cancel each other.

(6) Consider now our "three equations" as noted in (4) above. Suppose our initial pick for A^μ gives $\partial_\mu A^\mu = f \neq 0$. Then $\partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \Lambda = f + \square^2 \Lambda$. Then all we need do is pick Λ such that $\square^2 \Lambda = -f$ [a sort of Poisson equation in 4D] and we have achieved $\partial_\mu A'^\mu = 0$. So there is a subset of our initial "family" of solutions that give the same physical fields, which subset meets the Lorentz Condition. This smaller family of solutions is called **the Lorentz Gauge**.

(6A) If we are in the Lorentz Gauge, then

$$\partial_\nu F^{\mu\nu} = \partial_\nu \{ \partial^\mu A^\nu - \partial^\nu A^\mu \} = \partial^\mu (\partial_\nu A^\nu) - \square^2 A^\mu = -\square^2 A^\mu = -\partial_\mu F^{\mu\nu}$$

so that we can write $\partial_\mu F^{\mu\nu} = \square^2 A^\mu = (4\pi/c)J^\mu$.

(7) Once we are "in the Lorentz gauge", we can still do further transformations $A'^\mu = A^\mu + \partial^\mu \Lambda'$ where $\square^2 \Lambda' = 0$ [a sort of Laplace equation in 4D], and we "remain" in the Lorentz Gauge. These further transformations move us within the smaller family of solutions.

6.5B The Coulomb Gauge (aka the Transverse Gauge)

(8) Instead of requiring the condition $\partial_\mu A^\mu = 0$, we here require that $\nabla \cdot \mathbf{A} = 0$. The first thing you see on the top of page 180 is that electrostatics as we studied it just continues! We solve for ϕ in terms of ρ just as we always did, which is what 6.45 says. But now the "other" equation 6.33 has a junk term in it as shown 6.46.

It turns out that you can partition J as shown in 6.48. To prove this (I did it in problem 6.6), you first show that the expressions shown in 6.49 and 6.50 really do add up to J , this is a non-trivial piece of work requiring two vector identities. Secondly, you note that the required curl and div conditions on the two pieces are met because $\text{curl grad} = 0$ and $\text{div curl} = 0$. Again, this is a highly non-trivial decomposition.

At this point the rest is easy. That "junk term" I mentioned is just $4\pi J_t/c$ and you arrive at the very simple result 6.52 which says that, in this gauge, A is driven only by J_t . Of course figuring out what J_t actually is requires doing that messy integration 6.50, but it can be done. So in this gauge, you have again obtained decoupling of the A and ϕ equations. Specifically, ϕ is as in electrostatics with 6.45, driven instantaneously by the charge. Then you have only 6.52 to solve with 6.50 as the source. This too is an instantaneous integral.

Application of this gauge: no sources (no ρ and no J) implies that $\phi = 0$ and also $J_t = 0$, so your entire problem is the homo wave equation for A with no sources and 6.53.

6.6 The Green's function for the wave equation.

I remember doing this in detail, and there are several results to note. The original equation is 6.54 with a driving term shown and a -4π sitting there too -- a scalar time-dependent wave equation. The G definition also has the -4π as in 6.55, and then 6.56 says how we would use this G if we knew what it was to then solve the full problem. The solution for G in momentum space is 6.59, and when we 4-Fourier that back into real 4-space we get the famous result which is 6.64 which has the time-retarding delta function times

1/R. So if we look at how we use G to get a full solution, we have 6.65, and you see the famous $1/R$ appearing there. If you do the time integral, you get the even more famous solution which is 6.66. So: the solution of the time-dependent wave equation driven by function $-4\pi f$ is the space integral of f at retarded time, divided by R .

6.7 Solving the time-dependent scalar wave equation with and without sources f using the Green's Function Method: the Poisson initial-time solution, and the Kirchhoff Huygens integral.

Back on page 18 in electrostatics, we had showed that $1/R$ was the Green's Function of the Laplace, and we commented there that you could then add to your formal $1/R$ Green's function any solution $F(x, x')$ of the homogeneous Laplace equation, and that this extra freedom allows us to match boundary conditions, such as making $G = 0$ on some closed surface. This concept was simply adding homo solutions to the "particular" solution.

Here we have a similar situation. We have found that $\delta(\text{time})/R$ is the formal Green's function, and we ought to be able to add to it another function F which solves the homogeneous wave equation. We now have a hyperbolic ODE so the only way to have a win is specifying the solution ψ and ψ' on a finite open surface. In our 4D equation, so to speak, an "open surface" is usually taken to be all of 3D space plus the boundary $t = t_0$, some initial time. So the idea is to specify ψ and ψ' (normal derivative) on a closed 3D surface at this initial time. Remember that this was "too much" in the electrostatic case, but it is what we need here.

But Jackson does not "add something to the Green's Function" the way he did back in that electrostatics chapter. Instead, he sort of starts from scratch. Using Green's Theorem, and an infinite volume with bounding surface S at infinity, we get the result 6.70 where we can junk that last surface integral term because we just assume ψ drops off there I guess. There is obviously some condition to think about, but Jackson has his hands full with other details. At time $t_0 = 0$, suppose we have ψ and ψ' described by some functions F and D as show in 6.71. Then if you park yourself at the origin by setting $x = 0$, you can get result 6.73. The first term is our familiar particular solution from 6.66. Added to this, we see a surface term at radius ct . We have found a solution to the problem that meets the boundary conditions 6.71 and it is good at any time t . It's only weakness is that it is only at $x = 0$, but I imagine the references show the general result. Notice that you have to integrate the boundary value stuff over the entire closed 2D surface, all 4π steradians. This fancy 6.73 solution is called the **Poisson's solution**. It seems that the surface integral samples a shell of both boundary value functions. The more time passes t , the farther out is that shell being sampled. I am not sure exactly how one would use this, I need to see some sample problems, but I think it is beyond this book but we have some references.

Again, the boundary conditions F and D have to be known ONLY at time $t=0$, that is the "open surface" idea. After that, it takes off on its own according to 6.73. Notice that at a later time, it is sampling the surface functions *as they were* at $t=0$, but of course it has to go out to distance $r' = ct$ to find those values.

Going back to 6.70, suppose we can drop the initial value parts terms, and suppose there are no sources f , then we have *only* the surface term which is the last term in 6.70 which we dropped earlier. So we have then 6.74. By manipulation, Jackson converts this to 6.76 which gives the field ψ inside a volume in terms of a surface area integral around the volume of ψ and various derivatives of same (space and time). This is all being done in the time domain, so we have the "retarded" label on the guts. This is the **Kirchhoff diffraction-type integral formula**, but it looks quite foreign to me because I am used to seeing in with sine time dependence only, where we have just two terms and not three, and where we have $\exp(ikR)/R$ and not just $1/R$. I know that when you undo the retardation into a time integral with the

delta showing and put in the sine time dependence, that is how you pick up the $\exp(ikR)$ factor. How three terms become two terms is pretty hazy, I would have to do the conversion to k space in detail. // Well if we sneak a peak ahead at page 281, we have a repeat of our result 6.76, and Jackson takes this right to k space and now I am reminded that there really are 3 terms, though we usually drop one of them. This then serves as Jackson's derivation of the Kirchhoff scalar formula. When he does his vector Kirchhoff, he starts from scratch again.

6.8 Poynting's Theorem (1884): Conservation of Energy

This is a very good and quick derivation. How can we "do work" on a system of E and B fields?

Think of a point charge in motion $\mathbf{v}$. The force that an E, B system of fields exerts on this charge is $\mathbf{F} = q\mathbf{E} + (1/c) \mathbf{v} \times \mathbf{B}$. The work done by the fields on the charge is $dW = \mathbf{F} \cdot d\mathbf{x} = \{ q\mathbf{E} + (1/c) \mathbf{v} \times \mathbf{B} \} \cdot d\mathbf{x}$. The work per unit time is then given as $dW/dt = \{ q\mathbf{E} + (1/c) \mathbf{v} \times \mathbf{B} \} \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v}$. The important point is that the magnetic field can never do work on a particle because the magnetic force is always at right angles to the velocity vector of the particle! Now write $(dW/dt) dV = \mathbf{E} \cdot \mathbf{v} q \delta(\mathbf{r} - \mathbf{a}) dV = \mathbf{E} \cdot \mathbf{J} dV$. This is the work done per second in a small volume dV by the fields on a current. So this then takes us to Jackson's starting point:

$$\text{power applied by fields to sources} = \int \mathbf{E} \cdot \mathbf{J} dV.$$

Jackson then goofs around with Maxwell's equations and manages to rewrite the RHS above as two terms, shown in the RHS of 6.81. This 6.81 is the conservation of energy. It says that, in time interval dt ,

$$\begin{aligned} \text{decrease in stored energy} = \\ \text{energy lost by fields to sources} + \text{energy lost through bounding surface} \end{aligned}$$

In differential form we have 6.82. In either case we interpret the quantity $(c/4\pi) \mathbf{E} \times \mathbf{H}$ as the rate at which power flows out per unit area, and this is called $\mathbf{S}$, Poynting's vector. It is energy flow per unit area. We have not done any time averages here, this is all instantaneous. You could add an arbitrary curl of something to $\mathbf{S}$ and not change anything, but no one ever does that. Jackson has assumed that the media are linear, otherwise you have extra energy storage and loss mechanisms such as hysteresis.

6.9 Conservation of Linear Momentum

We take a similar starting point with $\mathbf{F} = q\mathbf{E} + (1/c) \mathbf{v} \times \mathbf{B}$ as the force of fields on a charge. But this is dp/dt , the momentum change put onto the charge. Integrate this as in 6.89 over dV to get the total momentum transferred from fields to charges. Think of this as an increase in $\mathbf{P}_{\text{mech}}$. Now the game is to fiddle with the RHS of 6.89 using the two Maxwells shown in 6.90. After purely mathematical shufflings, we end up with 6.100 as a restatement of our equation. Now one reason for the extra complication we encounter here is that the thing being conserved, linear momentum, is a vector, whereas in the previous section it was the scalar: energy. The resulting equation like 6.100 will be a vector equation. In order to maintain an elegant notation, the dyadic notation is used, but you can really just regard 6.100 as the three

component equations and for each component i we have the divergence of $T \cdot e_i = T_{ji}$ which we may regard as a vector in the index j . But the result really is tensor in nature, no getting around that.

So what does 6.100 say? The first issue is the identification of the linear momentum $\mathbf{P}_{\text{field}}$ of the E and B fields according to 6.94. Jackson has not mentioned photons, but I will, in order to confirm this result. Write:

$$\text{energy passing thru wall piece of area } dA \text{ in time } dt = (c/4\pi) \mathbf{E} \times \mathbf{B} \cdot d\mathbf{A} \, dt = S \, dA \, dt$$

$$\# \text{ photons passing thru wall piece of area } dA \text{ in time } dt = (c/4\pi) \mathbf{E} \times \mathbf{B} \cdot d\mathbf{A} \, dt / \hbar\omega$$

$$\text{momentum per photon} = \hbar\omega/c$$

$$\text{total mom passing thru wall piece of area } dA \text{ in time } dt = \{ (c/4\pi) \mathbf{E} \times \mathbf{B} \cdot d\mathbf{A} \, dt / \hbar\omega \} * (\hbar\omega/c) = \Delta P$$

$$\text{volume density of momentum } \mathcal{P} = \Delta P/dV = \Delta P/[dA(cdt)]$$

$$= \{ (c/4\pi) \mathbf{E} \times \mathbf{B} \cdot d\mathbf{A} \, dt / \hbar\omega \} * (\hbar\omega/c) / [dA(cdt)] = \{ (c/4\pi) \mathbf{E} \times \mathbf{B} \} * (1/c^2) = (1/4\pi c) \mathbf{E} \times \mathbf{B} = S/c^2$$

This confirms the factor of $1/c^2$ that he gets. One factor comes from momentum per photon, the other comes from the thickness of the volume.

So looking at the LHS of 6.93 or 6.100, we see that the left hand side represents the rate of increase in mechanical momentum of the charges plus the rate of increase in momentum stored in the field within our volume. Momentum is thus "coming in from somewhere" and adding to both these terms. Thus, in 6.100, we have to interpret the *vector* quantity $\mathbf{n} \cdot \mathbf{T}$ as a momentum flux on the surface that is feeding momentum in through the boundary! Just as we made a distinction between stored energy and energy flux through the boundary, here we make a distinction between stored momentum and momentum flux through the boundary. What symbols are we going to use :

$$u = \text{energy density}$$

$$\mathbf{S} = \text{energy flux}$$

$$\mathcal{P} = \text{momentum density}$$

$$\overleftrightarrow{\mathbf{T}} = \text{momentum flux (the Maxwell stress tensor)}$$

Now imagine that at our wall we have a complete absorber on the "right side". Then in time dt we know that $\hat{\mathbf{n}} \cdot \overleftrightarrow{\mathbf{T}} \, dA$ momentum sinks into the absorber, so the pressure on the surface will be

$$\text{pressure on surface} = \hat{\mathbf{n}} \cdot \overleftrightarrow{\mathbf{T}}$$

For example, if the surface is in the z -constant plane, we have

$$\text{Pressure} = \hat{\mathbf{z}} \cdot \overleftrightarrow{\mathbf{T}} = T_{3j} = \text{a vector}$$

$$\text{Pressure}_3 = T_{33} = (1/4\pi) [E_3 E_3 + B_3 B_3 - (1/2) (E^2 + B^2)]$$

This would be the normal pressure on the wall, pushing it back -- we call this radiation pressure. However, for general fields, there are going to also be tangential pressure components. Simple case is a

plane wave hitting an absorber at an angle. We expect tangential pressure pushing the wall sideways, in addition to the normal pressure component.

Application: put a black ball on a spindle and shine a laser at it at an grazing angle. We expect the laser to keep the ball rotating against the loss of friction, and we could compute the force on the ball using this T thing! [Hey guess what: I think that is what a radiometer is! One in vacuum should rotate away from the white or mirrored sides. But ones with gas go the other direction due to a thermal effect.]

Question: what are the absolute intensities of E fields in light of various kinds encountered every day? Think of sunlight as a kilowatt per square meter, integrated from IR through UV I suppose. Assume all at an average λ of 5000Å. Conversion says we have then $S = 10^6$ erg/sec /cm². This tells us that $E^2 = 10^{-3}$ erg/cm³, so that $E = 10^{-6}$ dyne/esu. Thus, field storage is on the order of milli-ergs/cc and radiation pressure is milli-dynes/cm². Suppose a radiometer vane weighs 1 gram and is 1 cm² in area. The linear acceleration on this could be 10^{-3} cm/sec² so in 1 second it is not going very fast. Too bad, web sites say that radiation pressure is not enough and that heat convection does it! The toy radiometer spins the wrong way from a radiation pressure point of view.

6.10 Macroscopic Equations

Jackson has already come up with expressions for u and S and $\mathcal{P}$ in terms of fields E, D, B, H . In this section, he thinks again about the macro fields being now spatial *and* temporal averages of micro fields (including this time ϕ and A). One important result is 6.112 which reminds us exactly about the effective charge due to the presence of a dielectric (usually on a boundary), as well as the effective current arising from magnetization M and also from P . This dP/dt term seems new to me. Naturally it was not present in the static equations! Jackson's main point in this section is that he is doing everything from scratch, using only Maxwell's equations. [So does this mean I can use $J = dP/dt$ to compute a dielectric radiation situation? I was always sort of stumped on how to start in the calculation of A when you could not do the little parts trick].

So, if we start with the energy conservation law in terms of the D and H fields and start stripping out the wrappings on these fields, the law appears as 6.117. The point is that we have now identified the extra terms due to M and P , and these are showing up as adders to the current as in 6.112. The point is that the field not only does work on free current J in the form $J \cdot E$, but it also does work (at least reactive work) on M and P exactly as shown in this equation! You basically have inductance and capacitive work staring you in the face here. Since these terms are reactive, and represent temporary storage and not power loss, Jackson argues that it is natural to incorporate them in the D and H field presentation.

Problems. Interestingly, the ones I have done or looked at below are the same ones (apart from 6) that Jackson had us do when I took the class.

Problem 6.6. See hand-written solution, shows that you can separate currents as claimed for the Coulomb Gauge.

Problem 6.8: statement of the main chapter conserved quantities

We have already derived the expressions for u . Keep in mind the asymmetrical way in which D and H are defined:

$$\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{H} &= (1/\mu) \mathbf{B} \end{aligned}$$

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 4\pi \rho_{\text{free}} \\ \nabla \times \mathbf{H} &= (4\pi/c) \mathbf{J}_{\text{free}} \end{aligned}$$

So the first energy formula here for u has $ED + BH$ inside.

The S we are happy with.

The thing he is calling $\mathcal{g}$ here is our momentum density $\mathcal{P}$, and seems that the result here is $\mathbf{D} \times \mathbf{B}$ which is a slightly unexpected combination. If you trace things through on page 192, this is in fact the result.

We could go trace out the derivation of the tensor (I have some pencil marks in the book from when I once did this), and the result is that you always have DE and HB at each quadratic location. So I am not really doing this problem, I am just reviewing the results that it states.

Problem 6.9: Angular Momentum of the Fields and its Conservation Law

I am not doing this problem, just studying its results. If we look at angular momentum instead of linear momentum, here are the things we find:

$$\mathcal{L} = \mathbf{r} \times \mathcal{P} = \mathbf{r} \times [(1/4\pi c) \mathbf{D} \times \mathbf{B}] \quad \text{since} \quad \mathcal{P} = (1/4\pi c) \mathbf{D} \times \mathbf{B}$$

For angular momentum (a vector!) we again have to ask what is the tensor that describes the flow of angular momentum through a boundary and the answer is $\vec{\mathbf{M}} = \vec{\mathbf{T}} \times \mathbf{r}$.

Now we know that in quantum mechanics we have

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = (\hbar/i) \mathbf{r} \times \nabla \quad \mathbf{p} = (\hbar/i) \nabla$$

These are differential operators. One should not confuse these operators with the $\mathcal{L}$ and $\mathcal{P}$ density functions for an EM field. You cannot take L_3 and apply it to something like E or B or A or ϕ and expect to magically get the same thing multiplied by ± 1 and claim that you have located a photon and this is its spin relative to the z axis. On the other hand, you should find this to be the case when you apply L_3 to a multipole field Y_{lm} , that is, you are diagonal with eigenvalues m . See next couple of problems.

Problem 6.11. Classical model for a photon group

The E field shown has some kind of unspecified slow (relative to λ) transverse cutoff function (could use a Gaussian, as in a Gaussian Beam), and has circular polarization, so you might try to think of this as some kind of semi-localized photon. When you set $\nabla \cdot \mathbf{E} = 0$, you get exactly 0 from the function shown. The ∂_z picks up ik from the exponent, and the z factors then cancel the ∂_x and ∂_y factors. The form for $\mathbf{B}$ comes from the $\nabla \times \mathbf{E}$ Maxwell equation where we agree to drop second derivatives of E_0 .

What does this set of fields look like? The $\mathbf{B}$ is always perp to $\mathbf{E}$ due to the 90 degree rotation of the i factor. The wave is going to the right. In one, the $\mathbf{B}$ leads by 90 degrees, and in the other it lags by same. There is a z component due to the transverse confinement of the beam, due to the divergence requirement.

Problem 6.12. Continuation of 6.11.

For L3, I guess we might compute u and $\mathcal{L}$ for an infinite cross section disk across the beam, of finite thickness. We would count on the E_0 function to converge this integration. We would do this for both u and $\mathcal{L}$ and then take the ratio. There are some things that puzzle me here, however. What is the meaning of the object $\mathbf{E} \cdot \mathbf{B}$ here? We know the transverse part is going to be 0, but this is not obvious with the -- well I guess it just works, yes, just do it, fine. Clearly the z direction cannot converge so don't try to integrate that way. Now let's assume we get the result shown, I have not done this calculation. If we interpret this plane wave as some number N of photons all in the same state, then $u/N = \hbar\omega$ and $\mathcal{L}/N = \pm\hbar$ and the ratio is then exactly as shown, so that would be our interpretation. As for the final question, if a field has cyl symmetry, then no transverse direction in space is picked out so you cannot have any transverse vector quantity be non-zero (is this right...?)

Imagine a contour that is moving in a direction $\mathbf{v}$. Imagine we are interested in the area integral over a surface bounded by this contour of some vector field $\mathbf{A}$,

$$I = \int_{C(t)} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{A}$$

If we want the total derivative of I , we get one contribution from $d\mathbf{A}/dt$, but we get another one because the contour moves. The first term is obvious. The second term can be written like this:

$$\left[\int_{C(t+dt)} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{A} - \int_{C(t)} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{A} \right] / dt$$

If you draw a picture of the contour moving in direction $\mathbf{v}$, you see that the values of $\mathbf{A}$ that are picked up at time $t+dt$ were the values that $\mathbf{A}$ had at t but at a distance shifted by $\mathbf{v} dt$. This is just because the entire contour and its attached surface has shifted by $\mathbf{v} dt$. Thus we can rewrite the above as:

$$\left[\int_{C(t)} \mathbf{A}(\mathbf{r} + \mathbf{v} dt) \cdot d\mathbf{A} - \int_{C(t)} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{A} \right] / dt = \int_{C(t)} (\mathbf{v} \cdot \nabla) \mathbf{A}(\mathbf{r}) \cdot d\mathbf{A}$$

where this $\delta \mathbf{A}$ first term expansion used here is obvious if you just write out the components.

$$\mathbf{A}(\mathbf{r} + \mathbf{a}) = \mathbf{A}(\mathbf{r}) + (\mathbf{a} \cdot \nabla) \mathbf{A}(\mathbf{r}) + \dots$$

Now we have this little vector identity,

$$\nabla \times \mathbf{A} \times \mathbf{v} = (\mathbf{v} \cdot \nabla) \mathbf{A} - \mathbf{v} (\nabla \cdot \mathbf{A})$$

where $\mathbf{v}$ is a constant so all gradients on it vanish, killing two of the usual terms in this identity. Now, if $\mathbf{A}$ happens to be $\mathbf{B}$, the magnetic field, we know that $\nabla \cdot \mathbf{B} = 0$ so we ignore the second term. So we rewrite our term as follows:

$$= \int_{C(t)} \nabla \times (\mathbf{A} \times \mathbf{v}) \cdot d\mathbf{A}$$

But Stokes lets us relate an area integral like this with a curl to line integral

$$= \int_{C(t)} (\mathbf{A} \times \mathbf{v}) \cdot d\mathbf{l}$$

and this then is the rightmost term in Jackson's equation (6.5) on page 172.

Problem 6.6

$$\vec{J}_L + \vec{J}_T = \frac{1}{4\pi} \left[\vec{\nabla} \times \vec{\nabla} \times \int \frac{\vec{J}(x')}{R} d^3x' - \vec{\nabla} \int \underbrace{(\vec{\nabla}' \cdot \vec{J}(x'))}_{(3)} \frac{1}{R} d^3x' \right]$$

all $\vec{\nabla}$'s shown act only on R , and $\vec{J}(x')$ is a constant vector.

So $\vec{\nabla} \times \vec{\nabla} \times \left[\frac{\vec{J}(x')}{R} \right] = \vec{\nabla} \underbrace{\left(\vec{\nabla}' \cdot \frac{\vec{J}(x')}{R} \right)}_{(1)} - \underbrace{\nabla^2 \left(\frac{\vec{J}(x')}{R} \right)}_{(2)}$

$(2) = -\nabla^2 \left(\frac{1}{R} \right) \vec{J}(x') = 4\pi \delta(x-x') \vec{J}(x')$, so $(2) \Rightarrow \vec{J}$, the answer

We now have to show that $(1) + (3) = 0$.

Want now to show:

$$\frac{1}{4\pi} \left[\vec{\nabla} \left(\vec{\nabla}' \cdot \int \frac{\vec{J}(x')}{R} d^3x' \right) - \vec{\nabla} \int \frac{[\vec{\nabla}' \cdot \vec{J}(x')]}{R} d^3x' \right] = 0.$$

Now use this fact:

$$\vec{\nabla}' \cdot \left(\frac{\vec{J}(x')}{R} \right) = \vec{J}(x') \cdot \vec{\nabla}' \left(\frac{1}{R} \right) + \frac{1}{R} (\vec{\nabla}' \cdot \vec{J}(x'))$$

Our proof now requires

$$\frac{1}{4\pi} \vec{\nabla} \left\{ \underbrace{\vec{\nabla}' \cdot \int \frac{\vec{J}(x')}{R} d^3x'}_{= -\int \vec{J}(x') \vec{\nabla}' \left(\frac{1}{R} \right) d^3x'} + \int \vec{J}(x') \cdot \vec{\nabla}' \left(\frac{1}{R} \right) - \underbrace{\int \vec{\nabla}' \cdot \left(\frac{\vec{J}(x')}{R} \right)}_{\equiv 0} \right\} = 0$$

QED.

Jackson, Chapter 8: "Wave Guides and Resonant Cavities."

8.1 Fields at Surface of and Within a Conductor.

1. Suppose you have a perfect conductor on one side of a plane surface. In electrostatics, you know that $\vec{E} = 0$ inside the conductor. Since there is no current flowing inside the conductor (statics), a magnetic field goes right through the boundary with no change. (ie, no surface current either). Thus if you put a magnet near a piece of metal, the B field goes through with no change at the surface.

The situation is a little different for time-varying fields. If slowly-varying, you still get $\vec{E} = 0$ inside conductor (some condition for this approx of course requires σ large enough; we assume $\sigma = \infty$). Also you cannot have a time-varying H-field inside a perfect conductor because it would induce circular currents, ie, it would induce an E field, but you are not allowed to have an E field.

So if you shine say a plane wave on a perfect conductor, there will be a certain surface charge density and current density so as to exactly cancel E and B inside the conductor. Just outside the surface, you know that $H_{\perp} = 0$ from $\text{div} \vec{H} = 0$, and you know that $E_{\parallel} = 0$ from $\text{curl} \vec{E} = \dots$. Figure 8.1 shows the situation.

If conductor has finite conductivity, you get certain H_c and E_c as in (8.9) and (8.10) which shows that $H_{\parallel}$ now penetrates to skin depth into the conductor, and that there is actually some small $E_{\parallel}$ also penetrating, though $E_{\parallel} \ll H_{\parallel}$. Thus, there is also some $E_{\perp}$ outside the surface, so some Poynting energy is flowing into the conductor, obvious ohmic losses.

As frequency increases, skin depth decreases and ohmic loss increases.

8.2 Cylindrical Cavities and Wave Guides.

Now he assumes perfect conduction again, and a uniform cross section on your metallic guide. Inside is some material with μ_e but no σ ; something like air. You write your Maxwells equations and insist that solution propagates down the guide with some k and frequency ω . This reduces wave equations to (8.19); notice that both the $\perp$ and $\parallel$ components of E and B obey this equation; notice that we have only specified the z and t dependence of the E_z and B_z fields.

It turns out that a "basis" for describing any wave consists of the TE and TM modes. A TE mode is one in which $E_z = 0$ identically, and B_z satisfies a certain boundary condition around the surface of some cross section, namely,  B_z has to hit each boundary with zero slope. The TM modes are those where $B_z = 0$ identically and E_z has BC namely $E_z = 0$ wherever it hits the boundary. (Note that once you know E_z and B_z for some mode, you can compute the E_t and B_t trivially from (8.19).)

Obviously for Tm modes, the electric field problem is like a particle in a two dimensional box, or better yet, a timpani drum surface. Clearly there are going to be eigen-values of k .

For a given geometry, you solve the 2-dim BV problem and you get some eigenvalues called k_λ . These then determine your eigen- k vectors k_λ as in (8.37). Defining the cutoff frequency obviously (for given mode) you of course get $k = k(\omega)$, ie, a dispersion relation. You learn that for a given frequency ω , there are only a finite number of useable (ie, k real) k_λ values, ie, a finite number of modes. Analogous to particle in potential well where there are only a finite number of bound states.

Usually, the lowest mode of all is the bottom TE mode. So if you choose geometry nicely, you can arrange for this mode to be the only "activated" or "propagating" mode at a given working frequency. For each mode, there is a frequency below which that mode will not go. That is the cutoff frequency of that mode.

In addition to the TE and TM modes of a waveguide, there is a special mode in which $E_z = 0$ and $B_z = 0$ identically. Ie, this is sort of a TE and TM mode, or TEM. Ie, this mode propagates completely transversely, as if it were in an infinite medium. However, the electric transverse field must satisfy an electrostatic BV problem, so this mode cannot go inside a hollow cavity. This TEM thing is the principle way energy goes down a coax line or parallel transmission line. In waveguides, on the other hand, you are talking the TE and TM stuff because TEM does not go at all.

8.4 Rectangular Waveguide.

Here your BV problem has eigenvalues indicated by m and n in the usual way. Modes are thus called TE_{mn} and TM_{mn} . First index refers to first dimension. If that dimension is largest, then $TE_{1,0}$ is the lowest mode (mode with the lowest cutoff frequency). By choosing $a \neq b$, you can make only this mode go; if $a=b$ then you are stuck with two degenerate modes, maybe harder to control at entrance and exit of waveguide.

The traditional scalar form is (9.67) with both terms in there (unlike Sommerfeld). G is shown, the usual thing. Since each component of E and B satisfies the same wave equation as ψ , we get (9.68) and the same for B . Jackson then launches into a whirlwind of vector identities with far surface integrals vanishing and when the dust settles, you have (9.75) and (9.76). I have no doubt he has done this all correctly, my job is to make sure I know how to interpret and use the result.

At this point, it helps to look back at page 236 where he discusses "what happens" at the surface of a conductor. You see that $n \cdot E$ at a surface is a measure of the surface charge σ there. And $n \times B$ is a measure of the surface current K (different dimensions from J). In other words, the presence of σ allows or supports a normal E field, while the presence of a surface current K allows a tangential B . The other two items are zero: tangential E and normal B . Inside the metal all four field pieces are 0. These conclusions arise trivially from the $\text{div} E = \rho$ and $\text{curl} B = J$ equations applied to a pillbox and loop straddling the surface. Of course things don't change instantaneously at the surface, so we have picture on page 239 showing that there is some skin thickness to all this activity just below the mathematical surface.

Now back to page 285 and the vector K formulas. The surface S in these formulas is a complete bounding surface in the Green sense. If you assume radiation field types, you can limit S to your finite surface S_1 , some closed boundary located at some localized spot in the middle of your volume. So this is what (9.77) is saying: he has taken (9.75) and included only the interior surface S_1 , and has changed the definition of n so it points out of the interior ball and into our "region of interest".

Jackson goes on to "interpret" the terms in (9.75) or (9.77). Since these are surface integrals, for those parts of the surface that are "metal", we can interpret $n \cdot E$ as a surface charge σ that must be there from our general surface rules. And $n \times B$ is a surface current. But then we have to interpret the other things like $n \times E$ and $n \cdot B$ as magical "magnetic" σ and K . Interpretation is free, what the formula says is what counts.

So, equations (9.77) and its counterpart for B are the equations to use if you are doing diffraction from some 3D localized object (such as a sphere, not necessarily metal). You have to integrate 9.77 over the entire surface of that object.

Next, Jackson wants to flat-adapt this general formula for use in the usual holes-in-metal applications. He imagines on page 286 that the internal surface S_1 is made into a pancake. The total internal surface in this picture is both sides, so he has now redefined the symbol S_1 so that $S_1(\text{in integral}) = S_1 + S_1'$. You have to integrate over both S_1 and S_1' . This is done in (9.68) but the two surface sides are so close that the normal vector geometry is the same, so things can be written as in (9.78). This is just a folding of the two sides into an integral over one side, nothing new has been done yet.

Now look at the three difference terms in (9.78) which are integrated only over the right-side surface S_1 . Think of the actual pancake right surface as metal with holes. The first term will be non-zero only in the holes, since it contains $n \times E$. Jackson wants to *make* the other two terms vanish everywhere on S_1 . If he can make (9.79) be true, then he achieves that goal. And (9.79) will be true and various required field conditions will also be true if he makes (9.80) be true. These equations are basically assumptions made for fields E' and B' on the *left* side of the pancake, sort of like doing image charges. In fact, we are really doing Sommerfeld's image charge method here, and we will end up with his result! Jackson imagines that currents and charges run around inside the pancake however they must in order to generate the E' and B' fields he is assuming. The pancake has a metal sheet only on the right, I guess, or if not, then he imagines surface charges and surface currents on the left surface of the metal. The idea is that these fields imagined on the left side will affect things on the left side, but we don't care about the left side. The fields there are

just construed to make the computation on the right side be simpler. (Jackson could have clarified this important point I think better than he does.) The end result is then (9.81) which you integrate only over the holes on S1. So this is the vector version of the Kirchhoff formula for flat surfaces that we most commonly use! He switches the ∇ operator to get (9.82) which seems easier to work with, since $G = e(ikR)/R$. He points out that this result is EXACT if you really know the tangential E field in your apertures.

Now, let's look carefully at (9.81). Since $G = \exp(ikR)/4\pi R$ where $R = |r - r'|$, we can compute:

$$\nabla G = (k^2/4\pi) e^{ikR} [i/kR - 1/(kR)^2] \hat{R} \quad \text{where } \nabla R = \hat{R}$$

Now we make the usual assumption that we are many λ away at our observation point, so $kR \gg 1$, or $R \gg \lambda$, so we get

$$\nabla G \approx ik \hat{R} G$$

Note that $\nabla' G = -\nabla G$, so we pick up a minus sign here! Next, let's put $\hat{n} = \hat{z}$ without giving up anything. We then get this exact result, apart from the approximation made above in computing ∇G :

$$E = -2ik/4\pi \int dA' (e^{ikR}/R^2) (z-z') \{ E_t - \hat{z} (E_t \cdot \mathbf{R})/(z-z') \} \quad \mathbf{R} = (x-x', y-y', z-z')$$

The integration coordinate is $r' = (x', y', z')$ and we might as well set $z'=0$ right now to get:

$$E(r) = 1/i\lambda \int dA' (e^{ikR}/R^2) \{ z E_t(r') - \hat{z} (E_t(r') \cdot \mathbf{R}) \} \quad \mathbf{R} = (x-x', y-y', z)$$

where E_t is the *tangential* field in the apertures,

$$E_t = (E_x, E_y, 0)$$

and the integral is only over the apertures. The second term is smaller than the first by factors like $(x-x')/z$ and this term can be neglected in the usual diffraction situation where we are far from a small diffractor, this is the $z \gg d$ business. If we select our coordinate system of integration such that $z' = 0$, we then get:

$$E(r) = 1/i\lambda \int dA' E_t(r') (e^{ikR}/R) (z/R)$$

and this is **exactly the Sommerfeld I formula** that I like, page 49 of Goodman. Notice that the field direction just passes through. Reminder: this final form is only correct for $z \gg d$. If you want to look closer in (or at large angles away from the z axis), then use the previous result above! A z component of the resulting field should be expected, since the radiated E is perp to the line R.

Of course once you have computed E as above, you can get B from the curl equation,

$$B = (i/k) \nabla \times E \quad // \text{ as shown as (9.5).}$$

Review of Jackson Chapter 14 "Radiation by Moving Charges"

Claim from an earlier chapter that:

$$A_{\mu}(x) = \frac{1}{c} \int d^3x' J_{\mu}(x') \frac{\delta(t' + \frac{R}{c} - t)}{R} \quad R = |\vec{x} - \vec{x}'|$$

You name your current, will tell you your field A_{μ} .
For a point charge of course one has:

$$J_{\mu}(x) = ec \beta_{\mu} \delta(\vec{x} - \vec{r}(t)) \quad \beta_0 = 1$$

Insert this current to get:

$$A_{\mu}(\vec{x}, t) = \frac{1}{c} \int dt' \int d^3x' [ec \beta_{\mu} \delta(\vec{x} - \vec{r}(t'))] \frac{\delta(t' + \frac{R}{c} - t)}{R}$$

Do the spatial and find, replace $\vec{x}' \rightarrow \vec{r}(t')$, so now $R = |\vec{x} - \vec{r}(t')|$. Then

$$A_{\mu}(\vec{x}, t) = e \beta_{\mu} \int dt' \beta_{\mu}(t') \delta(t' + \frac{R(t')}{c} - t) \frac{1}{R(t')}$$

Now use: $\delta(t' + \frac{R(t')}{c} - t) = \frac{1}{|1 + \frac{1}{c} \frac{\partial R}{\partial t'}|} \delta(t' - \tilde{t}) = \frac{\delta(t' - \tilde{t})}{1 - \hat{n} \cdot \vec{\beta}(t')}$

Thus, you get: $A_{\mu}(\vec{x}, t) = \frac{e \beta_{\mu}(\tilde{t})}{R(\tilde{t}) [1 - \hat{n} \cdot \vec{\beta}(\tilde{t})]}$

here, $\tilde{t}$ is the solution of: $t' + \frac{R(t')}{c} = t$ or t

Interp: if something happens at time t' , and position $\vec{r}(t')$, then at position $\vec{x}$ which is distance $R = |\vec{r}(t') - \vec{x}|$ away, the influence is felt not exactly at t' , but later, at $t' + R/c$. Hence, one:

Liénard-Wiechert potentials: $\phi(x, t) = e \left[\frac{1}{KR} \right]_{\text{ret}} \quad K = [1 - \hat{n} \cdot \vec{\beta}]$

Here, "ret" means to evaluate $\vec{\beta}$, K , and R at $\tilde{t}$ = solution of $t' + \frac{R(t')}{c} = t$.

$$\vec{A}(x, t) = e \left[\frac{\vec{\beta}}{KR} \right]_{\text{ret}}$$

In nonrelativistic, get usual results in the Lorentz gauge.

So, now that we have $\vec{A}$ and ϕ , it is mere computation to get $\vec{E}$ and $\vec{B}$.

See (14.8). Results:

$$\vec{E}(\vec{x}, t) = e \left[\frac{(\hat{n} - \vec{\beta})(1 - \beta^2)}{K^3 R^2} \right]_{\text{ret}} + \frac{e}{c} \left[\frac{\hat{n}}{K^3 R} \times \{(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}\} \right]_{\text{ret}}$$

independent of acceleration, $\frac{1}{r^2}$ the $\frac{1}{R}$ accel. field

And exactly true that $\vec{B} = \vec{n} \times \vec{E}$.

If we observe an accelerating charge from a frame in which $|\vec{\beta}| \ll 1$, then the 1/R portion of $\vec{E}$ is:

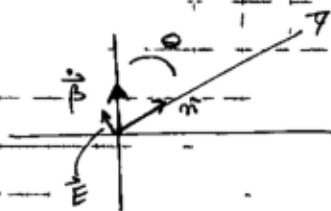
$$\vec{E}_a = \frac{e}{c} \left[\frac{\vec{n} \times (\vec{n} \times \dot{\vec{\beta}})}{R} \right]_{\text{ret.}}$$

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{c}{4\pi} |\vec{E}_a|^2 \vec{n}$$

$$dP = \vec{S} \cdot d\vec{A} = \vec{S} \cdot \vec{n} R^2 d\Omega \Rightarrow \frac{dP}{d\Omega} = \frac{c}{4\pi} |\vec{E}_a|^2 R^2$$

So the power versus angle is:

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{e^2}{4\pi c} |\vec{n} \times (\vec{n} \times \dot{\vec{\beta}})|^2 \\ &= \frac{e^2}{4\pi c} |\dot{\vec{\beta}}|^2 \sin^2 \Theta \end{aligned}$$



Integrate over solid angle to get: $P_{\text{rad}} = \frac{2}{3} \frac{e^2}{c^3} (\dot{\vec{r}})^2$ ← Larmor formula for total radiated power (in non-rel case).
Generalized result is:

$$P_{\text{rad}} = \frac{2}{3} \frac{e^2}{c} \gamma^6 [(\dot{\vec{\beta}})^2 - (\dot{\vec{\beta}} \times \dot{\vec{\beta}})^2] \leftarrow \text{Liénard's power formula.}$$

For circ accelerators there is a loss:

$$P = \frac{2}{3} \frac{e^2 c}{R^2} \beta^4 \gamma^4$$

$R \leftarrow \text{orbit radius}$

Example: in a $g = 10$ meter electron synchrotron, $E = 5$ BeV, but $\delta E \approx 5$ MeV/cycle.

A lot of energy:

$$\frac{P_{\text{rad/loss}}}{\text{sec}} = P_{\text{rad}} \approx \frac{2}{3} \frac{e^2 c}{g^2} \gamma^4 = \frac{2}{3} \frac{e^2 c}{g^2} \frac{E^4}{(mc^2)^4}$$

If $E = 5$ BeV, $mc^2 = .51$ MeV, then ratio is:

$$\gamma = \frac{E}{mc^2} = \frac{5 \times 10^3}{.51} = 10^4, \text{ so } P_{\text{rad}} \approx \frac{2}{3} \cdot (10^{16}) \cdot \frac{1}{10^2} \cdot 3 \times 10^8 \times$$

$$P_{\text{seg}} = \frac{2}{3} \frac{(5 \times 10^{10} \text{ eV})^2 (3 \times 10^{10} \text{ eV}) \cdot 10^{16}}{(10^3 \text{ eV})^2} \approx \frac{2 \times 10^{10} \times 25 \times 10^{20} \times 10^{16}}{10^6} = 50 \text{ ergs/sec.}$$

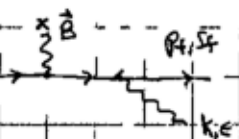
Well... This is a lot of loss per electron!

For a 10^{17} bunch, this is 50 watts/sec.

$$E_{\text{e}} = .51 \times 10^6 \text{ eV} \times \left[\frac{1.6 \times 10^{12} \text{ eV}}{1 \text{ eV}} \right] = 10^{-6} \text{ ergs}$$

What is synchrotron radiation?

Classical ED says it is caused by presence of $\vec{B}$, but I think QED would say it was bremsstrahlung in presence of a magnetic field.



Then we can ask about "spin flip" and so on.

The BD they do bremsstr. off a coulomb external potential and get a cross section:

$$d\sigma = \frac{Z^2 e^4 m^2}{2kV_i E_f E_i} \int \left[\frac{\vec{\epsilon} \cdot \vec{p}_f}{k \cdot \vec{p}_f} - \frac{\vec{\epsilon} \cdot \vec{p}_i}{k \cdot \vec{p}_i} \right]^2 \frac{|\bar{u}(p_f, s_f) \gamma_0 u(p_i, s_i)|^2}{|\vec{q}|^4} 2\pi \delta(E_f + k - E_i) \frac{d^3 k \cdot d^3 p_f}{(2\pi)^6}$$

You can see the spin dependence clearly. I can imagine a similar result where A_μ is a magnetic field. I do not see how one would handle this classically! How does the spin enter classically? Magnetic moment.

If, as in Schiff, you do a S.E. analysis with $\vec{B}$, you get

$$H = -\frac{\hbar^2}{2m} \nabla^2 + V(r) + g(r) \vec{L} \cdot \vec{S} + \epsilon (L_z + 2S_z) \quad \epsilon = \mu_B B.$$

So here the spin again enters. But we don't want atomic physics - we want the effect on a free electron in a $\vec{B}$ field. Still, $V(r)$ must be present to keep Dirac going in a circle. You could think of Dirac as a gigantic atom. What are the eigenstates of this "atom"? Then you could compute the matrix elements for transition.

$$\langle m | K | m_0 \rangle = \langle m | \epsilon (L_z + 2S_z) | m \rangle$$

So a scheme might be

- 1) compute eigenstates for this system? (maybe continuum)
- 2) compute EM transition rate, showing spin dependence.
- 3) then transform to a "lab" frame to get rel. answer.

So now, I think real QED would be easiest.

Sokolov's EM problem

In the E^2 machines, the e^+ beams build up a polarization of 92% in about 20 minutes; want a simple reason why this happens. Needs to include interference effect, see Phys Rev Letters for references.

As the phys letter states, the pol results from synchrotron radiation and leaves the electron polarized against the magnetic field lines.

$$|P| = \frac{8\sqrt{3}}{15} [1 - e^{-t/\tau}] \left[\frac{6\text{deg}}{3\text{deg} + 3\text{deg}} \right]$$

τ = vector line

$$\tau_0 = \left(\frac{1}{3\text{deg}} + \frac{1}{3\text{deg}} \right)^{-1} \text{ much less than } \tau_0$$

References:

Baier

Phys Lett Sov Phys
Proc. Int Sch Tauscheck ed. of Phys Enrico Fermi Conf
XLVI,
P1(1971)

Also, see Sov. Phys. Dokl. 8, 1203 (1964)
by Sokolov + Ternov

QC 7.86
V.295
1969
(out)

This ref is 2 pages. Let $n_1 = \# \text{ electrons w/ spin against } \vec{B}$
 $n_2 = \# \text{ electrons w/ spin with } \vec{B}$

$$n_1 + n_2 = n_0$$

$$\text{Claim: } 30n_1 = (15 + 8\sqrt{3})n_0 - [15(n_2 - n_1) + 8\sqrt{3}n_0] e^{-t/\tau}$$

$$30n_2 = (15 - 8\sqrt{3})n_0 + [15(n_2 - n_1) + 8\sqrt{3}n_0] e^{-t/\tau}$$

$$\text{where } \tau = \left[\frac{5\sqrt{3}}{8} \frac{1}{n_0 c R} \left(\frac{E}{n_0 c^2} \right)^5 \frac{e^2}{n_0 c R} \right]^{-1}$$

$$\text{Then, as } t \rightarrow \infty, \text{ get } \frac{n_1}{n_2} = \frac{15 + 8\sqrt{3}}{15 - 8\sqrt{3}} \approx 20$$

So most are against.

The actual work is given in other refs.

Now let's go back + radio waves! stuff relativistically. We had:

$$\vec{E}_a(\vec{r}, t) = \frac{e}{c} \left[\frac{\hat{n}}{K^3 R} \times \{ (\hat{n} + \vec{\beta}) \times \dot{\vec{\beta}} \} \right]_{\text{ret}}$$

$$[\vec{S} \cdot \hat{n}]_{\text{ret}} = \frac{e}{4\pi} \left[|\vec{E}_a|^2 \hat{n} \right]_{\text{ret}} = \frac{e^2}{4\pi c} \left[\frac{1}{K^6 R^2} |\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]|^2 \right]_{\text{ret}}$$

↑
notice

The power = energy / charge's time is

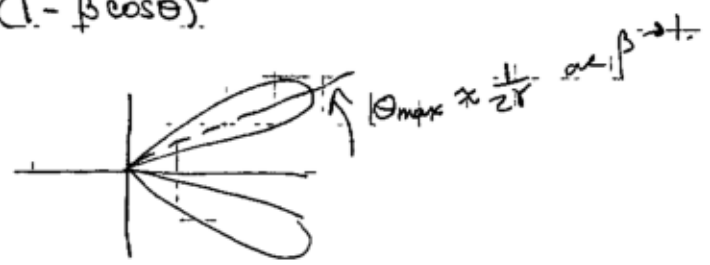
$$\frac{dP(t)}{d\Omega} \stackrel{\leftarrow \text{charge plane}}{=} K R^2 |\vec{S} \cdot \hat{n}| = \frac{e^2}{4\pi c} \frac{|\hat{n} \times \{ (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}} \}|^2}{(1 - \hat{n} \cdot \vec{\beta})^5}$$

This lowered K^6 to K^5 .

For linear motion: $\frac{dP(t)}{d\Omega} = \frac{e^2 (\dot{v})^2}{4\pi c^3} \cdot \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$

Thus, the lobes of sin² are bent:

And, $P(t)_{\text{tot}} = \frac{2}{3} \frac{e^2}{c^3} (\dot{v})^2 \cdot \gamma^6$



For circular motion

Chapter 16: Multipole Fields

This chapter shows how to solve Maxwell's equations in full glory in spherical coordinates. The usual $e^{i\omega t}$ time dependence is assumed. The general idea is that you expand the fields in each region of your problem in full generality (which means all ℓm values and both electric and magnetic modes), then you require that field boundary conditions be met at the boundaries. Several examples are given, and comments are made regarding both atomic and nuclear emissions. When sources are present, it is shown how to compute all the moments. In the general case there are 4 such moments in each ℓm harmonic. As an example of a problem with "sources", the center-fed antenna is treated with the usual King Smile assumption about the current shape. The result is compared with a full treatment in Chapter 9 that did all the multipoles at once, but again in the King Smile limit. The second example is scattering from a conducting sphere where we get heavily involved in the formalism. I have done a third example of scattering from a dielectric sphere in a separate document, a somewhat harder problem than the conducting sphere.

The reason that the multipole expansion can help solve problems is the usual one. The rotation group has "good quantum numbers" ℓ and m since each represents a conserved quantity. If the corresponding operators commute with the Hamiltonian, the solution can be diagonalized into the individual partial waves ℓm and there is no interaction between different partial waves. So the problem is simplified into a set of smaller problems, one in each partial wave. Within a partial wave, the general form of a solution is quite limited, there are only two radial function possibilities. If you do something like scatter off a cube, the boundary conditions will be functions of spherical angles, and this will mix adjacent harmonics of different ℓm , I think, this corresponding to that fact that the "potential" that this cube represents is no longer spherically symmetrical. For problems with radiating sources, the expansion is useful in long λ limit because the terms drop off quickly.

16.1 Solutions of the scalar wave equation in (r, θ, ϕ) . Back on page 54 where we had the Laplace equation (wave equation with $k^2 = 0$), we assumed $U(r)/r$ as the radial form and got (3.7) as the radial equation which led to r^ℓ and $r^{-\ell-1}$ as the solutions. We ended up with an expansion as in 3.61 on page 67. Here, we have that extra k^2 term, so the radial functions are going to be different. The expansion of a scalar wave equation solution is given in 16.4 and the radial functions turn out to be the *spherical* Bessel functions of various types. So one way to show the expansion is in 16.16 using the Hankels. Naturally, Jackson takes this opportunity to tell us *all about* these Bessel functions. Note that the Hankels are just those linear combinations of the j and n functions which match outgoing or incoming spherical waves at large argument.

Now, the Green's function in the presence of $k^2 \neq 0$ is e^{ikR}/R as in 16.18. Of course we want to expand this in the usual double-spherical-harmonic type formula and that turns out to be 16.22 which we can compare to the Laplace version in 3.70 page 69. We now have non-trivial radial functions in this thing, they are of course the new spherical Bessel functions. For $r >$ we have the Hankel which dies as e^{ix}/x , and for $r <$ we have the j type which is well-behaved at the origin.

We are now top of page 542. Here, we look again at the ODE for the $Y_{\ell m}$ functions. Jackson defines a vector of operators which include $\partial/\partial\phi$ and $\partial/\partial\theta$. He calls this vector by the name $\mathbf{L}$, and identifies this

thing squared as the LHS of that ODE! I know that this is just a specific representation of these operators, and that they have properties that are representation-independent. We get the raising and lower L operators as shown. Page 543 shows some commutators:

$$[L^2, L] = 0 \quad [p^2, L] = 0$$

Then 16.30 shows us that we really already know about L^2 .

Remember from quantum wave mechanics that $p = (\hbar/i)\nabla$ and $L = r \times p = (\hbar/i) r \times \nabla$. These are the representations of p and L in the "x,y,z" world, and you could compute the momentum say of a state by doing this: $\langle \psi | p | \psi \rangle = \langle \psi | x \rangle \langle x | p | x \rangle \langle x | \psi \rangle$ and the thing in the middle is $\delta(x-x')(\hbar/i)\nabla$ (I think). Here in E&M Jackson is using the same operators with $\hbar=1$ just because they are convenient for his manipulations which are going to involve the same Y_{lm} that appear in spherical coordinates in QM. In this chapter we are not "interpreting" L very much, just using it as a tool. If you write ∇^2 in sphericals, you get 16.30 where L^2 takes care of the angular stuff. So, I am happy with all this stuff on p 542-543.

16.2 Multipole Expansion of the Fields. First off the bat, Jackson writes two 3-equation sets which are claimed to be equivalent to Maxwell's 4 equations. In each case, the fourth is true because $\text{div curl} = 0$. So let's go with a B expansion, since either will do. Since B as a vector solves the wave equation, we get 16.35. BUT, this expansion as written only solves the wave equation member of our 3-equation set. We have to *impose* that $\nabla \cdot B = 0$ on the coefficients. This turns out to be a huge mess, and Jackson takes us through the gory details, and when the dust settles, we replace 16.35 with a sum with some as-yet-not shown coefficients over a set of basic fields B_{lm} given by 16.42. These have the form of any radial solution times not $Y_{lm}(\theta, \phi)$ but $L Y_{lm}(\theta, \phi)$. In 16.45 Jackson normalizes these things and calls them $X_{lm}(\theta, \phi)$. He selects X as a letter close to Y, and $\mathbf{X}$ is a vector.

Now here is a major point. Notice that $\mathbf{r} \cdot \mathbf{B}_{lm} = 0$ because $\mathbf{r} \cdot \mathbf{L} = 0$ in an operator sense from 16.27. (From mechanics, the angular momentum L is perpendicular to both $\mathbf{r}$ and $\mathbf{p}$, since $L = \mathbf{r} \times \mathbf{p}$.) This means that all the solutions $\mathbf{B}_{lm}$ in 16.42 are "transverse magnetic" or TM. The B field is exactly perpendicular to $\mathbf{r}$. Note that there is in general no reason why this must be true in a general solution, although it is in a plane wave solution of Maxwell's. But this set of solutions does have the TM property. And of course if you start with E instead, you get the $\mathbf{E}_{lm}$ shown in 15.44 and these will all be TE solutions. Luckily these two sets of solutions form a complete set! These $\mathbf{E}_{lm}$ and $\mathbf{B}_{lm}$ are called "the multipole fields".

Now TM is better called "electric" because we shall soon see that the driving source of such fields is electric charge, whereas the "magnetic" TE solutions are driven by magnetic charge. Recall right now from electric dipole radiation that E stays parallel to $\mathbf{p}$ no matter where you go, so this cannot be a TE mode. It turns out to be a TM mode and the dipole B field is going to be fully transverse.

So now we have the Grand Finale of this section which is expansion 16.47. Notice that B and E shown have all possible TM and TE components each. The f_l and g_l are not specific radial functions, they are some arbitrary linear combinations of the two radial basis functions. Notice that the same coefficients appear in each equation like a_E because that is the way the solutions are proportioned, just do it! Similarly, whatever the function f_l is, it must be the same function in both equations, and the same for g_l .

This discussion implies that you can and should study the individual modes by themselves. The big advantage here is that we can apply spherical system boundary conditions most conveniently in this form!

16.3 Properties of the Multipole Fields. The first order of business is to study these fields at very small and very large radius r .

First the small r situation. For electric, we get 15.61 which replicates our multipole expansion equation 4.1 from electrostatics, and 16.48 goes with it. The B field here is kr times smaller in size than the E field, another reason we might call this "electric" instead of TM. For the "magnetic" modes, opposite is true, and you can always reverse things with 16.52.

Now for large r with our usual outgoing boundary condition. In this case the radial function has to be the Hankel-1 which has the right large- r form. (Note that Hankel 2 is the complex conjugate and therefore goes as $\exp(-ikr)/r$ which is incoming, not outgoing) For the electric we get B as in 16.53 as the large- r limit, and then E is the usual far-field $B \times n$ result, so in the far zone B and E are equal in size. Note Well: all multipoles have solutions which go as $1/r$ and thus can exist in the large- r limit, there is nothing special about dipole.

Now we stay in this large- r limit and ask the questions: how much energy, and how much angular momentum is carted off by a given multipole field? On page 548 Jackson decides to look at the electric multipoles { I suspect the conclusions are the same for the magnetics, but I don't think he comments on this } and he computes the answers to the two questions. He gives the electric multipole its usual coefficient a_ℓ as shown in 16.57. We are going to assume that this is all we have at the moment.

Recall from page 200 that the instantaneous linear momentum density is $\mathcal{P} = \mu\epsilon S/c^2 = S/v^2$ where $S = (c/4\pi) E \times H$. Also recall from that page that the instantaneous angular momentum density is $\mathcal{L} = r \times \mathcal{P}$. If we use our time average theorem (see Chap 7 notes), then we add $1/2$ and put $*$ on the second field, so we then have $\mathcal{P} = (1/8\pi c) E \times H^*$ in this sense, and $\mathcal{L} = r \times \mathcal{P} = (1/8\pi c) r \times (E \times H^*)$. Instead of using the symbol $\mathcal{L}$, Jackson uses symbol m for this in 16.61. { He does not want to confuse $\mathcal{L}$ with L }

So, how much U and how much $\mathcal{L}$ is contained in a particular multipole field? The time averaged energy dU contained in a shell of thickness dr is given by 16.60. The time averaged $\mathcal{L}$ in the same shell is given in 16.64. In both cases, the angle stuff is exact, and we use large- r (radiation zone) for the radial stuff. Yes, only $\mathcal{L}_3$ does not vanish because of the raising/lowering business. We get the famous result then that $\mathcal{L}_3/U = m\hbar/\hbar\omega$ and $\mathcal{L}_1 = \mathcal{L}_2 = 0$. We are saying that an electric multipole field ℓm in the radiation zone only has angular momentum about the polar z axis and the amount is "as if" the field were made of "photons" having energy $\hbar\omega$ and having $L_3 = m\hbar$. This is some kind of special ℓm photon!

On page 549 Jackson tries to explain why we don't get an $\ell(\ell+1)$ form for $|M|$ the way we do in quantum mechanics, but his discussion evades me, he is really quoting from another source. This is one example of why engineers probably don't like Jackson's book much, because of this "real physics stuff".

Now, let's talk about how this " ℓm photon" is produced by quantum state transitions of matter, perhaps from atoms. Suppose the quantum states are $|JM\rangle$ and $|J'M'\rangle$. Then the selection rules are these:

J' must be contained within the series $J \otimes \ell = J-\ell$ up to $J+\ell$

$M' = M + m$ // Jackson has the other sign on m , I don't see why!

for parity conservation, must have

$\text{Parity}(|J'M'\rangle) = \text{Parity}(|JM\rangle) * (-1)^\ell$ for electric, $(-1)^{\ell+1}$ for magnetic

The first two rules are just conservation of angular momentum.

This process of emitting an ℓm photon is called a "multipole transition". I don't have any idea of what is really going on in such a transition. In general it must involve multiple true photons, since each photon only really has $j = 1$. Do I have a book where this subject is discussed? On the web I see a guy saying that in spherical geometry, you really do have ℓm photons. I have to imagine this as a coherent state of

multiple photons -- a multi-photon state that has quantum numbers ℓm . Still, various people do speak of the ℓm thing as "a photon". I will have to put this subject off to another day. I think there are interesting implications. Although I have many physics books, none of them seem to address this subject, but I have not looked in depth. (Question: in positronium we have s_1, s_2 and L combining to make J . In a 2-photon state, is there something like L ?)

The bottom line for me regarding multipoles is this: probably in a dielectric material like water, selection rules are probably going to cause all transitions other than electric dipole (electric, $\ell = 1$) to be very weak. If a material *can* emit $\ell=5$, that is fine, and this will radiate just as far as $\ell=1$. But this probably does not happen in normal materials.

16.4 Angular Distribution of the Multipole Fields. Recall that we had the general multipole expansion for B and E showing in Grand Finale 16.47. At that time, we were supposed to think of the f_ℓ type function as an arbitrary lin comb as in 16.43 with two more arbitrary coefficients that depend on ℓ . Since we have another coefficient layer on top of this (the a_ℓ for example), we could decide to normalize these lower level coefficients so the sum of squares is 1. In any event, in the far zone where we have only h_1 , we now give it a unity lower level coefficient. Doing this, we then arrive at 16.69 for the large- r form of the full fancy expansion! In this limit, the two fields have the simple $n \times$ relationship, and now we see the all-important upper level coefficients exposed in this limit.

Getting the power per angle function 16.70 is a trivial step. For a single pole term we get 16.71 in terms of the fancy X harmonics. If you expand that out in terms of the Y functions, you get a mixture or raising, lowering, and non-changing L operators, so we expect to see a mixture of adjacent m values. The result is stated in 16.72 without proof. Jackson's table on page 551 then shows the exact results of the X^2 object for $\ell=1$ and $\ell=2$. The dipole interpretation is this: $m=0$ term would come from a z -aligned dipole of size $p = a/k^3$ as I learn by comparing to 9.23. If you had an x -aligned, you would get a mix of $m = \pm 1$ in the usual combination.

We see the plots for the $\ell=1,2$ multiple cases (all of them). Jackson's comment is a good one on page 553: If you have some atoms radiating thermally, say, you would expect an incoherent sum of all the m multipoles for a given ℓ , and 16.73 shows that the result is isotropic as it of course has to be. But, if we stimulate atoms with a special field (plane wave, eg), this will not be the case, I suspect. He concludes by integrating the differential power over solid angle to get very simple results.

16.5 Sources of Multipole Radiation: the Moments.

Digression on magnetization M . Look back at page 151 in the magnetics area which I skipped on this reading. Equation 5.77 shows the notion of a magnetization density $M = n \langle m \rangle$ in exactly the way we talked about $P = n \langle p \rangle$ on page 118. As in the p case, you can have already-existing m dipoles in a material like iron which just get lined up, or you can induce your own m dipoles. Interestingly, Jackson does not have a little section on "models for M " in his book, the way he does for P . Portis does discuss this on page 243, however. Roughly setting $M = \chi B$ and studying a simple model for an electron, he gets result (24) for χ . This is "diamagnetism". Electrons tend to spin around in a B field in circles and make moments m , so in diamagnetism we are creating these moments. In paramagnetism these orbits already exist and we thermally line them up, giving 38. Spin gets into the act as well, so you can see that this is a complicated subject. Now, for our current purposes, the main point to note about M is that its presence in a material makes an effective current density which is $J_M = c \nabla \times M$ as on page 152. This is analogous to $\rho_P = -\nabla \cdot P$ as on page 112. Recall that you can say that $\nabla \cdot E = 4\pi(\rho_{free} + \rho_P)$ but $\nabla \cdot D = 4\pi\rho_{free}$ where D is

defined exactly so it only sees free charge. In similar fashion you define $\mathbf{H}$ so that $\nabla \times \mathbf{B} = (4\pi/c) (\mathbf{J}_{\text{free}} + \mathbf{J}_M)$ but $\nabla \times \mathbf{H} = (4\pi/c) \mathbf{J}_{\text{free}}$.

So, as we start out here, Jackson seems to be setting $\rho = \rho_{\text{free}} + \rho_P$, but he has $\mathbf{J} = \mathbf{J}_{\text{free}}$ and he breaks out the $\mathbf{M}$ stuff separately, calling it script $\mathbf{M}$. Next, Jackson has to define a new field variable $\mathbf{E}'$ as in 16.79 to incorporate the free $\mathbf{J}$ just to simplify the manipulations that are coming.

The first order of business is to now rewrite our "wave equation sets" by adding the sources, and this is done in 16.80 and 81. The result is an incredible mess! You wonder why people don't do the expansion in the vector potential $\mathbf{A}$ where the sources are simple, but I guess the reason is that we don't know how to put boundary conditions on $\mathbf{A}$ to find a solution. So the $\mathbf{B}$ wave equation is driven by $\nabla \times \mathbf{J}$ and a double curl on $\mathbf{M}$ which seems pretty clear. The $\mathbf{E}'$ equation is symmetrical in that it is driven by $\nabla \times \mathbf{M}$ and a double curl of $\mathbf{J}$. Hey, where did ρ go? Now we see the motivation for $\mathbf{E}'$! It is defined exactly so that $\nabla \cdot \mathbf{E}' = 0$. The $\mathbf{J}$ term is added to $\mathbf{E}'$ and then $\nabla \cdot \mathbf{J}$ is known in terms of ρ from continuity. So the answer is that the wave equations don't have ρ because they come from the Maxwell curls only. Then this $\mathbf{E}'$ trick gets rid of ρ as just mentioned. Fine.

Now things are going to get VERY serious! First, the new $\mathbf{B}$ and $\mathbf{E}'$ equations match the former $\mathbf{B}$ and $\mathbf{E}$ equations outside the sources, so there we know the Finale 16.47 applies for $\mathbf{B}$ and $\mathbf{E}'$ (another reason to do $\mathbf{E}'$ as done). So let's try something like 16.82 with asymptotic limits 16.83 to make things consistent with our general notation in the far zone. That is, this 16.83 is setting the scale on our functions f and g . Jackson then swirls around to come up with ODE's for f and g which are driven by complex source terms, see first two equations on page 555. The ODE left sides are exactly what we had earlier way back in 16.5 page 539 (the spherical Bessel operator, so to speak). We already computed the Green's function for this operator on page 541 16.21, and now we are going to use it! It is just a product of two Bessel functions, the inner one j , the outer one h . So, creating a shorthand notation K_E for the messy right side of 16.84, we instantly have the Green's solution 16.86 for f . Taking the large r limit, we are then able to identify the a_E coefficient, and this is then stated in 16.88. Same for a_M below that. Obviously, these things are "moments" of the source situation that drive the multipole fields.

The two results for these moments are then "simplified" and presented again on page 556. Each one is now an integral of source stuff against a Y_m function, the whole thing volume-integrated over the source region, and we notice that our friend ρ has reappeared in the first equation.

Let's pause to think about this. If we stimulate a water sphere with a plane wave, we can probably decompose the plane wave into multipole components (soon!). We can then think of these as somehow inducing a $\mathbf{J}$, ρ and $\mathbf{M}$ in the water droplet -- and that stage of course involves the "physics" of the matter known as water; perhaps there is no significant $\mathbf{M}$. Those induced sources then in turn imply some multipole moments according to the integrals shown on page 556, and these then give radiated multipole fields for $\mathbf{B}$ and $\mathbf{E}'$. We know what each moment field $\mathbf{E}'$ or $\mathbf{B}$ looks like for large r , and in theory we can compute the details at any r , but then we have to do the detailed integrals as in 16.86. This is because we need to know the f and g functions which appear in assumed 16.82. The good news is that we have a general solution to the entire problem in terms of some integrations. By the way, to carry out the induction program mentioned here, we have to assume the radiated fields are locally weak compared to the incoming field, something I want to see a proof of here at some point.

Now in the large λ limit (meaning $kd \gg 1$ where source has size d), things simplify because we can take limits of the Bessel functions. We end up with four kinds of moments called Q , Q' , M and M' as detailed on page 556-7. We recognize Q as our electrostatic moments! We have $a_E \sim Q + Q'$ and this second Q' thing can arise only if you have M , and in that case, $Q' \ll Q$. So the main idea is that a_E is

mainly driven by ρ , and that is why we called the TM modes "electric". We also have $a_M \sim M + M'$, where M is driven by the free current J , and M' comes only if there is M . M and M' are usually the same size so you have to worry about both of them if M is present. Now we see why he used script M -- to avoid confusion with these final long- λ moments!

So wow, again, we have a complete solution of things in a form geared to distant radiation problems from a localized source that is small relative to λ .

16.6 Radiation from Atoms and Nuclei. Another section very important to me. For both atoms and nuclei, we are going to use the same format "ball park" estimates for the electric and magnetic sources.

Rule 1: for EM radiation by either atoms or nuclei, we are usually in the long λ limit $kd \gg 1$. Certainly for atoms and visible light this is the case, compare 1A to 5000A. In our simple atomic model, our estimate for kd is $ka \sim (Z_{\text{eff}}/137)^2$. For nuclear, ka cover too large a range for us to say something about it.

To say that we have a ρ that is going to drive a *particular* multiple electric field is to say 16.99. Then in the integral 16.94 only one Q_m is generated. The dimensions require some kind of e/a^3 factor, where e is the charge of the "thing" that is radiating (electron in atom, proton in nucleus), while a is the size of the thing you are interested (atomic or nuclear radius). Jackson throws in a factor of 3 here probably just to make things look nicer down the road, no comment is given. Maybe each atom has 3 radiating electrons? I think it is just to make things simpler in our later comparisons. So I agree that 16.99 is reasonable for ρ .

Now 16.101 is trickier to understand. Here, he combines the M and M' integrands together. We dimly recall how magnetic dipoles are enhanced by g -factors, so we will allow such a dimensionless factor. The basic unit is going to be the Bohr magneton for atoms. In Livesey page 163 we see that this is the classical magnetic dipole moment of a circulating electron $m = e\hbar/2m_e$ usually called μ_B . The electron spin on its own has a similar moment, give or take a factor of 2. For the nucleus, the appropriate mass is m_p and again moments come from both "orbital" and "spin" presence in the nucleus.

So, in 16.101 we have the g factor, we have a $1/a^3$ for our volume dimension factor, and we have an appropriate magneton factor. The $1/r$ there must be due to our having $\nabla \cdot \mathbf{M}$ and not M . Note that in spherical coordinates, the divergence of a constant vector $\mathbf{A} = A\hat{\mathbf{r}}$ is $\nabla \cdot \mathbf{A} = A(2/r)$. All the divergence terms have a $1/r$ in them due to the $r^2 dr$ volume factor, so I am happy to have a $1/r$, and we need it as well just to have the dimensions be right. Finally, we have our usual Y_m for this assumed multipole.

Using these crude estimates for the driving factors of our moment integrals, we get the three simple results shown on page 558 for Q , Q' and $M+M'$. Right off the bat, we get a major result:

Rule 2: For either atoms or nuclei, $Q' \lll Q$ due to relation 16.103, so forget Q' (the "induced electric")

At the same time, we use 16.98 to talk about our transition rates for radiation between two quantum states. That is, we are simply dividing the total power (as computed in 16.97 for a multipole) by the photon energy which has dimensions 1/time. We associated $1/\tau$ with the rate of transition measured in number of photons/sec. Using this general formula and inserting our electric result 16.100 we get 16.104 for our transition rate.

Aside: { In answer to an earlier question I had, you can add $\boldsymbol{\mu} \cdot \mathbf{B}$ into a Hamiltonian and get quantum split levels due to this thing and I presume these would be M1 transitions. }

Now, in 16.105 we are comparing the rate of electric versus magnetic for the same order ℓ , and then in 16.106 we compare order ℓ to order $\ell+1$, which comparison is the same for either electric or magnetic.

Atomic Radiation. In this case, the factor in 16.105 is roughly $(Z_{\text{eff}}/137)^2$ which is a very small number unless you are talking X-rays from a deep inner shell of a very heavy atom. For water this is going to be on the order of 10^{-2} to 10^{-4} . So we get this rule:

Rule 3: In atomic radiation of order ℓ , magnetic is *weaker* than electric by 10^{-2} to 10^{-4} $[(Z_{\text{eff}}/137)^2]$. So you will see the magnetic only if the electric is somehow forbidden by selection rules.

Rule 4: In either atomic or nuclear radiation of either type (M or E), the transition rate in order $\ell+1$ is smaller than in order ℓ by $\sim (kd)^2 \sim (Z_{\text{eff}}/137)^2$, as in 16.108. Therefore, within a multipole family (M or E), the lowest allowed ℓ dominates. Usually this is electric dipole for the E family. So, when electric dipole is fully allowed, you don't have to think about higher order Q moments, assuming they are "stimulated" in a similar strength in an induced radiation situation. Thus, even if we find that a plane wave has large higher Q moments beyond $\ell = 1$, they are not going to matter when we scatter light off water drops.

Now some nuclear-only comments, starting bottom of page 559.

From the 16.104 estimate, you can plot lifetime τ versus ω (or $\hbar\omega = E$) as shown, and you get a different line for each order ℓ due to the power law you see sitting in 16.104, which is $\omega^{2\ell+1}$ so on log log plot we get $\log \tau = -(2\ell+1) \log E + \text{constant}(\ell)$. One point to be made here is this: for a fixed energy release E (draw a vertical line in the plot), you find that the higher order ℓ transitions have longer times and lower transition rates. Jackson claims that if you study a boatload of nuclear EM transitions, they fall in bands close to the lines shown in the figure.

Now for nuclear, what about ratio 16.105 for E versus M of the same order ℓ ? In general, electric are stronger by 25-120 times except for $\ell=1$ where the E1 is inhibited by a characteristic of nuclei (perhaps naively you don't see much a p moment in a ball of protons), so E1 and M1 are about the same in this one case.

Now a slightly different reading on ratio 16.106. For a given parity, if the lowest order of M is ℓ , then the lowest order in the E family for the same transition will be $\ell+1$, so 16.110 is a ratio you really care about. In nuclei, you might find that Q2 is 5% of M1 as being typical. So if the lowest allowed is M, then you might have *some* noticeable E in the next higher order. However, if the lowest allowed is E, then 16.111 shows that the next parity allowed transition $M_{\ell+1}$ is completely negligible.

16.7 The very famous center fed antenna. Total antenna length is d . Do NOT assume $kd \gg 1$. Study this case *very* carefully please. There is no $\mathbf{M}$, so ignore those terms in 16.91 and 16.92. Inside the antenna wires, $\mathbf{r} \times \mathbf{J} = 0$. This kills off the only remaining term in the $\mathbf{a}_M$ equation, so a major fast result:

(1) there are no magnetic modes at all, so all modes are going to be electrics such as electric dipole E1

Now for current, Jackson for the moment just assumes some unknown $I(r)$ that vanishes at the ends of the antenna wires. He can then write $\mathbf{J}$ as in 16.113 and then ρ as in 16.114, using delta functions. Since $\mathbf{J}$ is

in the $\hat{r}$ direction, and since we have $\mathbf{r} \cdot \mathbf{J}$ in the a_E equation, the integrand has no ϕ dependence, which means that:

(2) only the $m=0$ electric modes survive. We knew this anyway from symmetry.

Now, the integrand has no θ dependence either, apart from the $Y_{\ell m}$ function. Since the even ℓ $Y_{\ell 0}$ functions are odd, they are killed off as well, so

(3) only the $\ell = \text{odd}$ modes survive.

So we now want to compute up the surviving modes which are ℓ odd and $m=0$ using 16.118, and now we need a model for the current $I(r)$. The wave equation *suggests* $\exp(ikz)$ type dependence along the wires, as if we had a plane wave in space along the z direction. If you made a parallel wire transmission line, you would expect to see $\exp(ikz)$ along the line, but this is another subject Jackson does not treat in his book (well maybe he does). You could regard the conductors as mere boundary conditions and the wave travels along them at the full speed of light with $\exp(ikz)$ in the ideal case. When there are ohmic and radiative losses, you have to modify this result. So Jackson is saying lets try to ignore radiation loss and ignore ohmic loss and put in the idealized current shape which then is 16.119 since it has to vanish at the ends. Boom, you can now get the result as in 16.120! All done! This is the general result (with our assumptions as stated) regardless of the relationship between d and λ .

Jackson next considers the half-wave and full-wave antennas as examples. For the shorter antenna, the table shows that the E_1 is most of it, and the E_3 is about 5% and the E_5 is 0.1%, we are talking amplitude here. For the longer antenna, things converge more slowly. The E_3 is now 33% of E_1 and the E_5 is 3% in amplitude. Things get messier when they are large compared to λ , keep that in mind!

He goes on to compute the power distribution for the sum of $E_1 + E_3$ for these two antennas. The math is done on page 565 with some mildly messy numerical results.

Now back in Chapter 9 we solved the arbitrary length center-fed antenna with the same assumptions as here (same current assumed, same ignoring of radiative and ohmic losses), but there we used the full power far-zone result (9.8) so we computed all orders at once! Here we are just doing $E_1 + E_3$, so it is interesting to compare the results. Jackson did write down the half and full wave "exact" results on page 279, so we can compare our approximation here to that. The plots on page 566 are for this purpose.

Aside: I recall how my Harvard independent study guy King used to pooh-pooh books like Jackson's for assuming the current is not affected by the radiation. "Real" antenna people like King take this into consideration.. Recall also that I have a whole fat Transmission Line Theory book by this same King.

16.8 Expand a plane wave in sphericals. The result we want is 16.127 or 16.128, and Jackson derives these quickly by just taking the large r limit of the $\exp(ikR)/R$ result we already found in 16.22. He gets here to use his "sum rule" found earlier. (It is always nice to get future use out of an obscure result!)

Look at 16.129. For sure all ℓ orders appear here, although only with $m=0$. The z -axis here can be thought of as the k axis in which case we identify $\gamma = \theta$. There is lots of energy in lots of orders, so you cannot say a plane wave is a dipole E_1 or anything like that! In fact, a plane wave is going to try to induce all $Y_{\ell 0}$ modes into an object that will radiate. We should see this happen in the conducting sphere example to follow. Some day I will try to graphically add the terms to see how a plane wave arises here!

Now Jackson starts over without clearly explaining his motivation, but I am now aware of it. In this section, Jackson says: " Think about 16.128 but now add a unit vector to it so you have a real field

instead of just a scalar function. In fact, take the traditional linear combinations of those unit vectors that define the circular polarizations. We like these because they are going to allow us to use $L_{\pm}$ operators in our calculation which will greatly simplify things. Now using this, let's derive result (16.139). We are going to need this result in the next section in the problem of scattering from a conducting sphere! "

Sidetrack and Confusion: I tried to show these circular polarizations somehow diagonalize $L_z = -i\partial_{\phi}$ with ± 1 eigenvalues, but concluded that in fact, for a given t and z , a circ pol plane wave has no ϕ dependence, so $L_z = 0$. In fact, if I compute $\mathbf{L} e^{i\mathbf{k}\cdot\mathbf{r}}$ I find that $d\mathbf{L} = \mathbf{L} e^{i\mathbf{k}\cdot\mathbf{r}} dV = k e^{i\mathbf{k}\cdot\mathbf{r}} \sin\theta \hat{\phi} dV$ for a little volume dV at location r, θ, ϕ relative to $\mathbf{k}$. If I integrate this around the ϕ ring at r, θ , we get zero. So then the integral over the entire plane wave is also 0. So the total "angular momentum" of a plane wave is 0. BUT, I have here not really computed the $\mathbf{L}$ of a field, only of a scalar function $e^{i\mathbf{k}\cdot\mathbf{r}}$, and $\mathbf{L}$ can only be applied to a scalar function. Still, one might talk about $\mathbf{L} \cdot \mathbf{E}$ where we really mean $\mathbf{L} \cdot \mathbf{E}_i$ and then it is true. In a plane wave, we have photons traveling in the $\mathbf{k}$ direction, so the ring integral just described really is zero. I don't see how the circ pol changes this situation.

However: if we look in the problems on page 200, we see that Jackson has in fact pondered this question with a medium ϵ, μ . In the above paragraph, I was thinking of $\mathbf{L}$ as the operator $\mathbf{L} = (1/i) \mathbf{r} \times \nabla$. However, on page 200 we show that a EM field has an angular momentum density $\mathcal{L} = \mathbf{r} \times \mathcal{P}$ where this thing $\mathcal{P}$ is the linear-momentum density of the field, given by $\mathcal{P} = \mathbf{S}/v^2$ where $\mathbf{S}$ is the Poynting vector, and $v = \text{velocity} = c/\sqrt{\mu\epsilon}$, so $\mathcal{P} = \sqrt{\mu\epsilon} \mathbf{S}/c^2$. In problems 6.11 and 6.12, which I have now more or less done, we show that a localized plane wave (Gaussian beam like) with circ pol does in fact have a z component of angular momentum that is ± 1 when scaled in the usual way $N\hbar/\hbar\omega$. So my confusion was to confuse $\mathbf{p}$ with $\mathcal{P}$ and $\mathbf{L}$ with $\mathcal{L}$. Also, in the following when we expand a true plane wave in sphericals, we will again find that for each multipole component ℓ of such a plane wave, we still get a z component of angular momentum that is ± 1 . Here the units are $m = \pm 1$. There are no components of a plane wave that have $m > 1$.

Resume: So how does Jackson derive 16.139? He starts by assuming the expansion 16.131 for a plane wave of circular polarization. This is the Grand Finale 16.47 where we have assumed j for both radial functions. The reason for doing this is that anything else blows up at $r=0$, but plane waves do not blow up there. From this expansion, we can extract the coefficients as in 16.133,4. Now in 16.133 for $\mathbf{E}$ we put our simple form 16.130, the plane wave form $e^{i\mathbf{k}\cdot\mathbf{z}}$. This is the point at which our choice of circ pol gives us the $L_{\pm}$ operators in 16.135, resulting in 16.136. At this point we shove in our earlier expansion 16.127 for $e^{i\mathbf{k}\cdot\mathbf{z}}$ and out come simple results 16.137,8. Thus, we now know the coefficients in our original expansion 16.131 and the result is 16.139. Notice that the m sum went away because all the coefficients only exist for $m=1$ or $m=-1$, depending on which circ pol you are talking about.

If we think of our plane wave as photons, although we still get a sum over all ℓ (recall the ℓm photons discussed earlier), we only get $|m|=1$, suggesting that the plane wave does have a component of angular momentum about the z axis that is in fact ± 1 . This makes us think of a photon with two states of longitudinal polarization, but we know photons only have transverse polarization, so not quite clear how you fit the photon picture in with our plane wave expansion! Defer this to a rainy day.

16.9 The Conducting Sphere Scattering Problem!

This is a big problem, and is probably the closest thing in Jackson to my dielectric water sphere scattering problem, so pay close attention! See details of the solution at the end of this document! We assume

separate forms for incoming and outgoing fields. For the incoming fields, we use our plane wave expanded in the very fancy 16.139 that we just pondered in the previous section. For the outgoing we use 16.141 where NOW (unlike in the plane wave expansion in the last section) we put Hankel-1 everywhere because it alone has the right asymptotic behavior e^{ikr}/r .

Now, why do we only have $m = \pm 1$ in expansion 16.141? Jackson is uncharacteristically silent on this question. We suspect it has to do with conservation of L_3 . I will create my own argument. Suppose we localize the incoming plane wave to a very long but finite packet, by slightly mixing frequencies. Then we can talk about "before" and "after" the scattering process. The sphere might absorb some energy in the process, but maybe it is not going to absorb any angular momentum. If we knew this to be true, then we might argue that the outgoing wave must have the same L_3 as the incoming. But still this seems vague to me.

Let's see if Fitzpatrick has anything to say about this. This is a guy on the web who has made lots of Jackson-like notes. Here is the top level for his notes from one class:

<http://farside.ph.utexas.edu/~rfitzp/teaching/jk1/lectures/lectures.html>

This is the HTML presentation of his notes. He has another web page

<http://farside.ph.utexas.edu/~rfitzp/teaching.html>

which shows ALL his course, and also leads to PDF versions of his notes. These are much better since the equation numbers are correct. I have downloaded his multipole notes. His notes follow extremely closely to Jackson, same notation, he admits that Jackson is his main source.

He makes this comment: "a spherically symmetric scatterer cannot couple different m components". Well that is not much of a proof! I just asked Jim about this.

I think the real answer is this: you could assume all the m terms in the scattered wave and then you would have coefficients $a(\ell, m)$ in the outgoing equation. But when you then tried to have the total fields satisfy the boundary conditions at the sphere, you would find that the other m waves have to be zero or you cannot make the match. In other words, what you would get is something like this:

$$\hat{\mathbf{r}} \cdot \mathbf{B} = 0 \Rightarrow \sum f(\ell, m, r=a) \mathbf{X}_{\ell, m} = F(\ell, r=a) \mathbf{X}_{\ell, \pm 1}$$

and you would then say that since each $\mathbf{X}_{\ell, m}$ has its own characteristic angular dependence, the match requires that all the other m coefficients vanish. This would be true for any boundary conditions that were "spherically symmetric" because the same general argument above applies. So OK.

I now accept 16.141 as the general form of the scattered wave. The boundary conditions are extremely elegant, being 16.142 where $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. The form 16.143 is very helpful in arriving at the boundary conditions 16.144, along with the fact that $\hat{\mathbf{r}} \cdot \mathbf{X}_{\ell, m} = 0$. Notice that the $\hat{\mathbf{r}} \cdot \mathbf{B} = 0$ equation determines the α coefficients, and the $\hat{\mathbf{r}} \times \mathbf{E} = 0$ determines the β ones. If you write these coefficients only using hankels, you see that each ratio in 16.145 is just a phasor as shown in 16.146, so this makes for a very simple formula for the coefficients 16.148. The solution to the entire problem is in these little phase shifts called δ_ℓ and δ'_ℓ which are functions of k , a and ℓ . The large and small k limits are shown in 16.149, 150. Notice that α and δ are associated with the *magnetic* (TE) fields in 16.141. This is the coefficient called a_M in 16.47.

So the final result for the scattered wave is stated in 16.151 where the α and β are done as phasors times sines. Keep in mind: this is the exact result for this problem, even very close in to the sphere, so we really DO have this problem solved. In the radiation zone the limit for B is shown in 16.152. Since the phases in a partial wave are not likely to be equal, the polarization in each partial wave of the resulting wave is elliptical.

Jackson goes on to write down the differential cross section as in 16.155 and he remarks that it is "rather complicated". In the dielectric sphere problem, Jim thought you need 40 partial waves before you actually see the rainbow stuff. We can of course do the large and small k limits (relative to sphere radius a). The exact total cross section however can be done (as is often the case, recall unitarity) and you find that each partial wave contributes as $(2\ell+1)$ times the sum of the sines of the phase shifts.

The long λ (small k) limit is dominated by $\ell=1$. Notice, by the way, that $\ell=0$ plays no role in this problem. You see that $\ell=0$ is missing from 16.139, the incident wave. In fact, $X_{00} = LY_{00} \sim \mathbf{r} \times \nabla Y_{00} = 0$ because Y_{00} is just a constant. So anyway, the $\ell=1$ dominates and is stated in 16.157 which is restated in 16.160 and is the same for either circ pol incident state. Notice that both the δ and δ' terms contribute to this result, so it is a mix of E1 and M1. The plot shows that most of the scattering is backward in this limit. The front/back asymmetry arises from the $\cos\theta$ term which arises from the cross terms which come from interference between the E1 and M1.

Finally, the total cross section in this limit shows the Rayleigh's Law k^4 dependence. I agree that we have seen this arise in our current problem, but it is not obvious to me that this would be case for any scattering that has mainly $\ell=1$. No doubt you can prove it to be the case, and we know that it is the case in simple dipole radiation models as we have made earlier. For example, on page 271 we know that S the Poynting will be proportional to k^4 in any dipole problem, so I guess that clinches it!

In any event, the work done in this section was first done by Mie and Debye in 1908-1909.

References: Born and Wolf give the *general* sphere solution, allowing it to have dielectric and conducting properties! I might go check this out somewhere. // I just web-ordered 7th edition for \$64 hardback, I hope it still has that solution Jackson mentions. (he refs a 1959 Principles of Optics edition).

Problems: The dielectric sphere problem is assigned as problem 16.12, the last one. I don't see a solution to this messy problem on the web, I can imagine why, it is a mess.

I notice that our friend Fitzpatrick entitles this section of his "version of Jackson" as Mie scattering. In doing so, he notes that you can write out everything in terms of the coefficients without ever using the boundary conditions, then apply them at the last minute. He then does the same conductor as Jackson, so he does not attempt the general problem!

Appendix A. Details of Jackson's Boundary Conditions for the Conducting Sphere

The first thing to understand is this: the three vectors ($\hat{\mathbf{r}}$, $\mathbf{X}_{\ell m}$, $\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$) are mutually perpendicular. How do we know this? Well, $\mathbf{X}_{\ell m} = \alpha_{\ell} (LY_{\ell m})$ where $\alpha_{\ell} = (1/\sqrt{\ell(\ell+1)})$. We know for sure that $\hat{\mathbf{r}} \cdot \mathbf{X}_{\ell m} = 0$ because we know $\hat{\mathbf{r}} \cdot \mathbf{L} = 0$. This says that $\mathbf{X}_{\ell m}$ lies in a plane perpendicular to the unit vector $\hat{\mathbf{r}}$. The third vector here gives 0 when dotted with $\hat{\mathbf{r}}$ or $\mathbf{X}_{\ell m}$, since $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$, so it also lies in that same plane with $\mathbf{X}_{\ell m}$ but is perp to $\mathbf{X}_{\ell m}$. Of these three vectors, we are only claiming that $\hat{\mathbf{r}}$ has unit length. The

vector $\mathbf{X}_{\ell m}$ is a vector of functions of θ and ϕ so is likely not to be of unit length for all such angles. You can and should think of $\mathbf{X}_{\ell m}$ and $\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$ as the two perpendicular "transverse" vectors, as if you were thinking about polarization of a plane wave in the $\hat{\mathbf{r}}$ direction.

The second thing to understand is Jackson's 16.143. I have proven this by hand for an arbitrary radial function $f(r)$, the function doesn't have to be spherical Bessel's the way Jackson might imply. The important thing to realize is that this equation is showing that the quantity $\nabla \times [f(r) \mathbf{X}_{\ell m}]$ can be decomposed into a radial piece (the first term), and a transverse piece that is in the $\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$ direction.

Now about the boundary conditions. The two shown we understand: there can be no tangential E field, and there can be no normal B field. What about the other two possible boundary conditions here? What about normal E? The answer is (pill box straddles surface) that surface charge will arrange itself however it needs to in order to make normal E be what our solution makes it come out being! The same thing is true for tangential B, think of a loop and $B = 0$ inside the conductor. Surface currents $\mathbf{K}$ will flow however they must to make tangential B come as it comes out. So this is why we do not try to "impose" two more boundary conditions.

Here then are the fields as Jackson arranges them:

$$\mathbf{E}_{\text{inc}} = (1/2) \sum i^{\ell} \sqrt{4\pi(2\ell+1)} [2j_{\ell}(k_1 r) \mathbf{X}_{\ell \pm 1} \pm 2/k_1 \nabla \times \{ j_{\ell}(k_1 r) \mathbf{X}_{\ell \pm 1} \}]$$

$$\mathbf{B}'_{\text{inc}} = (1/2) \sum i^{\ell} \sqrt{4\pi(2\ell+1)} [\mp 2i j_{\ell}(k_1 r) \mathbf{X}_{\ell \pm 1} - 2i/k_1 \nabla \times \{ j_{\ell}(k_1 r) \mathbf{X}_{\ell \pm 1} \}]$$

$$\mathbf{E}_{\text{sc}} = (1/2) \sum i^{\ell} \sqrt{4\pi(2\ell+1)} [\alpha_{\pm}(\ell) h^{(1)}_{\ell}(k_1 r) \mathbf{X}_{\ell \pm 1} \pm \beta_{\pm}(\ell)/k_1 \nabla \times \{ h^{(1)}_{\ell}(k_1 r) \mathbf{X}_{\ell \pm 1} \}]$$

$$\mathbf{B}'_{\text{sc}} = (1/2) \sum i^{\ell} \sqrt{4\pi(2\ell+1)} [\mp i \beta_{\pm}(\ell) h^{(1)}_{\ell}(k_1 r) \mathbf{X}_{\ell \pm 1} - i\alpha_{\pm}(\ell)/k_1 \nabla \times \{ h^{(1)}_{\ell}(k_1 r) \mathbf{X}_{\ell \pm 1} \}]$$

Our boundary conditions are these:

$$\hat{\mathbf{r}} \cdot (\mathbf{B}'_{\text{inc}} + \mathbf{B}'_{\text{sc}}) = 0 \quad \#1$$

$$\hat{\mathbf{r}} \times (\mathbf{E}'_{\text{inc}} + \mathbf{E}'_{\text{sc}}) = 0 \quad \#2$$

For condition #1, notice that since $\hat{\mathbf{r}} \cdot \mathbf{X}_{\ell m} = 0$, we only pick off the second term in each B field expansion. Since the second term itself is then expanded as shown in 16.143, and since $\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}) = 0$, all that survives is the first term in 16.143 for each field. If we ignore the overall constant factors and the $Y_{\ell m}$ then we get this:

$$-2i/k_1 j_{\ell}(k_1 r) - i\alpha_{\pm}(\ell)/k_1 h^{(1)}_{\ell}(k_1 r) = 0 \quad \#1$$

and this is just what Jackson shows in 16.144.

Now consider condition #2; more terms survive. Out of each field we will now have an $\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$ term and an $\mathbf{X}_{\ell m}$ term. The first term in each field expansion obviously contributes to the $\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$ amount, but what about the second term. If we expand the second term using 16.143, the first term yields *nothing* since $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0$. We are left with a contribution from the second term which looks like $\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$. But this is easily shown to be $-\mathbf{X}_{\ell m}$. Now, since the two terms are along different (and mutually perp)

transverse direction vectors, the coefficients of the two terms must *separately* meet the boundary condition! Thus, we arrive at two conditions from condition #2:

$$\begin{aligned}\hat{\mathbf{r}} \times \mathbf{X}_{\text{em}} \text{ terms:} & \quad 2j_\ell(k_1 r) + \alpha_\pm(\ell) h_\ell^{(1)}(k_1 r) = 0 \\ -\mathbf{X}_{\text{em}}/r \text{ terms:} & \quad \pm 2/k_1 \partial_r [r j_\ell(k_1 r)] \pm \beta_\pm(\ell)/k_1 \partial_r [r h_\ell^{(1)}(k_1 r)] = 0\end{aligned}$$

Now it is lucky that the first condition here replicates the condition we already got from BC #1. The second condition is then as Jackson states in 16.144b.

Note that we have 2 equations in 2 unknowns (ignoring $\pm$ etc).

Appendix B. Changes in the presence of μ and ϵ

In order to even be able to attack the dielectric sphere problem (which I do in a separate document), you have to know how to "interpret" all of the Chapter 16 formulas in the case that you have a uniform ϵ and μ . I think I have found a good way to do it.

Section 16.2 on deriving the Grand Finale formula. You might think you could just replace B with H and E with D and have the $\mu\epsilon$ stuff cleared out of Maxwell's equations, but that is not the case. Looking at page 178, the problem is that the Faraday's law will then say that $\text{curl } \mathbf{D} = -1/c \partial_t \mathbf{H} * \mu\epsilon$. So we need another plan.

Go back to the starting point 16.31 and add $\mu\epsilon$ as shown based on 7.1 page 202 which shows the four Maxwell's in this case, notice there is a single insertion point. Go ahead and do the time derivatives and get 16.32. We are going to end up rescaling the B field, so the div equations won't change. We care about the curl equations. These are:

$$\begin{aligned}\nabla \times \mathbf{E} &= ikB & k &= \frac{\omega}{c} = \frac{\omega}{v} * \frac{v}{c} = k' \beta & \beta &= 1/\sqrt{\mu\epsilon} \\ \nabla \times \mathbf{B} &= -ik\mu\epsilon\mathbf{E} = -ik\beta^2 \mathbf{E} & & & \sqrt{\mu\epsilon} &= \beta^{-1}\end{aligned}$$

Now define the usual modified k (call it k') as shown above right. Then we get

$$\begin{aligned}\nabla \times \mathbf{E} &= ik'\beta\mathbf{B} & \nabla \times \mathbf{E} &= ik'(\beta\mathbf{B}) & \nabla \times \mathbf{E} &= ik'\mathbf{B}' \\ \nabla \times \mathbf{B} &= -ik'\beta^{-1}\mathbf{E} & \nabla \times (\beta\mathbf{B}) &= -ik'\mathbf{E} & \nabla \times \mathbf{B}' &= -ik'\mathbf{E}\end{aligned}$$

Therefore, we have shown that we can rewrite 16.32, our starting point, by making two replacements:

$$\begin{aligned}1) \text{ replace } k \text{ with } k' & & k' &= k / \beta & \beta &= 1/\sqrt{\mu\epsilon} & n &= \frac{1}{\beta} = \sqrt{\mu\epsilon} \\ 2) \text{ replace } \mathbf{B} \text{ with } \mathbf{B}' & & \mathbf{B}' &= \beta\mathbf{B} & & & &\end{aligned}$$

Once we have done this, all four equations look exactly the same. This is the main point!!!

The triplet equation groups page 543. These are as shown, except make the two replacements shown above.

The multipole fields page 545. These are as shown, except make the two replacements shown above. Notice that all Bessel functions will now have $(k'r)$ as argument.

Grand Finale 16.47 page 546. Put in k' for k everywhere, and the first equation is for B' . These are just the same changes made in the steps above.

Section 16.3 equations. Equations like 16.48 and 16.51 are just limits with constants not shown, so we are not bothered by not seeing $1/\epsilon$ in 16.51. The swap rule 16.52 of course should have our B' in place of B . I would put a B' in 16.53 and therefore also in 16.56. Same in 16.57 along with replacing k with k' there.

Now in 16.58 we have to make a different change! The true formula for u has ED and BH . So we would then write this as $\epsilon EE + (1/\mu) BB$. If we now rescale to our $B = B'/\beta$, we get $\epsilon EE + (1/\mu) \beta^{-2} B' B' = \epsilon [EE + B' B']$. In the radiation limit being done here, we have $|E| = |B'|$ from modified 16.56. Then we can write the energy $u = 2\epsilon (B')^2$. Therefore, 16.60 will have an extra ϵ out front.

Then in 16.61 we really should have H there and $H = B/\mu$. But then put in $B = B'/\beta$, and then 16.61 will have $1/(\mu\beta)$ out front. Using 16.57 with its actual k' and B' then gives 16.62 with both B' inside, and with $(1/\mu)$ out front. Thus, the result 16.65 will have this $1/\mu$ out front. The ratio in 16.66 will then say that $M/U(\text{modified}) = M/U(\text{shown}) * (1/\mu) * (1/\epsilon)$. So the ratio is then no longer as shown and I do not know how to interpret the modified result.

Section 16.4 on distribution. The true time averaged power will be $E \times H = (1/\mu) E \times B = (1/\mu\beta) E \times B' = \sqrt{\frac{\epsilon}{\mu}} E \times B'$. Thus, we should have this square root factor out in front of 16.70.

Step 1: Let's go through the general multipole expansion and add the presence of uniform μ and ϵ . Start with 7.1 page 202 which shows the four Maxwell's in this case, notice there is a single insertion point. So, I have marked up how the two 3-equation sets on page 543 are altered.

One alteration is that you have to think of k as ω/v instead of ω/c , but only in places that got their k from the wave equation! This means inside any Bessel function like $j_\ell(kr)$, the multiplying k in there must be the $k = \omega/v$ version of k . However, the external factors of k are unchanged, namely, the $1/k$ that you see in the multipole fields on 545 except you have to add one $\mu\epsilon$ factor as shown for E in the electric mode case only.

Looking then at the general expansion 16.47, we have to add same $\mu\epsilon$ factor that we added into 16.42. Note that this is NOT just a rescaling of the a_E coefficient because the two expansions are coupled.

On page 548 I added new factors as shown.

Next, how is the huge 16.139 going to be altered?

Section 16.5 on moments. Looking at 16.77, Jackson is starting with the full equations including H , and he has left the polarization charge grouped in with ρ . So I have no modifications to make to this entire section.

Section 16.7 on center-fed antenna. Everything goes exactly as stated. However, when we get to the radiated power formula 16.121, we should add the same $\sqrt{\frac{\epsilon}{\mu}}$ factor out front as on page 550 described above, and in all other power formulas in this section.

Section 16.8 on Plane Wave Expansions. First consider 16.130. As usual, $k = k'$ in the exponent. The second equation there should be modified according to 7.13 on page 204 which says $B_0 = \beta^{-1} E_0$. Thus the second equation is correct if we just replace B by B' . So we just make the usual 2 changes in 16.130.

As for 16.131, it is just a case of the Grand Finale 16.47, and we know to think of B' on the left of the second, and all k are k' . Luckily for us, all the solution for the coefficients is unchanged and we end up with 16.139 with just our usual 2 changes.

Section 16.9 on the Conducting Sphere. As usual, replace B with B' everywhere and k with k' . We do this in the assumed form of the outgoing waves as well as in 16.139. The boundary condition can be taken as $\mathbf{n} \cdot \mathbf{B}' = 0$ so nothing changes throughout. The final results 16.151 and 16.152 are the same except interpret k and B as k' and B' .

Now we come to the power formulas. There should be a $\sqrt{\frac{\epsilon}{\mu}}$ factor out front in 16.155 for the same reason as discussed above (really it is $\mathbf{E} \times \mathbf{H} = (1/\mu) \mathbf{E} \times \mathbf{B} = (1/\beta\mu) \mathbf{E} \times \mathbf{B}'$).

Proof of (16.143)

From (16.45) we know $\vec{X}_{lm} = \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_{lm}$

So take LHS of 16.143 and expand as follows.

$$\vec{\nabla} \times [f_l(r) \vec{X}] = \vec{\nabla} \times \left[(r f_l(r)) \left(\frac{1}{r} \vec{X} \right) \right] \quad \dots = k$$

Use vector identity to get

$$\begin{aligned} &= \underbrace{\vec{\nabla} (r f_l(r)) \times \left(\frac{1}{r} \vec{X} \right)}_{\text{Term 1}} + (r f_l(r)) \underbrace{\vec{\nabla} \times \left(\frac{1}{r} \vec{X} \right)}_{\text{Term 2}} = \textcircled{1} + \textcircled{2} \\ &= \frac{\partial}{\partial r} [r f_l(r)] \hat{r} \end{aligned}$$

$$\text{So } \textcircled{1} = \frac{\partial}{\partial r} [r f_l(r)] \underbrace{\hat{r} \times \left(\frac{1}{r} \vec{X} \right)}_{\frac{1}{r} \hat{r} \times \vec{X}} = \frac{1}{r} \frac{\partial}{\partial r} [r f_l(r)] \hat{r} \times \vec{X}.$$

But this is the second term in 16.143, so we now need to show that

$$\textcircled{2} = (r f_l(r)) \vec{\nabla} \times \left(\frac{1}{r} \vec{X} \right) = \frac{i}{r} \sqrt{l(l+1)} f_l(r) Y_{lm} \hat{r}$$

Well, on left write $\vec{X} = \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_{lm}$ to get.

[also, cancel $f_l(r)$ since a factor on both sides]

$$\begin{aligned} \vec{\nabla} \times \left(\frac{1}{r} \vec{L} Y_{lm} \right) &= i l(l+1) \cdot \frac{1}{r^2} Y_{lm} \hat{r} \quad \text{/need to show!} \\ &= \left(\frac{1}{i} \right) \vec{\nabla} \times \left(\frac{1}{r} \vec{r} \times \vec{\nabla} \right) Y_{lm} \\ &= (-i) \vec{\nabla} \times (\hat{r} \times \vec{\nabla}) Y_{lm} \end{aligned}$$

Well $\vec{A} \times (\vec{B} \times \vec{C}) = \dots$ but $\vec{\nabla}$ is an operator.

$$\begin{aligned} \epsilon_{ijk} \nabla_j [\epsilon_{klm} \hat{r}_l \partial_m] Y &= \epsilon_{ijk} \epsilon_{klm} \partial_j [\hat{r}_l \partial_m] Y \\ &= \epsilon_{ijk} \epsilon_{klm} \{ \hat{r}_l \partial_j \partial_m Y + (\partial_j \hat{r}_l) (\partial_m Y) \} \end{aligned}$$

Now use attached Lemma to see that

$$(-i) \vec{\nabla} \times [\hat{r} \times \vec{\nabla} Y_{lm}] = (-i) \hat{r} \nabla^2 Y_{lm}$$

and it remains to show then that

$$(-i) \hat{r} \nabla^2 Y_{lm} = \left(i \frac{1}{r^2} L^2 Y_{lm} \right) \hat{r}$$

Since we know $L^2 \rightarrow l(l+1)$ on Y_{lm} .

But 16.30 says $\nabla^2 = \frac{1}{r} \partial_r^2 - \frac{L^2}{r^2}$. But $\partial_r^2 Y_{lm} = 0$, so

$$\nabla^2 Y_{lm} = -\frac{1}{r^2} L^2 Y_{lm}$$

Therefore

$$(-i) \hat{r} \nabla^2 Y_{lm} = (-i) \hat{r} \left[-\frac{1}{r^2} L^2 Y_{lm} \right] = \left(i \frac{1}{r^2} L^2 Y_{lm} \right) \hat{r}$$

QED.

Lemma: $\vec{\nabla} \times [\hat{r} \times \vec{\nabla} f(\theta, \phi)] = \hat{r} \nabla^2 f(\theta, \phi)$

● Proof: use $\vec{\nabla} \times (\vec{A} \times \vec{B})$ identity to get

$$\text{LHS} = \underset{\textcircled{1}}{\hat{r} \nabla^2 f} - \underset{\textcircled{2}}{(\vec{\nabla} f)(\vec{\nabla} \cdot \hat{r})} + \underset{\textcircled{3}}{(\vec{\nabla} f \cdot \vec{\nabla}) \hat{r}} - \underset{\textcircled{4}}{(\hat{r} \cdot \vec{\nabla})(\vec{\nabla} f)}$$

Now ① is the answer, so want to show $\textcircled{2} + \textcircled{3} + \textcircled{4} = 0$.

$$\textcircled{2} = -(\vec{\nabla} f) \cdot \frac{\vec{r}}{r}$$

$$\textcircled{3}: \text{Note that } \vec{\nabla} f = \hat{\theta} \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi}$$

$$\begin{aligned} \text{So } \textcircled{3} &= (\nabla f)_{\theta} \nabla_{\theta} \hat{r} + (\nabla f)_{\phi} \nabla_{\phi} \hat{r} \\ &= (\nabla f)_{\theta} \frac{1}{r} \underbrace{\frac{\partial \hat{r}}{\partial \theta}}_{\hat{\theta}} + (\nabla f)_{\phi} \frac{1}{r \sin \theta} \underbrace{\frac{\partial \hat{r}}{\partial \phi}}_{\sin \theta \hat{\phi}} \\ &= \frac{1}{r} [(\nabla f)_{\theta} \hat{\theta} + (\nabla f)_{\phi} \hat{\phi}] = +\frac{1}{r} (\vec{\nabla} f) \end{aligned}$$

$$\textcircled{4} = -\nabla_r (\vec{\nabla} f) = \left(+\frac{1}{r}\right) (\vec{\nabla} f) \text{ since } \nabla_r \left(\frac{1}{r}\right) = -\frac{1}{r^2}$$

$$\text{Therefore } \textcircled{2} + \textcircled{3} + \textcircled{4} = (\vec{\nabla} f) \left[-\frac{2}{r} + \frac{1}{r} + \frac{1}{r}\right] = 0 \quad \underline{\underline{\text{QED}}}$$

$$\text{LHS} = \underbrace{\frac{1}{\delta \ell(\ell+1)}}_{k_\ell} \int j_\ell(kr) (\vec{L}\psi)^* \cdot (\vec{\nabla} \times \vec{A}) dV \quad (1)$$

$$= k_\ell \int dV j_\ell(kr) (\vec{L}\psi)^* \cdot \epsilon_{ijk} \partial_j A_k \quad (2)$$

[$\vec{X} = k_\ell \vec{L}\psi$ (16.45)]

(a) Do a parts to move ∂_j

$$= -k_\ell \int dV \partial_j [j_\ell(kr) (\vec{L}\psi)^*] \epsilon_{ijk} A_k$$

$$= +k_\ell \int dV A_k \epsilon_{kji} \partial_j [\quad]$$

$$= +k_\ell \int dV \vec{A} \cdot \vec{\nabla} \times [j_\ell(kr) (\vec{L}\psi)^*] \quad (3)$$

(b) Use v.I. that says

$$\vec{\nabla} \times [j_\ell(kr) (\vec{L}\psi)^*] = \underbrace{\vec{\nabla}(j_\ell(kr)) \times (\vec{L}\psi)^*}_{(1)} + \underbrace{j_\ell(kr) \vec{\nabla} \times (\vec{L}\psi)^*}_{(2)}$$

Look now at term ① which is

$$\vec{\nabla}(j_\ell(kr)) = [\partial_r j_\ell(kr)] \hat{r}$$

We now face this

$$\begin{aligned} i\vec{L} &= \vec{r} \times \vec{\nabla} \\ -i\vec{L}^* &= \vec{r} \times \vec{\nabla} \end{aligned}$$

$$-i\hat{r} \times (\vec{L}\psi)^* = \hat{r} \times (\vec{r} \times \vec{\nabla}\psi^*)$$

$$= (\hat{r} \cdot \vec{\nabla}\psi) \vec{r} - (\hat{r} \times \vec{r}) \cdot \vec{\nabla}\psi$$

$$= \underbrace{(\partial_r \psi) \vec{r}}_{=0} - \vec{r} \times \vec{\nabla}\psi$$

$$= -\vec{r} \times \vec{\nabla}\psi$$

Therefore,

$$\vec{\nabla} \times [j_l(kr) (\vec{L}\psi)^*] = + \frac{1}{i} [\partial_r j_l(kr)] \vec{\nabla} \psi^* + j_l(kr) \vec{\nabla} \times (\vec{L}\psi)^*$$

and so our result at this point is

$$\text{LHS} = k_l \int dV \vec{A} \cdot \left[\frac{1}{i} (\partial_r j_l) \vec{\nabla} \psi^* + j_l \vec{\nabla} \times (\vec{L}\psi)^* \right] \quad (4)$$

(c) Now use identity (16.49) so that

$$\begin{aligned} -i \vec{\nabla} \times (\vec{L}\psi)^* &= \vec{r} \nabla^2 \psi^* - \vec{\nabla} (1+r \partial_r) \psi^* \\ &\quad \uparrow = 0 \\ &= \vec{r} \nabla^2 \psi^* - \vec{\nabla} \psi^* \end{aligned}$$

But we know that

$$\nabla^2 \psi^* = -\frac{L^2}{r^2} \psi^* = -\frac{l(l+1)}{r^2} \psi^*,$$

so that

$$\vec{\nabla} \times (\vec{L}\psi)^* = \frac{-1}{i} \left[-\frac{l(l+1)}{r^2} \psi^* \vec{r} - \vec{\nabla} \psi^* \right]$$

which gives a new three-term result

$$\text{LHS} = -k_l \int dV \vec{A} \cdot \left[+i (\partial_r j_l) \vec{\nabla} \psi^* + \frac{i l(l+1)}{r^2} \psi^* \vec{r} j_l + i (\vec{\nabla} \psi^*) j_l \right]$$

Combine the two $\vec{\nabla} \psi$ terms to get

$$\text{LHS} = -k_l \int dV \vec{A} \cdot \left[\underbrace{\left(i j_l(kr) + \frac{1}{r} \partial_r (j_l(kr)) \right)}_{f(r)} \vec{\nabla} \psi^* + \frac{i l(l+1)}{r^2} j_l(kr) \psi^* \vec{r} \right] \quad (5)$$

(d) Now examine the first term in (5)

$$\begin{aligned} 1^{\text{st}} \text{ term (5)} &= \frac{1}{4\pi k\epsilon} \int dV \cdot \vec{A} \cdot [f(r) \vec{\nabla} \psi^*] \\ &= \frac{1}{4\pi k\epsilon} \int dV [A_i(r) f(r)] \partial_i \psi^* \end{aligned}$$

Do parts on this to get

$$\begin{aligned} &= \frac{1}{4\pi k\epsilon} \int dV \partial_i [A_i f] \psi^* \\ &= \frac{1}{4\pi k\epsilon} \int dV \{ (\partial_i A_i) f + A_i \partial_i f \} \psi^* \end{aligned}$$

where $f(r) = i j_l(kr) + r i \partial_r [j_l(kr)]$

Rewrite:

$$1^{\text{st}} \text{ term (5)} = \frac{1}{4\pi k\epsilon} \int dV \{ (\vec{\nabla} \cdot \vec{A}) f + (\vec{A} \cdot \vec{\nabla} f) \} \psi^*$$

Now: $\vec{\nabla} f = \vec{\nabla}(f(r)) = f'(r) \hat{r}$ so

$$(\vec{A} \cdot \vec{\nabla} f) = (\vec{A} \cdot \hat{r}) f'(r)$$

So again

$$1^{\text{st}} \text{ term (5)} = \frac{1}{4\pi k\epsilon} \int dV [(\vec{\nabla} \cdot \vec{A}) f + (\vec{A} \cdot \hat{r}) f'] \psi^*$$

Insert this into (5) to get

$$\text{LHS} = \frac{1}{4\pi k\epsilon} \int dV [(\vec{\nabla} \cdot \vec{A}) f + (\vec{A} \cdot \hat{r}) \left\{ f' - \frac{i l(l+1)}{r} j_l(kr) \right\}] \psi^*$$

and now at least this vaguely resembles the RHS of (16.90)

Look now at the $(\vec{A} \cdot \vec{r})$ coefficient

(4)

$$f(r) = i j_l(kr) + i r \partial_r [j_l(kr)]$$

$$f'(r) = i \partial_r [j_l(kr)] + i \partial_r \{ r \partial_r [j_l(kr)] \}$$

Shorthand from now on.

$$j_l(kr) = g(r) = g.$$

Then

$$\begin{aligned} f'(r) &= i g' + i \partial_r (r g') \\ &= i g' + i [r g'' + g'] = i r g'' + 2 i g' \end{aligned}$$

Then that coefficient is:

$$\begin{aligned} & \left\{ \cancel{i g'} + i r g'' + \cancel{i g'} - \frac{i l(l+1)}{r} g \right\} + 2 i g' \\ &= (r) \left\{ g'' + \frac{2}{r} g' - \frac{i}{r^2} g' + \frac{i l(l+1)}{r^3} g \right\} \\ &= \left\{ + i r g'' - \frac{i l(l+1)}{r} g \right\} + 2 i g' \\ &= (+i r) \left[g'' - \frac{l(l+1)}{r^2} g \right] + \frac{2 g'}{r} \end{aligned}$$

Now use the DE (16.7) which is solved by $g = j_l(kr)$
so that

$$\left[g'' + \frac{2}{r} g' + k^2 g - \frac{l(l+1)}{r^2} g \right] = 0$$

$$\Rightarrow \left[g'' - \frac{l(l+1)}{r^2} g \right] = \left[-\frac{2}{r} g' + k^2 g \right]$$

The coefficient has now become,

$$\{ \} = -in \left[\cancel{\frac{2}{r}} g' + k^2 g \right]$$

so we now have

$$LHS = +k_l \int dV \left[(\vec{\nabla} \cdot \vec{A}) f + (\vec{A} \cdot \hat{r}) (-in) \left\{ \cancel{\frac{2}{r}} g' + k^2 g \right\} \right] \psi^*$$

One more time

$$\begin{aligned} LHS &= +k_l \int dV \left[(\vec{\nabla} \cdot \vec{A}) [ig + in \partial_r g] - (\vec{A} \cdot \hat{r}) \left[\cancel{2i} \cancel{\frac{1}{r}} g + ik^2 r g \right] \right] \psi^* \\ &= \frac{+i}{\sqrt{l(l+1)}} \int dV \left\{ \underbrace{(\vec{\nabla} \cdot \vec{A}) [g + r \partial_r g]}_{\text{correct}} - (\vec{A} \cdot \hat{r}) \left[\cancel{\frac{2}{r}} g + \underbrace{k^2 g}_{\text{correct}} \right] \right\} \psi^* \end{aligned}$$

And this is (finally!) the correct
RHS of (18.90)

$$\begin{aligned} (16.88) \quad a_E &= \frac{4\pi i k}{c} \int g \vec{X}^* \cdot \vec{\nabla} \times \vec{\mathcal{E}} \\ &= \frac{-4\pi k}{\sqrt{c}} \int \psi^* \left[\underbrace{\vec{\nabla} \cdot \vec{\mathcal{E}} \partial_r(rg)}_{+i\omega g} - k^2 \vec{r} \cdot \vec{\mathcal{E}} g \right] dV \\ &= i \frac{4\pi k^2}{\sqrt{l(l+1)}} \int \psi^* g \partial_r(rg) - \left(\frac{k^2}{i\omega} (\vec{r} \cdot \vec{\mathcal{E}}) g \right) dV \\ &\quad + \frac{ik}{c} \end{aligned}$$

So we have now also

verified the first λ^2 terms in (18.91).

Vector Spherical Harmonics

Carleton

$$\vec{X} = \vec{r} \times \vec{\nabla} Y_{lm}$$

$$\vec{Y} = \hat{r} Y_{lm}$$

$$\vec{Z} = r \vec{\nabla} Y_{lm}$$

$$\vec{X} \cdot \vec{Y} = \vec{Y} \cdot \vec{Z} = \vec{Z} \cdot \vec{X} = 0$$

$$\vec{r} \cdot \vec{X} = 0$$

Jackson

$$\vec{X}_J = \frac{-i}{\sqrt{l(l+1)}} \vec{r} \times \vec{\nabla} Y_{lm} = \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_{lm}$$

M and N World

$$\vec{M} \equiv f_l(r) \vec{\nabla} \times (\vec{r} \psi) = -f_l(r) \vec{r} \times \vec{\nabla} \psi \quad (\vec{X}\text{-like})$$

$$\vec{N} \equiv \frac{1}{k} \vec{\nabla} \times \vec{M} = -\frac{1}{k} \vec{\nabla} \times \{f_l(r) \vec{r} \times \vec{\nabla} \psi\} \quad (\nabla\text{-2nd term})$$

$$\vec{M} = \frac{1}{k} \vec{\nabla} \times \vec{N}$$

$\psi = Y_{lm}$ or lincomb within l

$$\vec{\nabla} \cdot \vec{M} = \vec{\nabla} \cdot \vec{N} = 0$$

$$(\nabla^2 + k^2) \psi = 0 \Rightarrow (\nabla^2 + k^2) \vec{M} = 0 \text{ and } (\nabla^2 + k^2) \vec{N} = 0$$

Multipole Expansion

TM (electric)

$$\begin{cases} \vec{B}_{lm} = A_E(l,m) \vec{M}_{lm} \\ \vec{E}_{lm} = \frac{i}{k} \vec{\nabla} \times \vec{B} \\ \quad = i A_E(l,m) \vec{N}_{lm} \end{cases}$$

TE (magnetic)

$$\begin{cases} \vec{E}_{lm} = A_M(l,m) \vec{M}_{lm} \\ \vec{B}_{lm} = -\frac{i}{k} \vec{\nabla} \times \vec{E} \\ \quad = -i A_M(l,m) \vec{N}_{lm} \end{cases}$$

Vector Spherical Harmonics

This handout summarizes the vector spherical harmonic formalism discussed in lectures. This treatment was prepared by Professor Carleton DeTar and is taken from lecture notes by Professor Richard Price and Tracy Steelhammer.

Orthogonal Expansion Spherical harmonics provide an extremely useful orthonormal basis for expansion of scalar functions on a sphere. We would like to find a similar orthonormal basis for vector functions on a sphere. These functions are vector spherical harmonics (VSH's). There are a variety of choices for basis sets, but we shall choose a set that is closely related to that of Jackson, Chapter 16.

Define the vector harmonics

$$2 \quad X_{\ell m}(\theta, \phi) = \mathbf{r} \times \nabla Y_{\ell m}(\theta, \phi) \quad (1)$$

$$r \quad Y_{\ell m}(\theta, \phi) = \hat{r} Y_{\ell m}(\theta, \phi) \quad (2)$$

$$1 \quad Z_{\ell m}(\theta, \phi) = r \nabla Y_{\ell m}(\theta, \phi) \quad (3)$$

These vector functions form a complete basis for expanding a vector function of the spherical polar angles θ and ϕ . The expression for $X_{\ell m}$ differs from Jackson's by a factor of $i\sqrt{\ell(\ell+1)}$. He uses $\mathbf{L} = \frac{1}{i}(\mathbf{r} \times \nabla)$ [See Jackson, Eq (16.25-30) for a discussion of properties of $\mathbf{L}$.] The normalization of these harmonics is easily computed:

$$\int X_{\ell' m'}^* \cdot X_{\ell m} d\Omega = \ell(\ell+1) \delta_{\ell\ell'} \delta_{m'm} \quad (4) \checkmark$$

$$\int Y_{\ell' m'}^* \cdot Y_{\ell m} d\Omega = \delta_{\ell\ell'} \delta_{m'm} \quad (5) \checkmark \quad Y_{\ell m}'s$$

$$\int Z_{\ell' m'}^* \cdot Z_{\ell m} d\Omega = \ell(\ell+1) \delta_{\ell\ell'} \delta_{m'm} \quad (6) \checkmark$$

$$\int X_{\ell' m'}^* \cdot Z_{\ell m} d\Omega = 0. \quad (7) \checkmark$$

The harmonics are not just orthogonal in the sense of an integral over the sphere, but they are also orthogonal everywhere in the sense of the vector dot product:

$$X_{\ell' m'}^* \cdot Y_{\ell m} = 0 \quad (8) \checkmark$$

$$Y_{\ell' m'}^* \cdot Z_{\ell m} = 0. \quad (9) \checkmark$$

~~and $X \cdot Z = 0$ too!~~

$$\hat{r} \times (\vec{r} \times \vec{\nabla} \psi) = -r \vec{\nabla} \psi$$

An arbitrary (well-behaved) vector field $\mathbf{A}(r, \theta, \phi)$ can be expanded in terms of these VSH's as

$$\vec{\mathbf{A}}(r, \theta, \phi) = \sum_{\ell m} [A_{\ell m}^{(2)}(r) \vec{\mathbf{X}}_{\ell m} + A_{\ell m}^{(r)}(r) \vec{\mathbf{Y}}_{\ell m} + A_{\ell m}^{(1)}(r) \vec{\mathbf{Z}}_{\ell m}]. \quad (10) \checkmark$$

The radial coefficient functions are found from the orthogonality properties:

$$A_{\ell m}^{(2)}(r) = \frac{1}{\ell(\ell+1)} \int \mathbf{X}_{\ell m}^*(\theta, \phi) \cdot \mathbf{A}(r, \theta, \phi) d\Omega \quad (11)$$

$$A_{\ell m}^{(r)}(r) = \int \mathbf{Y}_{\ell m}^*(\theta, \phi) \cdot \mathbf{A}(r, \theta, \phi) d\Omega \quad (12)$$

$$A_{\ell m}^{(1)}(r) = \frac{1}{\ell(\ell+1)} \int \mathbf{Z}_{\ell m}^*(\theta, \phi) \cdot \mathbf{A}(r, \theta, \phi) d\Omega. \quad (13)$$

Calculus Exercises with Vector Spherical Harmonic Components

We plan to express Maxwell's equations in terms of the components of the VSH's, but first we must work out the expression for the divergence and curl of the harmonics. In the next section we present results. To get a feeling for the method of deriving them, we give a sample calculation here. Consider the divergence of the (r) component term:

$$\nabla \cdot [A_{\ell m}^{(r)}(r) \mathbf{Y}_{\ell m}] = \nabla \cdot [(A_{\ell m}^{(r)}/r) \mathbf{r} Y_{\ell m}(\theta, \phi)] \checkmark \quad (14)$$

$$= \left[\frac{d}{dr} (A_{\ell m}^{(r)}/r) \hat{\mathbf{r}} \cdot \mathbf{r} + 3 A_{\ell m}^{(r)}/r \right] Y_{\ell m}(\theta, \phi) \checkmark \quad (15)$$

$$= \frac{1}{r^2} \frac{d}{dr} (r^2 A_{\ell m}^{(r)}) Y_{\ell m}(\theta, \phi). \checkmark \quad (16)$$

Note that the term in $\mathbf{r} \cdot \nabla Y_{\ell m}(\theta, \phi)$ vanishes since the gradient of $Y_{\ell m}$ has no radial component. Some other useful identities may be found in Jackson, Chapter 16, including these:

$$(\mathbf{r} \times \nabla)^2 Y_{\ell m} = -\ell(\ell+1) Y_{\ell m} \checkmark \quad (17)$$

$$\nabla^2 Y_{\ell m} = -\frac{\ell(\ell+1)}{r^2} Y_{\ell m} \checkmark \quad (18)$$

$$\mathbf{r} \cdot (\mathbf{r} \times \nabla) = 0 \checkmark \quad (19)$$

$$\nabla \cdot (\mathbf{r} \times \nabla) = 0. \checkmark \quad (20)$$

Div and Curl of the Vector Spherical Harmonics As we will see what is particularly useful about the VSH is that the div and curl can be

$$\times \quad \vec{\nabla} \cdot \vec{\mathbf{L}} \psi = \vec{\nabla} \cdot (\vec{r} \times \vec{\nabla} \psi) = \vec{\nabla} \psi \cdot \vec{\nabla} \times \vec{r} - \vec{r} \cdot \vec{\nabla} \times \vec{\nabla} \psi = 0 + 0 = 0 \checkmark$$

$$2 \quad \vec{\nabla} \cdot r \vec{\nabla} \psi = \vec{\nabla} r \cdot \vec{\nabla} \psi + r \nabla^2 \psi = 0 + r \left[-\frac{\ell(\ell+1)}{r^2} \right] \psi = -\frac{\ell(\ell+1)}{r} \psi \checkmark$$

$$\gamma \quad \vec{\nabla} \times (\hat{r} r) = -\frac{1}{r} \hat{r} \times \vec{\nabla} r \quad \checkmark$$

$$Z \quad \vec{\nabla} \times (r \vec{\nabla} r) = \vec{\nabla} r \times \vec{\nabla} r + r \vec{\nabla} \times \vec{\nabla} r = \frac{1}{r} \hat{r} \times \vec{\nabla} r \quad \checkmark$$

$$X \quad \vec{\nabla} \times (\hat{r} \times \vec{\nabla} r) = -2 \hat{r} \times \vec{\nabla} r \quad \text{wrong!}$$

expressed in terms of the same ℓm components. There is no mixing between components of different ℓm . Taking $A(r, \theta, \phi)$ as expanded above, we have, after some calculation,

$$\nabla \cdot \mathbf{A} = \sum_{\ell m} \left[\frac{1}{r^2} \frac{d}{dr} (r^2 A_{\ell m}^{(r)}) - \frac{\ell(\ell+1)}{r} A_{\ell m}^{(1)} \right] Y_{\ell m} \quad \checkmark \quad (21)$$

Note that the terms in $\mathbf{X}_{\ell m}$ do not contribute to the divergence. The expression for the curl: $\nabla \times \mathbf{A} = \mathbf{B}$ is

$$\nabla \times \mathbf{A} = \sum_{\ell m} [B_{\ell m}^{(2)}(r) \mathbf{X}_{\ell m} + B_{\ell m}^{(r)}(r) \mathbf{Y}_{\ell m} + B_{\ell m}^{(1)}(r) \mathbf{Z}_{\ell m}], \quad (22) \quad \checkmark$$

where

$$B_{\ell m}^{(r)} = -\frac{\ell(\ell+1)}{r} A_{\ell m}^{(2)} \quad (23) \quad \checkmark$$

$$B_{\ell m}^{(1)} = -\frac{1}{r} \frac{d}{dr} (r A_{\ell m}^{(2)}) \quad (24) \quad \checkmark$$

$$B_{\ell m}^{(2)} = -\frac{1}{r} A_{\ell m}^{(r)} + \frac{1}{r} \frac{d}{dr} (r A_{\ell m}^{(1)}). \quad (25) \quad \checkmark$$

Parity of the Vector Spherical Harmonics Under spatial inversion (θ, ϕ) becomes $(\pi - \theta, \phi + \pi)$ and $Y_{\ell m}(\theta, \phi)$ becomes $Y_{\ell m}(\pi - \theta, \phi + \pi) = (-)^{\ell} Y_{\ell m}(\theta, \phi)$. Since vectors like $\hat{r}$ and ∇ reverse sign under parity, the parity transformations of the VSH can be summarized as follows:

$$\mathbf{X}_{\ell m}(\pi - \theta, \phi + \pi) = (-)^{\ell} \mathbf{X}_{\ell m}(\theta, \phi) \quad (26)$$

$$\mathbf{Y}_{\ell m}(\pi - \theta, \phi + \pi) = -(-)^{\ell} \mathbf{Y}_{\ell m}(\theta, \phi) \quad (27)$$

$$\mathbf{Z}_{\ell m}(\pi - \theta, \phi + \pi) = -(-)^{\ell} \mathbf{Z}_{\ell m}(\theta, \phi). \quad (28)$$

Since Maxwell's equations are form-invariant under parity transformations, the parity transformation properties of the VSH can be of help in reducing the number of components we need to consider in some applications.

Illustration: Electrostatic Multipole What follows is not the recommended way to solve an electrostatics problem, but it illustrates the way the VSH's work. We start with the requirement that $\nabla \times \mathbf{E} = 0$. In terms of the VSH components of $\mathbf{E}$: $E_{\ell m}^{(r)}$, $E_{\ell m}^{(1)}$, and $E_{\ell m}^{(2)}$ the vanishing of the curl implies that

$$E_{\ell m}^{(2)} = 0 \quad (29) \quad \checkmark$$

$$E_{\ell m}^{(r)} = \frac{d}{dr} (r E_{\ell m}^{(1)}), \quad (30) \quad \checkmark$$

and from $\nabla \cdot \mathbf{E} = 4\pi\rho$ and the expression for the divergence in terms of the VSH components we get

$$\frac{1}{r^2} \frac{d}{dr} (r^2 E_{\ell m}^{(r)}) - \frac{\ell(\ell+1)}{r} E_{\ell m}^{(1)} = 4\pi\rho_{\ell m} \quad (31) \checkmark$$

where $\rho_{\ell m}$ is the usual spherical multipole moment of the charge density ρ . Putting together these two equations gives the radial Poisson equation:

$$\partial_r^2(rE) + \frac{2}{r}\partial_r(rE) \leftarrow \left(\frac{1}{r^2} \frac{d}{dr} (r^2 \frac{d}{dr} [rE_{\ell m}^{(1)}]) - \frac{\ell(\ell+1)}{r^2} (rE_{\ell m}^{(1)}) \right) = 4\pi\rho_{\ell m}. \quad (32) \checkmark$$

Notice that the VSH component $rE_{\ell m}^{(1)}$ appears in this equation where we would expect to find the potential multipole component $-\phi_{\ell m}$. This result can be explained by noting that if we insert the multipole expansion for the scalar potential ϕ into the expression $-\nabla\phi$ we find

$$\vec{\nabla}\phi = \partial_r\phi \hat{r} \quad -\nabla(\phi_{\ell m} Y_{\ell m}) = -\frac{d}{dr}(\phi_{\ell m}) Y_{\ell m} \left[-\frac{1}{r} \phi_{\ell m} Z_{\ell m} \right] \quad (33) \checkmark$$

so comparing this result with the expansion of $\mathbf{E}$ in terms of VSH's we see that

$$E_{\ell m}^{(1)} = -\phi_{\ell m}/r \quad (34)$$

as expected. We also see that the (r) and (1) components are correctly correlated according to the requirement that $\nabla \times \mathbf{E} = 0$, and that the (2) component is missing, as also required.

see Pl model.

Illustration: Magnetostatic Multipole We are given a localized current distribution $\mathbf{J}$ and want to express the magnetic field $\mathbf{B}$ outside the source as a multipole field expansion. In analogy with the multipole expansion for the electric field we write $\mathbf{B} = -\nabla\Phi$ outside the source, where the magnetic scalar potential has the multipole expansion

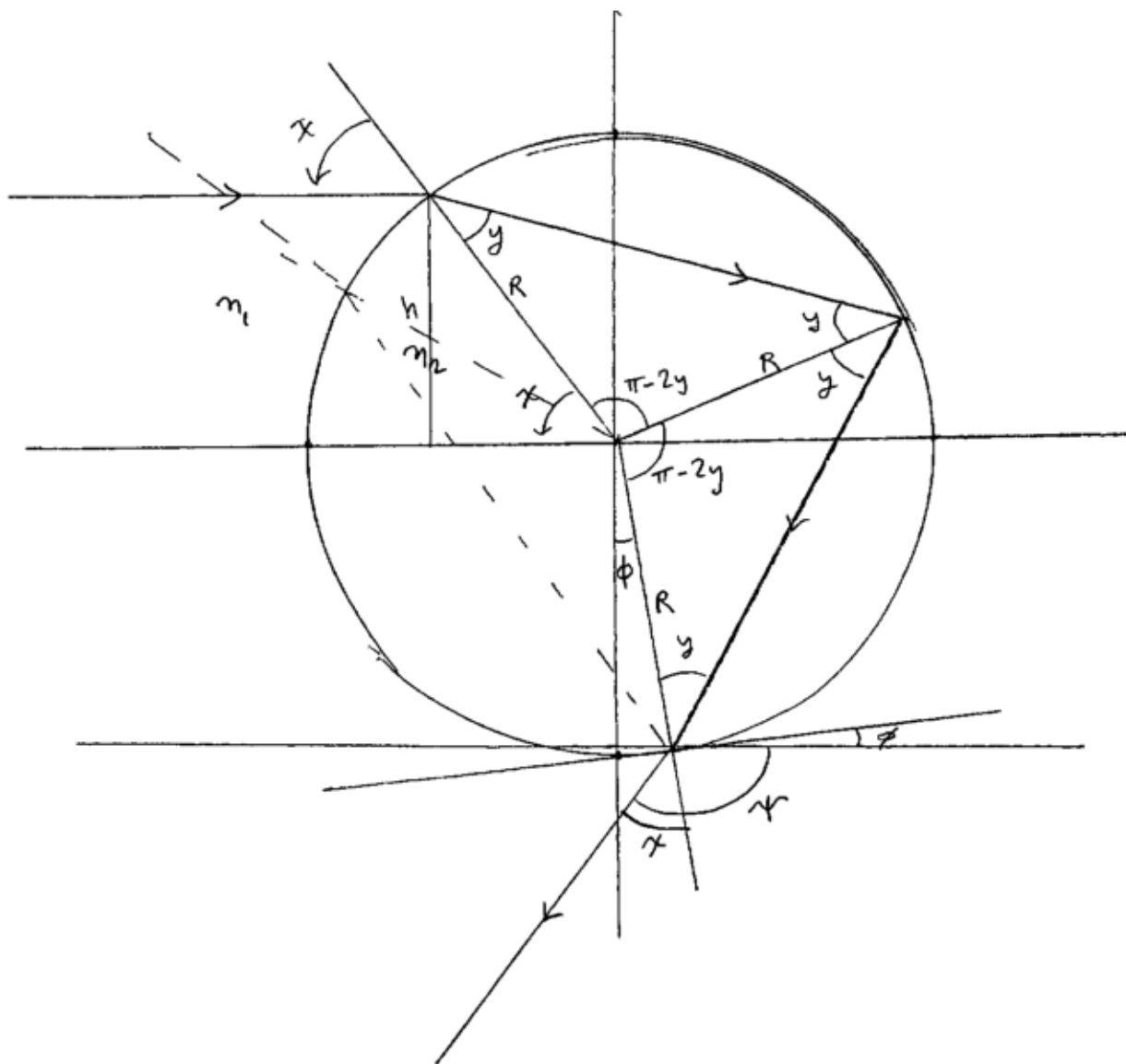
$$\Phi = \sum_{\ell m} \frac{4\pi}{2\ell+1} \frac{\mu_{\ell m}}{r^{\ell+1}} Y_{\ell m}. \quad (35) \checkmark$$

mag. moments.

Jackson never does this

So the problem is to find an expression for the magnetic multipole moments $\mu_{\ell m}$ in terms of $\mathbf{J}$. (Recall that the solution to this problem was not given in Jackson, Chapter 5.)

We want to solve Ampère's Law $\nabla \times \mathbf{B} = 4\pi/c\mathbf{J}$ and $\nabla \cdot \mathbf{B} = 0$. Our strategy is to expand both $\mathbf{B}$ and $\mathbf{J}$ in terms of VSH and use the above



① At center, see that $x + 2(\pi - 2y) + \phi = \frac{3\pi}{2}$

② At exit $\psi + \phi - x = \frac{\pi}{2}$

③ $n_1 \sin x = n_2 \sin y$

Therefore 1+2 $\Rightarrow \psi = \pi + 2x - 4y$ where $y = \sin^{-1} \left[\frac{n_1}{n_2} \sin x \right]$

x varies over range 0 to $\frac{\pi}{2}$

$$\frac{d\psi}{dx} = 0 \Rightarrow \cos x = \sqrt{\frac{1}{3} \left(\frac{n_2^2}{n_1^2} - 1 \right)} \quad \checkmark \Rightarrow \psi_{\min} = 137.48^\circ$$

$$\frac{d\psi}{dx} = \frac{2 - 4 \frac{n_1}{n_2} \cos x}{\sqrt{1 - \left(\frac{n_1}{n_2} \sin x \right)^2}}$$

expression for the curl to get a set of equations for the VSH components. The vanishing divergence of both $\mathbf{B}$ and $\mathbf{J}$ will give us extra conditions to determine all of the components. OK

First, let's write the VSH components of $\mathbf{B}$ outside the source in terms of the multipole moments $\mu_{\ell m}$ by calculating the gradient of the magnetic scalar potential explicitly. We get

$$B_{\ell m}^{(r)} = \frac{4\pi}{2\ell+1} \frac{(\ell+1)}{r^{\ell+2}} \mu_{\ell m} \quad (36) \checkmark$$

$$B_{\ell m}^{(1)} = -\frac{4\pi}{2\ell+1} \frac{1}{r^{\ell+2}} \mu_{\ell m} \quad (37) \checkmark$$

$$B_{\ell m}^{(2)} = 0 \quad (38) \checkmark$$

$\vec{\nabla} \Phi = \dots$

outside the source.

From Ampère's Law and the expression for the curl we have

$$\frac{4\pi}{c} J_{\ell m}^{(r)} = -\frac{\ell(\ell+1)}{r} B_{\ell m}^{(2)} \quad (39) \checkmark$$

$$\frac{4\pi}{c} J_{\ell m}^{(1)} = -\frac{1}{r} \frac{d}{dr} (r B_{\ell m}^{(2)}) \quad (40) \checkmark$$

$$\frac{4\pi}{c} J_{\ell m}^{(2)} = -\frac{1}{r} B_{\ell m}^{(r)} + \frac{1}{r} \frac{d}{dr} (r B_{\ell m}^{(1)}) \quad (41) \checkmark$$

also arises

$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$

Notice that the first two equations correlate the (r) and (1) components of $\mathbf{J}$ as required from $\nabla \cdot \mathbf{J} = 0$. Notice that outside the current distribution $B_{\ell m}^{(2)} = 0$. From $\nabla \cdot \mathbf{B} = 0$ we have

$$\frac{1}{r^2} \frac{d}{dr} (r^2 B_{\ell m}^{(r)}) - \frac{\ell(\ell+1)}{r} B_{\ell m}^{(1)} = 0 \quad (42) \checkmark$$

We solve this equation for $B_{\ell m}^{(1)}$ and plug into Eq (41) and divide by r to get

$$\partial_r^2 (rB) + \frac{2}{r} \partial_r (rB) \leftarrow \frac{1}{r} \frac{d^2}{dr^2} (r^2 B_{\ell m}^{(r)}) - \frac{\ell(\ell+1)}{r^2} (r B_{\ell m}^{(r)}) = \frac{4\pi}{c} \ell(\ell+1) J_{\ell m}^{(2)} \quad (43) \checkmark$$

This is again the radial Poisson equation. This time the role of the potential is played by $r B_{\ell m}^{(r)}$. Using the usual radial Green's function to get the solution, we find outside the source

$$r B_{\ell m}^{(r)} = -\frac{4\pi}{c} \frac{\ell(\ell+1)}{2\ell+1} \frac{1}{r^{\ell+1}} \int (J_{\ell m}^{(2)}(r'))^{\ell+2} dr'. \quad (44) \checkmark$$

$$g = \frac{k}{r^{\ell+1}} \Rightarrow rB = \int g \cdot \frac{4\pi}{c} \ell(\ell+1) J^2 dr.$$

Comparing with Eq (36), we get,

$$\mu_{\ell m} = -\frac{1}{c(\ell+1)} \int r^\ell (\mathbf{X}_{\ell m}^* \cdot \mathbf{J}) r^2 dr d\Omega \quad (45) \checkmark$$

or, in terms of spherical harmonics explicitly,

$$\mu_{\ell m} = -\frac{1}{c(\ell+1)} \int r^\ell (\mathbf{r} \times \nabla Y_{\ell m}^*) \cdot \mathbf{J} dV. \quad (46) \checkmark$$

This is the expression we wanted. see + modify Jackson 16.96, sign 1627.

Maxwell's Equations Let us write the harmonic form of Maxwell's equations in terms of VSH components. The equations we want to rewrite are

$$\nabla \cdot \mathbf{E} = 4\pi\rho \quad (47) \checkmark$$

$$\nabla \cdot \mathbf{B} = 0 \quad (48) \checkmark$$

$$\nabla \times \mathbf{E} = i k \mathbf{B} \quad (49) \checkmark$$

$$\nabla \times \mathbf{B} = -i k \mathbf{E} + \frac{4\pi}{c} \mathbf{J} \quad (50) \checkmark$$

In terms of VSH components the first two equations read

$$\nabla \cdot \mathbf{E} \quad \frac{1}{r^2} \frac{d}{dr} (r^2 E_{\ell m}^{(r)}) - \frac{\ell(\ell+1)}{r} E_{\ell m}^{(1)} = 4\pi\rho_{\ell m} \quad \alpha \quad (51) \checkmark \quad \text{see (21)}$$

$$\nabla \cdot \mathbf{B} \quad \frac{1}{r^2} \frac{d}{dr} (r^2 B_{\ell m}^{(r)}) - \frac{\ell(\ell+1)}{r} B_{\ell m}^{(1)} = 0 \quad \beta \quad (52) \checkmark$$

and the second two Maxwell equations read

$$\begin{aligned} \nabla \times \mathbf{E} \quad \left\{ \begin{aligned} -\frac{\ell(\ell+1)}{r} E_{\ell m}^{(2)} &= i k B_{\ell m}^{(r)} & \beta \quad (53) \checkmark & r \\ -\frac{1}{r} \frac{d}{dr} (r E_{\ell m}^{(2)}) &= i k B_{\ell m}^{(1)} & \beta \quad (54) \checkmark & 1 \\ -\frac{1}{r} E_{\ell m}^{(r)} + \frac{1}{r} \frac{d}{dr} (r E_{\ell m}^{(1)}) &= i k B_{\ell m}^{(2)} & \alpha \quad (55) \checkmark & 2 \end{aligned} \right. \\ \nabla \times \mathbf{B} \quad \left\{ \begin{aligned} -\frac{\ell(\ell+1)}{r} B_{\ell m}^{(2)} &= -i k E_{\ell m}^{(r)} + \frac{4\pi}{c} J_{\ell m}^{(r)} & \alpha \quad (56) \checkmark & r \\ -\frac{1}{r} \frac{d}{dr} (r B_{\ell m}^{(2)}) &= -i k E_{\ell m}^{(1)} + \frac{4\pi}{c} J_{\ell m}^{(1)} & \alpha \quad (57) \checkmark & 1 \\ -\frac{1}{r} B_{\ell m}^{(r)} + \frac{1}{r} \frac{d}{dr} (r B_{\ell m}^{(1)}) &= -i k E_{\ell m}^{(2)} + \frac{4\pi}{c} J_{\ell m}^{(2)} & \beta \quad (58) \checkmark & 2 \end{aligned} \right. \end{aligned}$$

$\alpha, \beta = \text{set}$

$$\alpha = E_1, E_r, B_2 \quad \beta = B_1, B_r, E_2$$

These eight equations divide neatly into two independent sets of four equations, one set involving only $B_{\ell m}^{(2)}$, $E_{\ell m}^{(1)}$, and $E_{\ell m}^{(r)}$ and the other involving only $E_{\ell m}^{(2)}$, $B_{\ell m}^{(1)}$, and $B_{\ell m}^{(r)}$. The wave equation for $B_{\ell m}^{(2)}$ is obtained by substituting Eqs (56) and (57) into Eq (55) to eliminate $E_{\ell m}^{(1)}$ and $E_{\ell m}^{(r)}$. A similar manipulation yields a wave equation for $E_{\ell m}^{(2)}$ with the result:

$$\left\{ \begin{aligned} \frac{1}{r} \frac{d^2}{dr^2} (r B_{\ell m}^{(2)}) + [k^2 - \frac{\ell(\ell+1)}{r^2}] B_{\ell m}^{(2)} &= \frac{4\pi}{rc} [J_{\ell m}^{(r)} - \frac{d}{dr} (r J_{\ell m}^{(1)})] \quad (59) \quad \checkmark \quad \text{electric TM} \\ \frac{1}{r} \frac{d^2}{dr^2} (r E_{\ell m}^{(2)}) + [k^2 - \frac{\ell(\ell+1)}{r^2}] E_{\ell m}^{(2)} &= -ik \frac{4\pi}{c} J_{\ell m}^{(2)} \quad (60) \quad \checkmark \quad \text{magnetic TE} \end{aligned} \right.$$

see Jackson 16.5

These are inhomogeneous radial Helmholtz wave equations for $B_{\ell m}^{(2)}$ and $E_{\ell m}^{(2)}$. The first equation yields "electric" multipole waves and the second, "magnetic" multipole waves. These equations can be solved using the radial Green's functions for the Helmholtz wave equation, discussed in Jackson, Chapter 16, and summarized in the section after the next below.

$$= \frac{4\pi}{rc} \left[\frac{-ik r^2}{\ell m} \right]$$

Resonances in a Conducting Spherical Cavity We now consider the homogeneous solutions of the wave equations for the interior of a hollow spherical perfectly conducting shell of radius a . We want solutions that are regular at the origin and satisfy the perfect conductor boundary conditions: $\hat{r} \cdot \mathbf{B} = 0$ and $\hat{r} \times \mathbf{E} = 0$. Looking at the form of the VSH terms, we see that these conditions require

$$B_{\ell m}^{(r)}(a) = E_{\ell m}^{(1)}(a) = E_{\ell m}^{(2)}(a) = 0. \quad (61) \quad \checkmark$$

For zero current, the equations (59) and (60) are just the equations for the spherical Bessel functions. The requirement that the solution be regular at the origin gives the solutions

$$B_{\ell m}^{(2)} = b_{\ell m}^2 j_{\ell}(kr) \quad (62) \quad \checkmark$$

$$E_{\ell m}^{(2)} = e_{\ell m}^2 j_{\ell}(kr). \quad (63) \quad \checkmark$$

The boundary condition on the surface permits two types of solutions, just as in a wave guide: electric multipole, or "TM", or "even" with $B_{\ell m}^{(r)} = 0$ everywhere and magnetic multipole, or "TE", or "odd" with $E_{\ell m}^{(r)} = 0$ everywhere.

Jackson p539

For the electric multipole solutions $B_{\ell m}^{(r)} = 0$ everywhere implies that $E_{\ell m}^{(2)} = B_{\ell m}^{(1)} = 0$ everywhere, as can be seen from Eqs (53) and (56). The boundary condition $E_{\ell m}^{(1)}(a) = 0$ together with Eq. (57) requires that

$$\frac{d}{dr}[rB_{\ell m}^{(2)}(r)]|_{r=a} = 0 \quad (64) \checkmark$$

at the surface for the electric multipole solutions. In terms of the spherical Bessel function, we require that

$$\frac{d}{dr}[rj_{\ell}(kr)]|_{r=a} = 0. \quad (65) \checkmark \text{ electric TM}$$

This condition can be satisfied only for special values of $k = \omega/c$ and determines the eigenfrequencies.

For the magnetic multipole solutions the requirement that $E_{\ell m}^{(r)} = 0$ everywhere implies similarly that $E_{\ell m}^{(1)} = B_{\ell m}^{(2)} = 0$ everywhere, and the boundary condition $E_{\ell m}^{(2)}(a) = 0$ requires that (63)

$$j_{\ell}(ka) = 0. \quad (66) \checkmark \text{ my TE}$$

Radiation from a Localized Source Equations (59) and (60) may be solved to give electric and magnetic dipole radiation fields outside a localized source. The relevant Green's function satisfies the equation

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2}\right]g_{\ell}(r, r') = -\frac{1}{r^2}\delta(r - r') \quad (67) \quad (16.20)$$

The solution that is regular at the origin and has outgoing-wave boundary conditions as $r \rightarrow \infty$ is given by Jackson in his Eq (16.21):

$$g_{\ell}(r, r') = ikj_{\ell}(kr_{<})h_{\ell}^{(1)}(kr_{>}). \quad (68) \quad (16.21)$$

Thus we may solve Eq. (59) for the electric multipole field:

$r > a$ "outside the source" $\approx$ OK if $\vec{J}$ is local.

$$B_{\ell m}^{(2)}(r) = -\frac{4\pi ik}{c}h_{\ell}^{(1)}(kr) \int_0^{\infty} \frac{1}{r'} [J_{\ell m}^{(r)} - \frac{d}{dr'}(r'J_{\ell m}^{(1)})] j_{\ell}(kr')(r')^2 dr'. \quad (69) \checkmark$$

Comparing the integrand with the expression for the curl in terms of VSH, Eq. (25), we recognize the (2)-component of the curl of $\vec{J}$. The contribution of this multipole to the magnetic and electric fields can be found by

$$\begin{aligned} &= -\int \frac{d\Omega'}{\ell(\ell+1)} \chi_{\ell m}^*(\theta', \phi') \cdot \vec{\nabla}' \times \vec{J}(\theta', \phi') \quad \text{from (10) and (11)} \\ &= \text{the 2 coefficient of the } \vec{\nabla} \times \vec{J} = \dots \text{ expansion} \end{aligned}$$

$$\vec{X}_{\text{Jackson}} = \frac{1}{i\sqrt{\ell(\ell+1)}} \vec{r} \times \vec{\nabla} Y_{\ell m}$$

$$= \frac{1}{i\sqrt{\ell(\ell+1)}} \vec{X}_{\text{Cartesian}}$$

comparing with Eqs (10) and (50). We may therefore rewrite this solution outside the source as

$$\mathbf{B}(\mathbf{r}) = a_E(\ell, m) h_\ell^{(1)}(kr) \mathbf{X}_{\ell m}(\theta, \phi) \quad \text{TM!} \quad (70)$$

$$\mathbf{E}(\mathbf{r}) = \frac{i}{k} \nabla \times \mathbf{B} \quad (71) \quad \text{TM} \quad (50)$$

where

$$a_E(\ell, m) = \frac{4\pi i k}{\ell(\ell+1)c} \int j_\ell(kr') d^3 r' \mathbf{X}_{\ell m}^*(\theta', \phi') \cdot \nabla' \times \mathbf{J}(\mathbf{r}'). \quad (72) \quad \text{Jackson (16.88) rescale } \vec{X} \text{ and } a.$$

Another expression for $a_E(\ell, m)$ can be obtained by integrating by parts and using the current conservation condition relating the charge density to the current density: $i\omega\rho = \nabla \cdot \mathbf{J}$. After some computation we get

$$a_E(\ell, m) = -\frac{4\pi k^2}{\ell(\ell+1)} \int Y_{\ell m}^*(\theta, \phi) \left\{ \rho(\mathbf{r}) \frac{d}{dr} [r j_\ell(kr)] + \frac{ik}{c} \mathbf{r} \cdot \mathbf{J}(\mathbf{r}) j_\ell(kr) \right\} d^3 r. \quad (73) \quad \text{Jackson (16.91)}$$

A similar analysis of the wave equation for the magnetic multipole fields, Eq (60), can be done with the result outside the source:

$$\mathbf{E}(\mathbf{r}) = a_M(\ell, m) h_\ell^{(1)}(kr) \mathbf{X}_{\ell m}(\theta, \phi) \quad (74)$$

$$\mathbf{B}(\mathbf{r}) = -\frac{i}{k} \nabla \times \mathbf{E} \quad (75) \quad \text{TE}$$

where

$$a_M(\ell, m) = -\frac{4\pi k^2}{\ell(\ell+1)c} \int j_\ell(kr') d^3 r' \mathbf{X}_{\ell m}^*(\theta', \phi') \cdot \mathbf{J}(\mathbf{r}'). \quad (76)$$

These results may be compared with Jackson (16.46), (16.89), and (16.91). (The normalization of $\mathbf{X}_{\ell m}$ is different here).

If the source has definite parity, then there are restrictions on the multipole moments that can be generated. Consider Eq (73). If the current density is odd under parity, then the charge density must be even, and only even ℓ electric multipole moments are generated. Thus the lowest possible electric multipole moment is $\ell = 2$, or electric quadrupole. Likewise, in view of Eq (76) such a source generates only odd ℓ magnetic multipole moments. The lowest term is magnetic dipole. Conversely, if the current density is even under parity, so that the charge density is odd, then we get only odd ℓ electric multipole moments (electric dipole and higher) and even ℓ magnetic multipole moments (magnetic quadrupole and higher). These requirements

$$a_E^{\text{car}} \vec{X}^{\text{car}} = a_E^{\text{JAC}} \vec{X}^{\text{JAC}} = a_E^{\text{JAC}} \frac{1}{i\sqrt{\ell(\ell+1)}} \vec{X}^{\text{car}}$$

are summarized in the table below. If the source does not have definite parity, then both types of multipole fields can be generated.

multipole type	parity of ρ	parity of $\mathbf{J}$
electric	$(-)^{\ell}$	$-(-)^{\ell}$
magnetic	$-(-)^{\ell}$	$(-)^{\ell}$

Radiation from a Small Source If the size of a localized source d is considerably less than the wavelength of the radiation, i.e. $kd \ll 1$, the expressions for the electric and magnetic dipole coefficients can be simplified further. The simplification makes use of the small argument approximation for the spherical Bessel function:

$$j_{\ell}(kr) \underset{r \rightarrow 0}{\sim} \frac{(kr)^{\ell}}{(2\ell + 1)!!} \quad (77)$$

For a small source the arguments of the Bessel functions in the expressions for $a_E(\ell, m)$ and $a_M(\ell, m)$ are small in the region where the integrand is non-vanishing. The ℓ th electric dipole moment can be seen to be of the order $(kd)^{\ell}$. Since the current density is typically of the order $kcd\rho(r)$, the magnetic multipole moments are of the order $(kd)^{\ell+1}$. The lowest non-vanishing moment is the dominant one in each series. A magnetic dipole moment is expected to be of the same size as an electric quadrupole moment. In the expression for the electric multipole coefficient $a_E(\ell, m)$, the dominant term is the first one involving the charge density, since the $\mathbf{J} \cdot \mathbf{r}$ term is usually smaller by two powers of kd . When the spherical Bessel function is replaced by a simple power of r the integrals for the multipole moments then become proportional to the expressions for the static multipole moments:

$$a_E(\ell, m) = \frac{4\pi k^{\ell+2}}{\ell(2\ell + 1)!!} q_{\ell m} \quad (78)$$

$$a_M(\ell, m) = \frac{4\pi k^{\ell+2}}{\ell(2\ell + 1)!!} \mu_{\ell m} \quad (79)$$

where the static electric multipole moment is

$$q_{\ell, m} = \int Y_{\ell m}^* r^{\ell} \rho(r) d^3r \quad (80)$$

and the static magnetic multipole moment is given by Eq (45).

Relationship to Cartesian Multipole Moments For low-order multipole moments it is often convenient to express the fields in terms of the Cartesian multipole moments, instead of ℓm moments. The static electric dipole, magnetic dipole, and electric quadrupole moments are given respectively by

$$\mathbf{p} = \int \mathbf{r} \rho(\mathbf{r}) d^3 r \quad (81)$$

$$\boldsymbol{\mu} = \frac{1}{2c} \int \mathbf{r} \times \mathbf{J}(\mathbf{r}) d^3 r \quad (82)$$

$$Q_{\alpha\beta} = \int (3r_\alpha r_\beta - r^2 \delta_{\alpha\beta}) \rho(\mathbf{r}) d^3 r. \quad (83)$$

To illustrate how we rewrite the expressions for the corresponding low-order multipole fields in terms of these Cartesian moments, consider the electric dipole fields obtained by substituting (78) into (70).

$$\mathbf{B} = \sum_{m=-1}^1 \frac{4\pi k^3}{3} q_{1m} h_1^{(1)}(kr) (\mathbf{r} \times \nabla) Y_{1m}(\theta, \phi) \quad (84)$$

$$\mathbf{E} = \frac{i}{k} \nabla \times \mathbf{B}. \quad (85)$$

The sum over m involves only $q_{1m} Y_{1m}$. Using the helpful identity

$$\frac{4\pi}{3} \sum_{m=-1}^1 Y_{1m}^*(\theta', \phi') Y_{1m}(\theta, \phi) = \hat{\mathbf{r}}' \cdot \hat{\mathbf{r}}, \quad (86)$$

where $\hat{\mathbf{r}}'$ has spherical polar angles θ', ϕ' and $\mathbf{r}$ has angles θ, ϕ , we get

$$\frac{4\pi}{3} \sum q_{1m} Y_{1m}(\theta, \phi) = \mathbf{p} \cdot \mathbf{r}. \quad (87)$$

Substituting this result into Eq (84) and using the explicit expression for $h_1(kr)$ gives

$$\mathbf{B} = k^2 (\hat{\mathbf{r}} \times \mathbf{p}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right) \quad (88)$$

$$\mathbf{E} = k^2 (\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}} \frac{e^{ikr}}{r} + [3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}) - \mathbf{p}] \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr}. \quad (89)$$

A similar treatment for the magnetic dipole fields gives the analogous expression

$$\mathbf{E} = -k^2 (\hat{\mathbf{r}} \times \boldsymbol{\mu}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right) \quad (90)$$

$$\mathbf{B} = k^2(\hat{\mathbf{r}} \times \boldsymbol{\mu}) \times \hat{\mathbf{r}} \frac{e^{ikr}}{r} + [3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \boldsymbol{\mu}) - \boldsymbol{\mu}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}. \quad (91)$$

To convert the expression for the electric quadrupole fields into Cartesian moments, we use the identity

$$\frac{8\pi}{5} \sum_{m=-2}^2 q_{2m} Y_{2m}(\theta, \phi) = \sum_{\alpha, \beta} \hat{r}_\alpha Q_{\alpha\beta} \hat{r}_\beta. \quad (92)$$

The fields are then written in terms of the vector

$$Q_\alpha(\hat{\mathbf{r}}) = \sum_\beta Q_{\alpha\beta} \hat{r}_\beta. \quad (93)$$

The result is

$$\mathbf{B} = \frac{k^4}{6} h_2^{(1)}(kr) \hat{\mathbf{r}} \times \mathbf{Q}(\mathbf{r}) \quad (94)$$

$$\mathbf{E} = \frac{i}{k} \nabla \times \mathbf{B}. \quad (95)$$

In the radiation zone the fields have the form

$$\mathbf{B} = \frac{-ik^3}{6} \frac{e^{ikr}}{r} \hat{\mathbf{r}} \times \mathbf{Q}(\hat{\mathbf{r}}) \quad (96)$$

$$\mathbf{E} = \frac{-ik^3}{6} \frac{e^{ikr}}{r} [\hat{\mathbf{r}} \times \mathbf{Q}(\hat{\mathbf{r}})] \times \hat{\mathbf{r}}. \quad (97)$$

For higher multipole moments the expressions in terms of Cartesian moments is clumsy, and the ℓm moments are preferred.

Exercises

1. Derive the div and curl expressions (21), (23), (24), (25).
2. Derive the wave equations (59) and (60).
3. Given a conducting spherical resonant cavity of radius a , write the complete expression for the electric and magnetic fields corresponding to the lowest electric dipole and lowest magnetic dipole resonance, specifying the resonant frequencies in both cases.
4. Prove the identities (86) and (92).
5. Derive (90) and (91).
6. Derive (94).

These are my *extended* notes on the 1995 Physics 712 handout on the subject of VSH which was assembled by Carleton from notes of Richard and Tracy. Lots of calculations are recorded in my appendix. I plan to summarize this set of notes in *another* document, leaving out all the details, and trying to focus on the main points.

1. The Scalar Function ψ defines an axis rotation.

Consider the three vectors as follows:

$$\mathbf{r} \times \nabla \psi \quad \hat{\mathbf{r}} \psi \quad r \nabla \psi$$

where ψ is any *complex* function $\psi(\theta, \phi)$. Notice that all three vectors are dimensionless. These three vectors are mutually perpendicular:

$$\mathbf{r} \cdot \nabla \psi = r \partial_r \psi = 0 \quad \text{since } \psi \text{ is not a function of } r$$

$$\mathbf{r} \cdot (\mathbf{r} \times \nabla \psi) = 0 \quad \text{since } \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0 \text{ for any vectors}$$

$$\nabla \psi \cdot (\mathbf{r} \times \nabla \psi) = 0 \quad \text{same reason}$$

What are these vectors? The radial one is just of magnitude ψ pointing radially outward, that is pretty clear. The $r \nabla \psi$ vector points in a direction *tangent* to the sphere's surface in the direction in which ψ has its maximum increase. And $\mathbf{r} \times \nabla \psi$ is also tangent also and is perp to the other two vectors.

Now let's try to expand an arbitrary complex function $A(\theta, \phi)$ as follows:

$$A = \alpha r \nabla \psi + \beta \hat{\mathbf{r}} \psi + \gamma \mathbf{r} \times \nabla \psi$$

where α , β and γ are complex functions of θ and ϕ . Is this a completely general expansion of A ? I will argue that it is, and here are the components:

$$\alpha = A \cdot (r \nabla \psi) \quad \beta = (A \cdot \hat{\mathbf{r}}) \psi \quad \gamma = A \cdot (\mathbf{r} \times \nabla \psi)$$

All we are doing is changing from x, y, z to x', y', z' where the new coordinate system is a function of our pilot function ψ .

2. Specific choice for ψ as $Y_{lm}(\theta, \phi)$ and some reasons for this choice.

We could, for example, set $\psi = Y_{lm}(\theta, \phi)$, then we get:

$$\mathbf{r} \cdot \nabla Y_{lm} = 0$$

$$\mathbf{r} \cdot (\mathbf{r} \times \nabla Y_{\ell m}) = 0$$

$$\nabla Y_{\ell m} \cdot (\mathbf{r} \times \nabla Y_{\ell m}) = 0$$

Let's now give these three vector functions Carleton's names,

$$\mathbf{X}_{\ell m} = \mathbf{r} \times \nabla Y_{\ell m}$$

$$\mathbf{Y}_{\ell m} = \hat{\mathbf{r}} Y_{\ell m}$$

$$\mathbf{Z}_{\ell m} = r \nabla Y_{\ell m}$$

In passing, we note that Jackson also uses an X function, but his is scaled differently, see (16.45).

$$\mathbf{X}_{\ell m}^{\text{JACKSON}} = \frac{1}{i\sqrt{\ell(\ell+1)}} \mathbf{X}_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}} \mathbf{L} Y_{\ell m} \quad i\mathbf{L} = \mathbf{r} \times \nabla$$

The i here comes from Jackson's inclusion of an i in his definition of $\mathbf{L}$, making it consistent with the $\mathbf{L}$ that we use in quantum mechanics. For our purposes here, however, this factor causes only trouble.

I might have called these more suggestive names, like $\mathbf{M}_{\ell m}$, $\mathbf{R}_{\ell m}$ and $\mathbf{G}_{\ell m}$ (angular Momentum, Radial, Gradient). One problem is that you can never remember which letter goes with which function. In retrospect, I think I like the simple X,Y,Z choice.

Now, for each choice of values ℓm , we have a little perpendicular vector triad where one vector is radial and the other two are tangential to the surface of the sphere. Notice, however, that the orientation of the two tangential vectors relative to the sphere is going to be dependent on the choice of ℓ and m , whereas the radial vector is always radial.

Why are we interested in these vectors with $\psi = Y_{\ell m}$? Well, first off, in our appendix at the end of this paper we painstakingly calculate many things, and in doing so we learn many things. One fact we find is that application of any of the standard operators like curl, div, grad, lapl, $\mathbf{r} \cdot$ and $\mathbf{r} \times$ to the three basis functions produces either a linear combination of the same basis functions (which reinforces our idea of them spanning the 3-space), or produces a multiple of $Y_{\ell m}$ in the case of an operation that produces a scalar result like div. In these proofs, we sometimes encounter the ∇^2 operator and we replace it with the operator $-1/r^2 \mathbf{L}^2$ when it acts on a function of θ and ϕ , where $i\mathbf{L}$ is $\mathbf{r} \times \nabla$. It is *at this point* that we make use of the specific properties of the $Y_{\ell m}$ functions, that $\mathbf{L}^2 Y_{\ell m} = \ell(\ell+1) Y_{\ell m}$. It is this fact that keeps everything closed within the little ℓm manifold. For example, you never encounter an $\mathbf{L}_+$ type operator that tries to raise you to $Y_{\ell m+1}$, for example.

Let's look more closely at this claim that nothing can push us out of the ℓm space. No matter how much vector stuff we apply on the left, when we work through all the vector identities, these are the only dimensionless vector things we can "end up with":

$\hat{\mathbf{r}} Y_{\ell m}$	$\mathbf{Y}$
$r \nabla Y_{\ell m}$	$\mathbf{Z}$
$\mathbf{r} \times \nabla Y_{\ell m}$	$\mathbf{X}$
$\hat{\mathbf{r}} (\mathbf{r} \cdot \nabla Y_{\ell m})$	0
$r^2 \nabla \times \nabla Y_{\ell m}$	0
$\hat{\mathbf{r}} r^2 \nabla \cdot \nabla Y_{\ell m}$	$\hat{\mathbf{r}} r^2 \nabla^2 Y_{\ell m} = -\hat{\mathbf{r}} \mathbf{L}^2 Y_{\ell m} = -\ell(\ell+1) \hat{\mathbf{r}} Y_{\ell m} = -\ell(\ell+1) \mathbf{Y}$

The only things we have to "play with" are the vector $\mathbf{r}$ or $\hat{\mathbf{r}}$, the scalar r , and the ∇ operator. Now for any pilot function $\psi(\theta, \phi)$ in place of Y , the first 6 items would still be true, and our space would be closed under these operations. It is the last item that is critical. With the generic ψ function we would end up with $r^2 \nabla^2 \psi = r^2 g$, some new function, and we could then end up with $\hat{\mathbf{r}} g$ as a new vector.

As an example, we know (by explicit calc in our appendix) that ∇^2 applied to any of the vector basis functions just yields a linear combination of those basis functions. We also know that any purely- r operator can only shuffle the powers of r around in some way, without affecting the $Y_{\ell m}$ -- certainly we will not shift any ℓm by applying combinations of ∂_r and r . Therefore, it is true that the L^2 operator when acting on any of the vector basis functions will give at worst a linear combination of same. Since $[L^2, \mathbf{L}] = 0$, we can see that $L^2 \mathbf{X} = \ell(\ell+1) \mathbf{X}$. For $\mathbf{Y}$ and $\mathbf{Z}$ I don't think this is true, but L^2 will certainly yield a linear combination of the vector basis functions as per our discussion above.

In particular, the vector operators in Maxwell's Equations won't take you out of ℓm , so each ℓm manifold represents a "separate problem", as is the goal in any diagonalization. Of course the underlying fact is that Maxwell's equations are invariant under $SO(3)$ and this is why $L^2 L_3$ diagonalization is possible. The Hamiltonian or Lagrangian would be something like $F^{\mu\nu} F_{\mu\nu} + A^\mu J_\mu$ and is clearly $SO(3)$ invariant, not to mention $SO(3,1)$ invariant. So these comments I think provide a little extra motivation for the underpinning of the vector multipole expansion.

Before going on, we should note that Jackson's "grand finale multipole expansion of $\mathbf{E}$ or $\mathbf{B}$ " 16.47 has two vector terms, but according to 16.143, the second term itself breaks into two other terms, and you then end up with three vector terms in each equation, and these are precisely the vector basis functions we are here talking about. Jackson however does not really discuss the fact that these three functions span the space; he is just silent about that and grinds out the results.

3. Orthogonality Proofs

Let's think of orthogonality not just in terms of dot products, but in terms of complex dot products with the left item complex conjugated, and integrated over all solid angle $d\Omega$. A special "inner product".

Y with Y. Orthogonality of the $\mathbf{Y}$'s is obvious from the ψ 's, since $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$. That is, we know that our choice of $\psi = Y_{\ell m}(\theta, \phi)$ precisely have this orthonormality, namely (stated as (5) in the paper)

$$\int d\Omega \hat{\mathbf{r}} Y_{\ell m}^*(\theta, \phi) \cdot \hat{\mathbf{r}} Y_{\ell' m'}(\theta, \phi) = \int d\Omega Y_{\ell m}^*(\theta, \phi) Y_{\ell' m'}(\theta, \phi) = \delta_{\ell, \ell'} \delta_{m, m'}$$

X with X. Orthogonality of the $\mathbf{X}$'s requires the following work: (there is probably an easier way, see below)

$$\mathbf{L} = \hat{\mathbf{x}} L_x + \hat{\mathbf{y}} L_y + \hat{\mathbf{z}} L_z$$

Define the following

$$\boldsymbol{\varepsilon}_\pm = \hat{\mathbf{x}} \pm i \hat{\mathbf{y}} \quad \Rightarrow \quad \boldsymbol{\varepsilon}_\pm \cdot \boldsymbol{\varepsilon}_\pm = 2 \quad \text{and} \quad \boldsymbol{\varepsilon}_\pm \cdot \boldsymbol{\varepsilon}_\mp = 0$$

$$L_{\pm} = L_x \pm L_y$$

You then find that

$$L = (1/2i)[\epsilon_+ L_+ + \epsilon_- L_-] + \hat{z} L_z$$

and then

$$(L \cdot) \cdot (L \cdot) = (1/2) \{ L_+ \cdot L_+ + L_- \cdot L_- \} + L_z \cdot L_z$$

where I have left space for you to write in $Y_{\ell m'}$ and $Y_{\ell m}$ in the spaces. The two raising or two lowering operators do the same change to the two Y functions, so the final delta form is not changed, but you pick up the square of the raising algebra coefficients Jackson shows on page 542. Then added to the last squared coefficient m^2 you get $\ell(\ell+1)$, as Carleton shows in his (4). So,

$$\int d\Omega L Y_{\ell m'}(\theta, \phi) \cdot L Y_{\ell m}(\theta, \phi) = \ell(\ell+1) \delta_{\ell, \ell} \delta_{m', m}$$

Notice that this proof does not work with just any old ψ functions! Here is another proof that is more to the point:

$$\begin{aligned} \int d\Omega L Y_{\ell m'}(\theta, \phi) \cdot L Y_{\ell m}(\theta, \phi) &= \int d\Omega \langle \ell' m' | \mathcal{L} | \theta \phi \rangle \cdot \langle \theta \phi | \mathcal{L} | \ell m \rangle = \langle \ell' m' | \mathcal{L} \cdot \mathcal{L} | \ell m \rangle \\ &= \langle \ell' m' | \mathcal{L}^2 | \ell m \rangle = \ell(\ell+1) \langle \ell' m' | \ell m \rangle = \ell(\ell+1) \delta_{\ell, \ell} \delta_{m', m} \end{aligned}$$

where $\mathcal{L}$ is the abstract operator in the usual Hilbert space we deal with in these matters. Again, it is the Casimir operator of the group and all that stuff.

Z with Z. Finally, what about orthogonality of the Z's? I got stuck on that for a while, but then I realized this theorem

$$\int d\Omega (\nabla \cdot \mathbf{f}) = 0$$

as long as $f(\theta, \phi)$ is finite at $\theta = \pm\pi$, and is periodic in ϕ . See proof on scratch. You basically treat the divergence as two terms and in each term you have a perfect differential with respect to one of the two angle variables and then the assumptions given make the result zero. Using a simple vector identity, a corollary to this theorem is

$$\int d\Omega (\nabla f \cdot \nabla g) = - \int d\Omega (f \nabla^2 g)$$

just as if you were doing a parts integration and throwing away the parts. We know from Jackson p 543 that $-\nabla^2 g = (1/r^2) L^2 g$ and if g is a $Y_{\ell m}$ then L^2 becomes $\ell(\ell+1)$. This then proves orthogonality of the Z 's, equation (6),

$$r^2 \int d\Omega \nabla Y_{\ell m}^*(\theta, \phi) \cdot \nabla Y_{\ell' m'}(\theta, \phi) = \ell(\ell+1) \delta_{\ell, \ell'} \delta_{m, m'}$$

Y with Z and Y with X. These function sets are orthogonal simply because the vectors are orthogonal from our earlier observations, namely, that

$$\mathbf{r} \cdot \nabla Y_{\ell m} = 0 \quad \Rightarrow Y \text{ with } Z = 0$$

$$\mathbf{r} \cdot (\mathbf{r} \times \nabla Y_{\ell m}) = 0 \quad \Rightarrow Y \text{ with } X = 0$$

Z with X. This is the one remaining combination, and it is not *trivially* true because, although

$$\nabla Y_{\ell m} \cdot (\mathbf{r} \times \nabla Y_{\ell m}) = 0$$

we have

$$\nabla Y_{\ell m}^* \cdot (\mathbf{r} \times \nabla Y_{\ell m}) \neq 0$$

Here is a proof that with the angular integral included, Z is orthogonal to X . Start with

$$\nabla \cdot [f \mathbf{L} g] = \nabla f \cdot \mathbf{L} g + f \nabla \cdot \mathbf{L} g \quad // \text{ just a vector identity, thinking } f = Y_{\ell m}^*$$

Integration over solid angle gives 0 for the left factor from our theorem above, assuming f and g are finite etc. The first term on the right is the one we *want* to show is 0 (it is $\sim \mathbf{Z}^* \cdot \mathbf{X}$), so we will have proven (7) if we can show that the *second* term on the right is zero. But

$$\nabla \cdot \mathbf{L} g = \nabla \cdot (\mathbf{r} \times \nabla g) = \nabla g \cdot (\nabla \times \mathbf{r}) - \mathbf{r} \cdot (\nabla \times \nabla g) = 0 + 0 = 0$$

since $(\nabla \times \mathbf{r}) = 0$ and $\nabla \times (\nabla g) = 0$, QED.

4. The Expansion Theorem

We can combine all the orthogonality conditions proven above involving the X, Y, Z into a single statement. To do this, let's redefine the vector functions as follows

$$\mathbf{X}_{\ell m}^1 = k_{\ell} \mathbf{X}_{\ell m} = k_{\ell} \mathbf{r} \times \nabla Y_{\ell m} = i k_{\ell} \mathbf{L} Y_{\ell m}$$

$$\mathbf{X}_{\ell m}^2 = Y_{\ell m} = \hat{\mathbf{r}} \cdot \nabla Y_{\ell m}$$

$$\mathbf{X}_{\ell m}^3 = k_{\ell} \mathbf{Z}_{\ell m} = k_{\ell} \mathbf{r} \cdot \nabla Y_{\ell m}$$

where we have defined

$$k_\ell \equiv 1/\sqrt{\ell(\ell+1)}$$

Then in the previous section we have *proven* the following fact:

$$\int d\Omega \mathbf{X}^{n'}_{\ell m'}(\theta, \phi)^* \bullet \mathbf{X}^n_{\ell m}(\theta, \phi) = \delta_{\ell, \ell'} \delta_{m', m} \delta_{n', n} \quad \text{orthogonality}$$

Furthermore, we have shown that the functions $\mathbf{X}^n_{\ell m}(\theta, \phi)$ as vectors in 3-space form a triad of perpendicular vectors which, if normalized to unity, would just be some rotation of the canonical unit vectors, with the specific rotation being a function of ℓ , m , θ and ϕ . Therefore, in terms of 3-space, we know that the vectors $\mathbf{X}^n_{\ell m}(\theta, \phi)$ clearly "span" the 3-space. Any vector could be written as a linear combination of these three vectors.

Now, we want to make this conjecture: any vector function of θ and ϕ can be expanded as follows:

$$\mathbf{A}(\theta, \phi) = \sum_{\ell m n} A_{\ell m n} \mathbf{X}^n_{\ell m}(\theta, \phi) \quad (1)$$

The radial projection of this equation is,

$$\hat{\mathbf{r}} \bullet \mathbf{A}(\theta, \phi) = A_r(\theta, \phi) = \sum_{\ell m n} A_{\ell m n} (\hat{\mathbf{r}} \bullet \mathbf{X}^n_{\ell m}(\theta, \phi)) = \sum_{\ell m} A_{\ell m 2} Y_{\ell m}(\theta, \phi)$$

and we know from scalar theory that this is in fact a "complete" expansion for the scalar function $A_r(\theta, \phi)$. So at least the conjecture seems possibly true. The orthogonality makes us think it is true! At least within each partial wave, we can see that we are spanning the full 3-vector space with our three basis functions.

If we assume the above expansion is true, we can use the orthonormality condition as follows. Start with the expansion,

$$\mathbf{A}(\theta, \phi) = \sum_{\ell m n} A_{\ell m n} \mathbf{X}^n_{\ell m}(\theta, \phi)$$

Peruse the orthonormality,

$$\int d\Omega \mathbf{X}^{n'}_{\ell' m'}(\theta, \phi)^* \bullet \mathbf{X}^n_{\ell m}(\theta, \phi) = \delta_{\ell, \ell'} \delta_{m', m} \delta_{n', n}$$

Apply integral operator to both sides

$$\begin{aligned} \int d\Omega \mathbf{X}^{n'}_{\ell' m'}(\theta, \phi)^* \bullet \mathbf{A}(\theta, \phi) &= \sum_{\ell m n} A_{\ell m n} \int d\Omega \mathbf{X}^{n'}_{\ell' m'}(\theta, \phi)^* \bullet \mathbf{X}^n_{\ell m}(\theta, \phi) \\ &= \sum_{\ell m n} A_{\ell m n} \delta_{\ell, \ell'} \delta_{m', m} \delta_{n', n} = A_{\ell' m' n'} \end{aligned}$$

And there is our projection,

$$A_{\ell' m' n'} = \int d\Omega \mathbf{X}^{n'}_{\ell' m'}(\theta, \phi)^* \bullet \mathbf{A}(\theta, \phi)$$

I have proven nothing of course, but it does seem reasonable. Here is the implied completeness condition:

$$\sum_{\ell m n} [X_{\ell m}^n(\theta, \phi)]_i [X_{\ell m}^n(\theta', \phi')]_r = \delta_{i,r} \langle \theta | \theta' \rangle = \delta_{i,r} \delta(\Omega - \Omega')$$

which would be called an addition theorem for the X 's. If we could prove this directly, I think our expansion conjecture would then be justified. However, I think the proof would be very difficult. The usual proof is to show completeness some other way, then the above has to be true!

In a different notation we might say,

$$\langle n\ell m | i\theta \phi \rangle \langle i\theta \phi | n'\ell' m' \rangle = \langle n\ell m | n'\ell' m' \rangle \quad \text{orthogonality}$$

$$\langle i\theta \phi | n\ell m \rangle \langle n\ell m | i'\theta' \phi' \rangle = \langle i\theta \phi | i'\theta' \phi' \rangle \quad \text{completeness}$$

Finally, if $A = A(r, \theta, \phi)$, treat r as a parameter being carried along, and we then get

$$A(r, \theta, \phi) = \sum_{\ell m n} A_{\ell m n}(r) X_{\ell m}^n(\theta, \phi)$$

$$A_{\ell m n}(r) = \int d\Omega X_{\ell m}^n(\theta, \phi)^* \bullet A(r, \theta, \phi)$$

This is really a multipole or partial wave expansion of a vector field, and we are going to apply it to the electric and magnetic fields soon. What we are doing is similar to Jackson Chapter 16 but maybe a little more organized. We shall find that we can deal with each partial wave separately and thereby simplify our attack on certain problems.

5. Apply divergence to the expansion

In these notes, I mistakenly used the symbol τ instead of r as the label for the coefficient of the *radial* function Y . I then stayed consistent with that use. The two transverse vectors Z and X get labels 1 and 2.

Since I want to check things against Carleton's writeup, let's use his form of the expansion

$$\mathbf{A} = \sum_{\ell m} [A_2 \mathbf{X} + A_\tau \mathbf{Y} + A_1 \mathbf{Z}]$$

$$\nabla \bullet \mathbf{A} = \sum_{\ell m} [\nabla \bullet (A_2 \mathbf{X}) + \nabla \bullet (A_\tau \mathbf{Y}) + \nabla \bullet (A_1 \mathbf{Z})] = \sum_{\ell m} [1 + 2 + 3]$$

From our appendix we have

$$\begin{array}{ll} \nabla \bullet \mathbf{X}_{\ell m} = 0 & \hat{\mathbf{r}} \bullet \mathbf{X}_{\ell m} = 0 \\ \nabla \bullet \mathbf{Y}_{\ell m} = (2/r) Y_{\ell m} & \hat{\mathbf{r}} \bullet \mathbf{Y}_{\ell m} = Y_{\ell m} \\ \nabla \bullet \mathbf{Z}_{\ell m} = -\ell(\ell+1) (1/r) Y_{\ell m} & \hat{\mathbf{r}} \bullet \mathbf{Z}_{\ell m} = 0 \end{array}$$

So we get

$$1 = \nabla \cdot (A_2 \mathbf{X}) = \nabla A_2 \cdot \mathbf{X} + A_2 \nabla \cdot \mathbf{X} = \partial_r A_2 (\hat{\mathbf{r}} \cdot \mathbf{X}) + A_2 (\nabla \cdot \mathbf{X}) \\ = \partial_r A_2 (\hat{\mathbf{r}} \cdot \mathbf{X}) + A_2 (\nabla \cdot \mathbf{X}) = \partial_r A_2 (0) + A_2 (0) = 0$$

$$2 = \nabla \cdot (A_r \mathbf{Y}) = \partial_r A_r (\hat{\mathbf{r}} \cdot \mathbf{Y}) + A_r (\nabla \cdot \mathbf{Y}) = (\partial_r A_r) Y_{\ell m} + A_r (2/r) Y_{\ell m} \\ = 1/r^2 \partial_r (r^2 A_r) Y_{\ell m}$$

$$3 = \nabla \cdot (A_1 \mathbf{Z}) = \partial_r A_1 (\hat{\mathbf{r}} \cdot \mathbf{Z}) + A_1 (\nabla \cdot \mathbf{Z}) = \partial_r A_1 (0) + A_1 (-\ell(\ell+1)(1/r) Y_{\ell m}) \\ = -A_1 \ell(\ell+1)(1/r) Y_{\ell m}$$

So here is the result

$$\mathbf{A} = \sum_{\ell m} [A_2 \mathbf{X} + A_r \mathbf{Y} + A_1 \mathbf{Z}]$$

$$\nabla \cdot \mathbf{A} = \sum_{\ell m} [\nabla \cdot (A_2 \mathbf{X}) + \nabla \cdot (A_r \mathbf{Y}) + \nabla \cdot (A_1 \mathbf{Z})]$$

$$= \sum_{\ell m} \{ 1/r^2 \partial_r (r^2 A_r) - A_1 \ell(\ell+1)(1/r) \} Y_{\ell m} \quad \text{which agrees with (21)}$$

Now, suppose someone imposes this requirement,

$$\nabla \cdot \mathbf{A} = 4\pi\rho$$

Then this really must be satisfied in each partial wave, so we find that

$$\{ 1/r^2 \partial_r (r^2 A_r) - A_1 \ell(\ell+1)(1/r) \} = 4\pi\rho_{\ell m}$$

Thus, this divergence condition (perhaps $\nabla \cdot \mathbf{B} = 0$ for a magnetic field, or $\nabla \cdot \mathbf{E} = \rho$ for E field) puts no restriction at all on the function A_2 , but requires the other two functions to be related by this little DE. This begins to show how we might apply Maxwell's equations and end up with constraints on the multipole coefficients of the fields within each partial wave.

5. Apply curl to the expansion

Since I want to check things against Carleton's writeup, let's use his form of the expansion

$$\mathbf{A} = \sum_{\ell m} [A_2 \mathbf{X} + A_r \mathbf{Y} + A_1 \mathbf{Z}]$$

$$\nabla \times \mathbf{A} = \sum_{\ell m} [\nabla \times (A_2 \mathbf{X}) + \nabla \times (A_r \mathbf{Y}) + \nabla \times (A_1 \mathbf{Z})] = \sum_{\ell m} [1 + 2 + 3]$$

From our appendix we have

$$\begin{aligned} \nabla \times \mathbf{X}_{\ell m} &= -[\ell(\ell+1)/r] \mathbf{Y}_{\ell m} - (1/r) \mathbf{Z}_{\ell m} & \hat{\mathbf{r}} \times \mathbf{X}_{\ell m} &= -\mathbf{Z}_{\ell m} \\ \nabla \times \mathbf{Y}_{\ell m} &= - (1/r) \mathbf{X}_{\ell m} & \hat{\mathbf{r}} \times \mathbf{Y}_{\ell m} &= 0 \end{aligned}$$

$$\nabla \times \mathbf{Z}_{\ell m} = (1/r) \mathbf{X}_{\ell m}$$

$$\hat{\mathbf{r}} \times \mathbf{Z}_{\ell m} = \mathbf{X}_{\ell m}$$

So we get

$$\begin{aligned} 1 = \nabla \times (A_2 \mathbf{X}) &= \nabla A_2 \times \mathbf{X} + A_2 \nabla \times \mathbf{X} = \partial_r A_2 (\hat{\mathbf{r}} \times \mathbf{X}) + A_2 (\nabla \times \mathbf{X}) \\ &= \partial_r A_2 (-\mathbf{Z}_{\ell m}) + A_2 (-[\ell(\ell+1)/r] \mathbf{Y}_{\ell m} - (1/r) \mathbf{Z}_{\ell m}) \\ &= -[\partial_r A_2 + (1/r)] \mathbf{Z}_{\ell m} - A_2 \ell(\ell+1)/r \mathbf{Y}_{\ell m} \\ &= -(1/r) \partial_r (r A_2) \mathbf{Z}_{\ell m} - A_2 \ell(\ell+1)/r \mathbf{Y}_{\ell m} \end{aligned}$$

$$\begin{aligned} 2 = \nabla \times (A_\tau \mathbf{Y}) &= \partial_r A_\tau (\hat{\mathbf{r}} \times \mathbf{Y}) + A_\tau (\nabla \times \mathbf{Y}) = (\partial_r A_\tau) \mathbf{0} + A_\tau (-1/r) \mathbf{X}_{\ell m} \\ &= -A_\tau (1/r) \mathbf{X}_{\ell m} \end{aligned}$$

$$\begin{aligned} 3 = \nabla \times (A_1 \mathbf{Z}) &= \partial_r A_1 (\hat{\mathbf{r}} \times \mathbf{Z}) + A_1 (\nabla \times \mathbf{Z}) = \partial_r A_1 (\mathbf{X}_{\ell m}) + A_1 (1/r) \mathbf{X}_{\ell m} \\ &= [\partial_r A_1 + A_1/r] \mathbf{X}_{\ell m} = (1/r) (\partial_r (r A_1)) \mathbf{X}_{\ell m} \end{aligned}$$

So here is the result

$$\mathbf{A} = \sum_{\ell m} [A_2 \mathbf{X} + A_\tau \mathbf{Y} + A_1 \mathbf{Z}]$$

$$\begin{aligned} \nabla \times \mathbf{A} &= \sum_{\ell m} [\nabla \times (A_2 \mathbf{X}) + \nabla \times (A_\tau \mathbf{Y}) + \nabla \times (A_1 \mathbf{Z})] \\ &= \sum_{\ell m} \left\{ \underbrace{-(1/r) \partial_r (r A_2) \mathbf{Z}_{\ell m}}_1 - \underbrace{\ell(\ell+1)/r A_2 \mathbf{Y}_{\ell m}}_\tau + \underbrace{[(1/r) (\partial_r (r A_1)) - A_\tau/r] \mathbf{X}_{\ell m}}_2 \right\} \end{aligned}$$

and amazingly enough, this exactly agrees with (22)-(25).

Now, suppose someone imposes this requirement,

$$\nabla \times \mathbf{A} = \mathbf{0}$$

Then this really must be satisfied in each partial wave, and for each component! So we get

$$\begin{aligned} -(1/r) \partial_r (r A_2) &= 0 & \Rightarrow \text{nothing since } A_2 = 0 \text{ from next equation} \\ -\ell(\ell+1)/r A_2 &= 0 & \Rightarrow A_2 = 0 \\ (1/r) (\partial_r (r A_1)) - A_\tau/r &= 0 & \Rightarrow \partial_r (r A_1) = A_\tau \end{aligned}$$

So with this condition, in each partial wave we have knocked out the A_2 component! And the other two components are related by this simple differential equation!

6. Doing Electrostatics with VSH

We have our two equations already noted, repeat them here:

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

which implies

$$\{ 1/r^2 \partial_r (r^2 E_r) - E_1 \ell(\ell+1) (1/r) \} = 4\pi\rho_{\ell m} \quad (a) \quad \text{verifies (31)}$$

and

$$\nabla \times \mathbf{E} = 0$$

which implies

$$\begin{aligned} -\ell(\ell+1)/r E_2 &= 0 & \Rightarrow E_2 &= 0 \\ (1/r)(\partial_r(r E_1) - E_r/r) &= 0 & \Rightarrow \partial_r(r E_1) &= E_r \end{aligned} \quad (b)$$

The two DE's both involved E_1 and E_r . If we combine these we get (32) as a single DE for E_1 , and if we then define $\phi = -r E_1$ then equation (31) reads as follows:

$$\partial_r^2 \phi + (2/r) \partial_r \phi - \ell(\ell+1)/r^2 \phi = -4\pi\rho_{\ell m}$$

Compare this to Jackson 16.5 with $k^2 = 0$ and a source added, and you see that we have discovered our friend the Poisson equation where the "potential" seems to be $\phi = -r E_1$. So think of this as saying that we solve the Poisson equation for ϕ , and then we have $E_1 = -\phi/r$, $E_r = -\partial_r \phi$, $E_2 = 0$. Then our E field must be,

$$\begin{aligned} \mathbf{E} &= \sum_{\ell m} [E_r \mathbf{Y} + E_1 \mathbf{Z}] = \sum_{\ell m} [-\partial_r \phi \mathbf{Y} - \phi/r \mathbf{Z}] = \\ &= -\sum_{\ell m} [\partial_r \phi \hat{\mathbf{r}} Y_{\ell m} + \phi \nabla Y_{\ell m}] \end{aligned}$$

where now we have shown the subscripts on ϕ which we were too lazy to add before. Do we recognize this solution? Go to Jackson (4.1) on the multipole expansion for ϕ , and assume we have found some solution

$$\phi(r, \theta, \phi) = \sum_{\ell m} (4\pi/(2\ell+1)) q_{\ell m} 1/r^{\ell+1} Y_{\ell m} = \sum_{\ell m} \phi_{\ell m}(r) Y_{\ell m}(\theta, \phi)$$

where

$$\phi_{\ell m}(r) = (4\pi/(2\ell+1)) q_{\ell m} 1/r^{\ell+1}$$

Then the electric field should be

$$\mathbf{E} = -\nabla \phi = -\sum_{\ell m} \nabla [\phi_{\ell m}(r) Y_{\ell m}(\theta, \phi)] = -\sum_{\ell m} \{ \nabla [\phi_{\ell m}(r)] Y_{\ell m}(\theta, \phi) + \phi_{\ell m}(r) \nabla Y_{\ell m}(\theta, \phi) \}$$

$$= - \sum_{\ell m} \{ \partial_r \phi_{\ell m}(r) \hat{r} Y_{\ell m}(\theta, \phi) + \phi_{\ell m}(r)/r \ r \nabla Y_{\ell m}(\theta, \phi) \}$$

$$= - \sum_{\ell m} \{ \partial_r \phi_{\ell m}(r) \mathbf{Y} + \phi_{\ell m}(r)/r \mathbf{Z} \}$$

which agrees with our VSH result.

7. Doing Magnetostatics with VSH

Before looking at Maxwell's here, (35) is just the electrostatic expansion restated for magnetic quantities, including the magnetic potential. If we set $\mathbf{B} = -\nabla\phi$ we instantly end up with (36) through (38), no problem here. Now, let's look at Maxwell's equations.

We have our two equations already noted, repeat them here:

$$\nabla \cdot \mathbf{B} = 0$$

which implies

$$\{ 1/r^2 \partial_r (r^2 B_r) - B_\ell \ell(\ell+1) (1/r) \} = 0 \quad \text{see (42)}$$

and

$$\nabla \times \mathbf{B} = 4\pi/c \mathbf{J}$$

which implies

$$-(1/r) \partial_r (r B_\ell) = 4\pi/c J_\ell \quad / Z \quad (40)$$

$$- \ell(\ell+1)/r B_r = 4\pi/c J_r \quad / Y \quad (39)$$

$$(1/r)(\partial_r (r B_\ell) - B_r/r) = 4\pi/c J_\ell \quad / X \quad (41)$$

The two DE's both involved B_ℓ and B_r . I have combined these to get (43) for B^ℓ driven by J^ℓ .

Now derivation of (44) requires a bit of a digression! Jackson (3.120) shows the definition of the Green's function for this equation and the form of the solution. I abbreviate as follows:

$$\text{op}[g] = -4\pi/r^2 \delta(r-r') \quad (3.120)$$

Then if you have

$$\text{op}[F] = s$$

the solution will be

$$F(r) = - \int_0^a g(r,r') s(r') (r'^2/4\pi) dr'$$

as you know by applying op to both sides. We show the source function s as localized. For r outside this region, we have $r > r'$ and we know that $g(r,r') = B'(r')/r^{\ell+1}$. Similarly, if we look at very small r near $r=0$, we conclude that for $r < r'$ we have $g(r,r') = A(r')r^\ell$. From the required symmetry of g , this tells us that

$$g(r,r') = C [r <^\ell] [1 / r >^{\ell+1}] = 4\pi/(2\ell+1) r <^\ell / r >^{\ell+1}$$

We determine the constant according to page 80 and we find that $C = 4\pi/(2\ell+1)$ as shown above. The solution of our posed problem is then

$$F(r) = - 1/(2\ell+1) * 1/r^{\ell+1} * \int_0^a s(r') r'^{\ell+2} dr'$$

Now in our particular problem we have $s(r) = 4\pi/c \ell(\ell+1) J_2(r)$ and $F = rB_r$ and this gives (44) !!

So the point is that we did our VHS thing for this magnetic problem and we ended up with an equation (43) for B_r driven by J_2 and we have solved it in (44) for B_r . Comparing (44) to (36) gives

$$\mu_{em} = -\ell/c \int_0^a J_{2em}(r') r'^{\ell+2} dr'$$

but of course we are still in ℓm partial waves here. We can use (11) to replace $J_{2em}(r')$ with the projection integral, and this then gives (45) which can then be rewritten easily as (46).

This result (46) claims to tell us the magnetic moment of a current distribution. In Jackson this appears in (16.95), but the forms don't exactly match. Here is how we fix things up. Consider:

$$\nabla \cdot [r^\ell Y_{\ell m} \mathbf{r} \times \mathbf{J}] = \mathbf{r} \times \mathbf{J} \cdot \nabla [r^\ell Y_{\ell m}] + r^\ell Y_{\ell m} \nabla \cdot (\mathbf{r} \times \mathbf{J}) = 1 + 2$$

$$1 = \mathbf{r} \times \mathbf{J} \cdot \nabla (r^\ell) Y_{\ell m} + \mathbf{r} \times \mathbf{J} \cdot r^\ell \nabla Y_{\ell m} = 0 + r^\ell \nabla Y_{\ell m} \cdot \mathbf{r} \times \mathbf{J}$$

So we get

$$\nabla \cdot [r^\ell Y_{\ell m} \mathbf{r} \times \mathbf{J}] = r^\ell Y_{\ell m} \nabla \cdot (\mathbf{r} \times \mathbf{J}) + r^\ell \nabla Y_{\ell m} \cdot \mathbf{r} \times \mathbf{J}$$

In (16.96 a) the divergence integral will vanish, so we get $0 = \text{Jackson} + \text{Carleton}$. The only disagreement is a - sign (both have minus signs). It is a little hazy how Jackson defined his moments, perhaps this minus arises because a_M has a minus sign in 16.47.

Ignoring this minor detail, Carleton's exercise here was to consider magnetostatics, use a potential to define the moments, and then solve for the B field using the VSH and do a comparison to end up with a classic result for the moments.

8. Maxwell's Equations with VSH.

We list off the four fully coupled standard Maxwell equations with sources, and then we write down all 8 VSH equations. These follow trivially from my div and curl results above, all are correct. I then did exactly what Carleton said, and got the two driven wave equations shown in (59) and (60). But we hold now to do another exercise section.

9. Fields in conducting cavity.

(1) Duplicate stuff from above, but set to $\mathbf{B}$ and replace ∇ with $\hat{\mathbf{r}}$ since we want $\hat{\mathbf{r}} \cdot \mathbf{B} = 0$

$$\mathbf{B} = \sum_{\ell m} [B_2 \mathbf{X} + B_r \mathbf{Y} + B_1 \mathbf{Z}]$$

$$\hat{\mathbf{r}} \cdot \mathbf{B} = \sum_{\ell m} [\hat{\mathbf{r}} \cdot (B_2 \mathbf{X}) + \hat{\mathbf{r}} \cdot (B_r \mathbf{Y}) + \hat{\mathbf{r}} \cdot (B_1 \mathbf{Z})] = \sum_{\ell m} B_r Y_{\ell m} \\ \Rightarrow B_r = 0$$

using

$$\nabla \cdot \mathbf{X}_{\ell m} = 0$$

$$\hat{\mathbf{r}} \cdot \mathbf{X}_{\ell m} = 0$$

$$\nabla \cdot \mathbf{Y}_{\ell m} = (2/r) Y_{\ell m}$$

$$\hat{\mathbf{r}} \cdot \mathbf{Y}_{\ell m} = Y_{\ell m}$$

$$\nabla \cdot \mathbf{Z}_{\ell m} = -\ell(\ell+1)(1/r) Y_{\ell m}$$

$$\hat{\mathbf{r}} \cdot \mathbf{Z}_{\ell m} = 0$$

(2) Duplicate stuff from above, but set to $\mathbf{E}$ and replace ∇ with $\hat{\mathbf{r}}$ since we want $\hat{\mathbf{r}} \times \mathbf{E} = 0$

$$\mathbf{E} = \sum_{\ell m} [E_2 \mathbf{X} + E_r \mathbf{Y} + E_1 \mathbf{Z}]$$

$$\hat{\mathbf{r}} \times \mathbf{E} = \sum_{\ell m} [\hat{\mathbf{r}} \times (E_2 \mathbf{X}) + \hat{\mathbf{r}} \times (E_r \mathbf{Y}) + \hat{\mathbf{r}} \times (E_1 \mathbf{Z})] = \sum_{\ell m} [-E_2 \mathbf{Z}_{\ell m} + E_1 \mathbf{X}_{\ell m}] \\ \Rightarrow E_2 = E_1 = 0$$

using

$$\nabla \times \mathbf{X}_{\ell m} = -[\ell(\ell+1)/r] \mathbf{Y}_{\ell m} - (1/r) \mathbf{Z}_{\ell m}$$

$$\hat{\mathbf{r}} \times \mathbf{X}_{\ell m} = -\mathbf{Z}_{\ell m}$$

$$\nabla \times \mathbf{Y}_{\ell m} = -(1/r) \mathbf{X}_{\ell m}$$

$$\hat{\mathbf{r}} \times \mathbf{Y}_{\ell m} = 0$$

$$\nabla \times \mathbf{Z}_{\ell m} = (1/r) \mathbf{X}_{\ell m}$$

$$\hat{\mathbf{r}} \times \mathbf{Z}_{\ell m} = \mathbf{X}_{\ell m}$$

So, the boundary conditions for either a spherical conducting cavity or a spherical conducting sphere are very simple:

$$\hat{\mathbf{r}} \cdot \mathbf{B} = 0 \Rightarrow B_r(a) = 0$$

$$\hat{\mathbf{r}} \times \mathbf{E} = 0 \Rightarrow E_2(a) = E_1(a) = 0$$

where these fields are evaluated at the surface of the sphere, $r = a$. Certainly a simple result!

Now if there are no sources, we are just looking for stand-alone solutions that can exist, they refer to these as resonances. They will of course have surface currents and charges, but these are not the "free" sources implied by the equations.

Now we are going to "look for solutions" of the 8 equations (51)-(58) subject to the boundary conditions (61), everything is sourceless. Recall that E_r and B_r refer to RADIAL components of these fields. For each field, we know that the other two components are TRANSVERSE. So NOW finally we are seeing the meaning of the indices 1,2 and r -- better late than never. 1 and 2 are the two transverse directions that we associate with the functions Z and X .

Our equations divide into the α group and the β group as shown.

Suppose in the α group we look for solutions with $E_r = 0$. These would be "transverse electric" or TE, since there is no radial E in a solution of this sort. In this case, looking at the α group, we get $E_1 = 0$ from (51) and then $B_2 = 0$ from (55). Then equation (56) and (57) are both $0 = 0$ and add nothing new. The surviving fields are E_2 and B_1 and B_r which are all in the β group. So the TE solution means that all three fields in the α group are null everywhere.

TE = magnetic:	$\alpha: E_r = E_1 = B_2 = 0$	$\beta: B_r, B_1, E_2$ all active
TM = electric:	$\beta: B_r = B_1 = E_2 = 0$	$\alpha: E_r, E_1, B_2$ all active

For TE, to find the non-zero β fields, we see that E_2 must solve (60), and for the cavity case, it has to be a $j_k(kr)$. Notice that of the three conditions (61), the middle one is in the α group, the other two are in the β . So for our TM solution in the cavity which involves β fields only, the condition $E_2(a) = 0$ implies $j_k(ka) = 0$ which gives the k mode eigenvalues from the zeros of this function. But we also have that $B_r(a) = 0$ to think about, but equation (53) shows that this is satisfied at once if $E_2(a) = 0$.

Suppose we look for TM solutions with the whole α group being zero. Again, just assuming $B_r = 0$ forces $E_2 = 0$ from (53) and then $B_1 = 0$ from 54. In this case, we take in the α group that B_2 is a $j_k(kr)$ for the same reason, finite at the origin. But now the BC is that $E_1(a) = 0$ and from (57) this puts the condition that $\partial_r (r B_2(a)) = 0 = \partial_r (r j_k(kr))|^{r=a}$ as claimed in (65).

So from this little exercise, we see the power of the entire formalism here. We have the two decoupled equation sets which we can think of completely independently, this is why we have the famous TE and TM "modes" in all these problems, I never quite understood that before. In Jackson page 543, he produced two 3-equation sets but then he had to do some arm-waving on the next page.

Solving the conducting cavity problem was easy! The boundary conditions were trivial, we thought about each of the two modes, and for each mode were able to write down "by inspection" the solution for the primary active field, either E_2 or B_2 . These are the X or $L\psi$ components, really $E_{2tm}(r)$ keep in mind. These are the primary E and M fields in Jackson's 16.47 grand finale. Now in either case, once you have the primary field, the only other existing fields are of the other type. For example, in TE you find that E_2 is your Bessel function, and then the only other existing fields are B_1 and B_r . These are trivial to write down using (53) and (54). And similarly for the other case. So again, once you go into partial waves, this spherically symmetric problem is trivial to solve, and you do get eigenvalues for k . This is a very excellent problem to consider.

10. Radiation from a Localized Source

So now we put the sources back in and we try to solve (59), (60) using the Green's Function method. From Jackson we know that the Green's is (68), and this gives solution (69). You can always verify a result like this by applying $\nabla \times$ to both sides which I just did. I modified (69) with an upper endpoint on the integral, but now see not necessary.

Now you have to recognize the J-function there as being the 2-coefficient (the X coefficient) in the expansion you could get for $\nabla \times J$. You could then use the inversion formula for this coefficient as I have shown after equation (69). Inserting this inversion into (69) then gives the result (70) with (72). And (71) is trivial from (50).

Stop! We have now derived the standard Jackson formula for an electric or TM partial wave multipole field! We are reminded that it is called electric since ρ appears in the coefficient.

I just spent about 4 hours deriving Jackson's identity (16.90). It took 5 pages, and I made about 5 mistakes along the way, each of which had to be identified and corrected. Very painful, retain in notes. So I am now happy with Carleton's (72) and (73). You have to realize that both Carleton's X and his a_E are scaled relative to Jackson.

Now look back at (59) and (60). We just used the Green's method to solve (59) for the TM modes and our result was (70)-(73). Solving (60) instead for the TE mode is simpler since the J structure is more straightforward. In analogy with (69) we get the square current bracket replaced with $[-ikr J^2]$ (which cancels the first $1/r'$ factor), and of course B^2 is replaced with E^2 . Therefore, our result will be (72) with curl J replaced by $-ikJ$, and this gives precisely (76).

I just read the trailing part of the handout on dipoles and quadrupole and long λ limits and Cartesian moments and all that, it is fine, but this is not what interests me in this handout, so no notes.

Done! Now I would like to write a "summary" of what all happened here! Forest for trees. Amazingly it took me 18 pages of notes to digest this little handout which is only 12 pages!

Appendix 1: Horrendous operations on the Basis Functions

$$\mathbf{r} \cdot \mathbf{X}_{\ell m}^1 \sim \hat{\mathbf{r}} \cdot \mathbf{X}_{\ell m} \sim \hat{\mathbf{r}} \cdot \mathbf{L} Y_{\ell m} \sim \mathbf{r} \cdot (\mathbf{r} \times \nabla) Y_{\ell m} = 0$$

$$\mathbf{r} \cdot \mathbf{X}_{\ell m}^2 = \mathbf{r} \cdot \mathbf{Y}_{\ell m} = \mathbf{r} \cdot \hat{\mathbf{r}} Y_{\ell m} = r Y_{\ell m}$$

$$\mathbf{r} \cdot \mathbf{X}_{\ell m}^3 = \sim \mathbf{r} \cdot \mathbf{Z}_{\ell m} = \sim \mathbf{r} \cdot \nabla Y_{\ell m} \sim r \nabla_r Y_{\ell m} = 0 \text{ because } Y_{\ell m} \text{ is not a function of } r$$

$$\begin{aligned} \mathbf{r} \times \mathbf{X}_{\ell m}^1 &= k_\ell \mathbf{r} \times \mathbf{X}_{\ell m} = k_\ell \mathbf{r} \times \mathbf{r} \times \nabla Y_{\ell m} = k_\ell [(\mathbf{r} \cdot \nabla Y_{\ell m}) \mathbf{r} - r^2 \nabla Y_{\ell m}] = -k_\ell r^2 \nabla Y_{\ell m} \\ &= -k_\ell r \mathbf{Z}_{\ell m} = -r \mathbf{X}_{\ell m}^3 \end{aligned}$$

$$\mathbf{r} \times \mathbf{X}_{\ell m}^2 = \mathbf{r} \times \mathbf{Y}_{\ell m} = \mathbf{r} \times \hat{\mathbf{r}} Y_{\ell m} = 0$$

$$\mathbf{r} \times \mathbf{X}_{\ell m}^3 = k_\ell \mathbf{r} \times \mathbf{Z}_{\ell m} = k_\ell \mathbf{r} \times (r \nabla Y_{\ell m}) = k_\ell r \mathbf{r} \times \nabla Y_{\ell m} = k_\ell r \mathbf{X}_{\ell m} = r \mathbf{X}_{\ell m}^1$$

$$\begin{aligned}
\nabla \bullet \mathbf{X}_{\ell m}^1 &= k_\ell \nabla \bullet \mathbf{X}_{\ell m} = k_\ell \nabla \bullet (\mathbf{r} \times \nabla Y_{\ell m}) = k_\ell [\nabla Y_{\ell m} \bullet (\nabla \times \mathbf{r}) - \mathbf{r} \bullet (\nabla \times \nabla Y_{\ell m})] = 0 \\
\nabla \bullet \mathbf{X}_{\ell m}^2 &= \nabla \bullet \mathbf{Y}_{\ell m} = \nabla \bullet (\hat{\mathbf{r}} Y_{\ell m}) = [\nabla Y_{\ell m} \bullet \hat{\mathbf{r}} + Y_{\ell m} (\nabla \bullet \hat{\mathbf{r}})] = Y_{\ell m} (2/r) = (2/r) Y_{\ell m} \\
\nabla \bullet \mathbf{X}_{\ell m}^3 &= k_\ell \nabla \bullet \mathbf{Z}_{\ell m} = k_\ell \nabla \bullet (\mathbf{r} \nabla Y_{\ell m}) = k_\ell [\nabla \mathbf{r} \bullet \nabla Y_{\ell m} + \mathbf{r} \bullet (\nabla \times \nabla Y_{\ell m})] = \mathbf{r} \bullet \nabla^2 Y_{\ell m} \\
&= -k_\ell \ell(\ell+1) (1/r) Y_{\ell m} = -(1/r k_\ell) Y_{\ell m}
\end{aligned}$$

$$\nabla \times \mathbf{X}_{\ell m}^1 = k_\ell \nabla \times \mathbf{X}_{\ell m} = k_\ell \nabla \times (\mathbf{r} \times \nabla Y_{\ell m}) = ? \quad // \text{ This is a harder one to do, to wit: } f = f(\theta, \phi)$$

$$\nabla \times (\mathbf{r} \times \nabla f) = \mathbf{r} (\nabla \bullet \nabla f) - \nabla f (\nabla \bullet \mathbf{r}) + (\nabla f \bullet \nabla) \mathbf{r} - (\mathbf{r} \bullet \nabla) \nabla f = 1 + 2 + 3 + 4$$

$$1 = \mathbf{r} \nabla^2 f \text{ any } f$$

$$2 = -3 \nabla f \text{ any } f$$

$$3 = \partial_i f \partial_i r_j = \partial_i f \delta_{ij} = \partial_j f = \nabla f \text{ any } f$$

$$\begin{aligned}
4 &= -r_i \partial_i \partial_j f = -r_i \partial_j \partial_i f = -\partial_j (r_i \partial_i f) + (\partial_j r_i) (\partial_i f) \\
&= -\partial_j (\mathbf{r} \bullet \nabla f) + \partial_j f = 0 + \nabla f = \nabla f \quad \text{true only for } f = f(\theta, \phi)
\end{aligned}$$

$$\text{So } \nabla \times (\mathbf{r} \times \nabla f) = \mathbf{r} \nabla^2 f - \nabla f \quad \text{and for } f = Y_{\ell m} \text{ get } = -[\ell(\ell+1)/r] Y_{\ell m} \hat{\mathbf{r}} - \nabla Y_{\ell m} \text{ so repeat:}$$

$$\begin{aligned}
\nabla \times \mathbf{X}_{\ell m}^1 &= k_\ell \nabla \times \mathbf{X}_{\ell m} = k_\ell \nabla \times (\mathbf{r} \times \nabla Y_{\ell m}) = k_\ell \{ -[\ell(\ell+1)/r] Y_{\ell m} \hat{\mathbf{r}} - \nabla Y_{\ell m} \} \\
&= k_\ell \{ -[\ell(\ell+1)/r] Y_{\ell m} - (1/r) \mathbf{Z}_{\ell m} \} \\
&= -[k_\ell \ell(\ell+1)/r] \mathbf{X}_{\ell m}^2 - (1/r) \mathbf{X}_{\ell m}^3
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{X}_{\ell m}^2 &= \nabla \times \mathbf{Y}_{\ell m} = \nabla \times (\hat{\mathbf{r}} Y_{\ell m}) = \nabla Y_{\ell m} \times \hat{\mathbf{r}} + Y_{\ell m} (\nabla \times \hat{\mathbf{r}}) = -(1/r) \mathbf{r} \times \nabla Y_{\ell m} \\
&= -(1/r) \mathbf{X}_{\ell m} = (-1/k_\ell r) \mathbf{X}_{\ell m}^1
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{X}_{\ell m}^3 &= \nabla \times k_\ell \mathbf{Z}_{\ell m} = k_\ell \nabla \times (\mathbf{r} \nabla Y_{\ell m}) = k_\ell [\nabla \mathbf{r} \times \nabla Y_{\ell m} + \mathbf{r} (\nabla \times \nabla Y_{\ell m})] = \\
&k_\ell [\hat{\mathbf{r}} \times \nabla Y_{\ell m} + 0] = k_\ell (1/r) \mathbf{r} \times \nabla Y_{\ell m} = k_\ell (1/r) \mathbf{X}_{\ell m} = (1/r) \mathbf{X}_{\ell m}^1
\end{aligned}$$

Now lets engage the curl of curl battle!

$$\begin{aligned}
\nabla \times \nabla \times \mathbf{X}_{\ell m}^1 &= \nabla \times (-[k_\ell \ell(\ell+1)/r] \mathbf{X}_{\ell m}^2 - (1/r) \mathbf{X}_{\ell m}^3) \\
&= -k_\ell \ell(\ell+1) \nabla \times [(1/r) \mathbf{X}_{\ell m}^2] - \nabla \times [(1/r) \mathbf{X}_{\ell m}^3] = -k_\ell \ell(\ell+1) [1] - [2]
\end{aligned}$$

$$\begin{aligned}
[1] &= \nabla \times [(1/r) \mathbf{X}_{\ell m}^2] = \nabla (1/r) \times \mathbf{X}_{\ell m}^2 + (1/r) \nabla \times \mathbf{X}_{\ell m}^2 \\
&= (-1/r^2) \{ \hat{\mathbf{r}} \times \mathbf{X}_{\ell m}^2 \} + (1/r) \{ \nabla \times \mathbf{X}_{\ell m}^2 \} \\
&= (-1/r^2) \{ \mathbf{0} \} + (1/r) \{ -(1/k_\ell r) \mathbf{X}_{\ell m}^1 \} \\
&= (-1/k_\ell r^2) \mathbf{X}_{\ell m}^1
\end{aligned}$$

$$\begin{aligned}
[2] &= \nabla \times [(1/r) \mathbf{X}_{\ell m}^3] = \nabla (1/r) \times \mathbf{X}_{\ell m}^3 + (1/r) \nabla \times \mathbf{X}_{\ell m}^3 \\
&= (-1/r^2) \{ \hat{\mathbf{r}} \times \mathbf{X}_{\ell m}^3 \} + (1/r) \{ \nabla \times \mathbf{X}_{\ell m}^3 \} \\
&= (-1/r^2) \{ \mathbf{X}_{\ell m}^1 \} + (1/r) \{ (1/r) \mathbf{X}_{\ell m}^1 \} \\
&= 0
\end{aligned}$$

$$\Rightarrow \nabla \times \nabla \times \mathbf{X}_{\ell m}^1 = -k_\ell \ell(\ell+1) [1] - [2] = -k_\ell \ell(\ell+1) (-1/k_\ell r^2) \mathbf{X}_{\ell m}^1 = \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^1$$

$$\Rightarrow \nabla \times \nabla \times \mathbf{X}_{\ell m}^1 = \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^1$$

$$\begin{aligned} \nabla \times \{ \nabla \times \mathbf{X}_{\ell m}^2 \} &= \nabla \times \{ -(1/k_\ell r) \mathbf{X}_{\ell m}^1 \} = (-1/k_\ell) \nabla \times \{ (1/r) \mathbf{X}_{\ell m}^1 \} \\ &= (-1/k_\ell) [(-1/r^2) \{ \hat{\mathbf{r}} \times \mathbf{X}_{\ell m}^1 \} + (1/r) \{ \nabla \times \mathbf{X}_{\ell m}^1 \}] \\ &= (-1/k_\ell) [(-1/r^2) \{ -\mathbf{X}_{\ell m}^3 \} + (1/r) \{ -[k_\ell \ell(\ell+1)/r] \mathbf{X}_{\ell m}^2 - (1/r) \mathbf{X}_{\ell m}^3 \}] \\ &= (-1/k_\ell) [(1/r) \{ -[k_\ell \ell(\ell+1)/r] \mathbf{X}_{\ell m}^2 \}] \\ &= \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \quad // \text{ interesting!} \end{aligned}$$

$$\begin{aligned} \nabla \times \{ \nabla \times \mathbf{X}_{\ell m}^3 \} &= \nabla \times \{ (1/r) \mathbf{X}_{\ell m}^1 \} = \\ &= [(-1/r^2) \{ \hat{\mathbf{r}} \times \mathbf{X}_{\ell m}^1 \} + (1/r) \{ \nabla \times \mathbf{X}_{\ell m}^1 \}] \\ &= [(1/r) \{ -[k_\ell \ell(\ell+1)/r] \mathbf{X}_{\ell m}^2 \}] \quad \text{from previous calculation} \\ &= -k_\ell \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \quad // \text{ breaking the rule set by the previous two!} \end{aligned}$$

Summarize these double-curl results:

$$\begin{aligned} \nabla \times \{ \nabla \times \mathbf{X}_{\ell m}^1 \} &= \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^1 \\ \nabla \times \{ \nabla \times \mathbf{X}_{\ell m}^2 \} &= \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \\ \nabla \times \{ \nabla \times \mathbf{X}_{\ell m}^3 \} &= -k_\ell \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \end{aligned}$$

Now let's try some Laplacians:

$$\begin{aligned} \nabla^2 \mathbf{X}_{\ell m}^1 &= \nabla \{ \nabla \cdot \mathbf{X}_{\ell m}^1 \} - \{ \nabla \times \nabla \times \mathbf{X}_{\ell m}^1 \} \\ &= \nabla \{ 0 \} - \{ \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^1 \} \\ &= -\ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^1 \end{aligned}$$

$$\begin{aligned} \nabla^2 \mathbf{X}_{\ell m}^2 &= \nabla \{ \nabla \cdot \mathbf{X}_{\ell m}^2 \} - \{ \nabla \times \nabla \times \mathbf{X}_{\ell m}^2 \} \\ &= \nabla \{ (2/r) Y_{\ell m} \} - \{ \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \} \\ &= 2 \{ \nabla [(1/r) Y_{\ell m}] \} - \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \\ &= 2 \{ (-1/r^2) \hat{\mathbf{r}} Y_{\ell m} + (1/r) \nabla Y_{\ell m} \} - \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \\ &= 2 \{ (-1/r^2) \mathbf{X}_{\ell m}^2 + (1/r^2) \mathbf{X}_{\ell m}^3 / k_\ell \} - \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \\ &= (2/k_\ell r^2) \mathbf{X}_{\ell m}^3 - [\ell(\ell+1) + 2] / r^2 \mathbf{X}_{\ell m}^2 \quad // \text{ not very nice} \end{aligned}$$

$$\begin{aligned} \nabla^2 \mathbf{X}_{\ell m}^3 &= \nabla \{ \nabla \cdot \mathbf{X}_{\ell m}^3 \} - \{ \nabla \times \nabla \times \mathbf{X}_{\ell m}^3 \} \\ &= \nabla \{ -k_\ell \ell(\ell+1) (1/r) Y_{\ell m} \} - \{ -k_\ell \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \} \\ &= -k_\ell \ell(\ell+1) \{ \nabla [(1/r) Y_{\ell m}] \} + k_\ell \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \\ &= -k_\ell \ell(\ell+1) \{ (-1/r^2) \hat{\mathbf{r}} Y_{\ell m} + (1/r) \nabla Y_{\ell m} \} + k_\ell \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \\ &= -k_\ell \ell(\ell+1) \{ (-1/r^2) \mathbf{X}_{\ell m}^2 + (1/r^2) \mathbf{X}_{\ell m}^3 / k_\ell \} + k_\ell \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 \\ &= 2 k_\ell \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^2 - \ell(\ell+1)/r^2 \mathbf{X}_{\ell m}^3 \end{aligned}$$

I am getting tired of all this stuff. Wondering if Maple can do it? No, but I found a package that I can add to my Maple, looks interesting... Well, the problem is related to "inert" versus "do it". I don't really want the thing to be calculated, I want it to remain in symbolic form, like ∇Y . Back to hand work!

The Laplacians look suspicious, I bet they should all come out the same, but there is so much algebra here I have no real way to check things, and early errors propagate! At least I have a written derivation record so I should be able to repair things at some point.

Let's summarize some of these result in a table:

$$X_{\ell m}^1 = k_{\ell} X_{\ell m} = k_{\ell} r \times \nabla Y_{\ell m} = i k_{\ell} L Y_{\ell m}$$

$$X_{\ell m}^2 = Y_{\ell m} = \hat{r} \cdot Y_{\ell m}$$

$$X_{\ell m}^3 = k_{\ell} Z_{\ell m} = k_{\ell} r \cdot \nabla Y_{\ell m}$$

$$X_{\ell m} = r \times \nabla Y_{\ell m}$$

$$Y_{\ell m} = \hat{r} \cdot Y_{\ell m}$$

$$Z_{\ell m} = r \cdot \nabla Y_{\ell m}$$

$$\hat{r} \cdot X_{\ell m}^1 = 0$$

$$\hat{r} \cdot X_{\ell m}^2 = Y_{\ell m}$$

$$\hat{r} \cdot X_{\ell m}^3 = 0$$

$$\hat{r} \cdot X_{\ell m} = 0$$

$$\hat{r} \cdot Y_{\ell m} = Y_{\ell m}$$

$$\hat{r} \cdot Z_{\ell m} = 0$$

$$\hat{r} \times X_{\ell m}^1 = -X_{\ell m}^3$$

$$\hat{r} \times X_{\ell m}^2 = 0$$

$$\hat{r} \times X_{\ell m}^3 = X_{\ell m}^1$$

$$\hat{r} \times X_{\ell m} = -Z_{\ell m}$$

$$\hat{r} \times Y_{\ell m} = 0$$

$$\hat{r} \times Z_{\ell m} = X_{\ell m}$$

$$\hat{r} \times \hat{r} \times X_{\ell m}^1 = -\hat{r} \times X_{\ell m}^3 = -X_{\ell m}^1$$

$$\hat{r} \times \hat{r} \times X_{\ell m}^2 = 0$$

$$\hat{r} \times \hat{r} \times X_{\ell m}^3 = \hat{r} \times X_{\ell m}^1 = -X_{\ell m}^3$$

$$\hat{r} \times \hat{r} \times X_{\ell m} = -X_{\ell m}$$

$$\hat{r} \times \hat{r} \times Y_{\ell m} = 0$$

$$\hat{r} \times \hat{r} \times Z_{\ell m} = -Z_{\ell m}$$

$$\nabla \cdot X_{\ell m}^1 = 0$$

$$\nabla \cdot X_{\ell m}^2 = (2/r) Y_{\ell m}$$

$$\nabla \cdot X_{\ell m}^3 = -k_{\ell} \ell(\ell+1) (1/r) Y_{\ell m}$$

$$\nabla \cdot X_{\ell m} = 0$$

$$\nabla \cdot Y_{\ell m} = (2/r) Y_{\ell m}$$

$$\nabla \cdot Z_{\ell m} = -\ell(\ell+1) (1/r) Y_{\ell m}$$

$$\nabla \times X_{\ell m}^1 = -[k_{\ell} \ell(\ell+1)/r] X_{\ell m}^2 - (1/r) X_{\ell m}^3$$

$$\nabla \times X_{\ell m}^2 = - (1/k_{\ell} r) X_{\ell m}^1$$

$$\nabla \times X_{\ell m}^3 = (1/r) X_{\ell m}^1$$

$$\nabla \times X_{\ell m} = -[\ell(\ell+1)/r] Y_{\ell m} - (1/r) Z_{\ell m}$$

$$\nabla \times Y_{\ell m} = - (1/r) X_{\ell m}$$

$$\nabla \times Z_{\ell m} = (1/r) X_{\ell m}$$

$$\nabla \cdot (r \times \nabla f) = 0 \text{ for any } f$$

The Basis Functions and Expansion Theorem

Define these three vector functions on the surface of a sphere θ, ϕ which has radius r :

$$\begin{array}{lll} \mathbf{X}_{\ell m} = \mathbf{r} \times \nabla Y_{\ell m} & 2 & = \mathbf{X}_{\ell m}^1 \sqrt{\ell(\ell+1)} \\ \mathbf{Y}_{\ell m} = \hat{\mathbf{r}} Y_{\ell m} & r & = \mathbf{X}_{\ell m}^2 \\ \mathbf{Z}_{\ell m} = r \nabla Y_{\ell m} & 1 & = \mathbf{X}_{\ell m}^3 \sqrt{\ell(\ell+1)} \end{array}$$

where $Y_{\ell m}$ is the usual spherical harmonic as in Jackson. These vectors are mutually perpendicular, with the $\mathbf{Y}$ function being radial and $\mathbf{Z}$ and $\mathbf{X}$ being transverse to the sphere. In an E&M problem, one is familiar with the two transverse directions, and 1 and 2 are exactly those directions.

Within the ℓm manifold (partial wave) of the multipole expansion of a vector function $\mathbf{A}(\theta, \phi)$, the three vectors above form a true set of basis functions, and we end up with the following expansion theorem including the projection formula

$$\begin{array}{ll} \mathbf{A}(\theta, \phi) = \sum_{\ell m n} A_{\ell m n} \mathbf{X}_{\ell m}^n(\theta, \phi) & A_{\ell m n} = \int d\Omega \mathbf{X}_{\ell m}^{n*}(\theta, \phi) \cdot \mathbf{A}(\theta, \phi) \\ \text{expansion} & \text{projection} \end{array}$$

These imply the following orthogonality and completeness relations:

$$\begin{array}{ll} \int d\Omega \mathbf{X}_{\ell' m'}^{n'}(\theta, \phi)^* \cdot \mathbf{X}_{\ell m}^n(\theta, \phi) = \delta_{\ell', \ell} \delta_{m', m} \delta_{n', n} & \text{orthogonality} \\ \sum_{\ell m n} [\mathbf{X}_{\ell m}^n(\theta, \phi)]_i [\mathbf{X}_{\ell' m'}^{n'}(\theta', \phi')]_j = \delta_{i, j} \delta(\Omega - \Omega') & \text{completeness} \end{array}$$

In the detailed notes, we prove orthogonality, and the fact that the three vectors are perpendicular to each other, but we have to simply accept as being awfully reasonable the fact that the basis functions are complete, so the expansion is valid for any reasonable vector function. I could not find a trivial proof of this fact, but there probably is one.

The above result is then trivially extended to vector functions of r, θ, ϕ by adding r as a ride-along parameter, and we then get,

$$\begin{array}{ll} \mathbf{A}(r, \theta, \phi) = \sum_{\ell m n} A_{\ell m n}(r) \mathbf{X}_{\ell m}^n(\theta, \phi) & A_{\ell m n}(r) = \int d\Omega \mathbf{X}_{\ell m}^{n*}(\theta, \phi) \cdot \mathbf{A}(r, \theta, \phi) \end{array}$$

In our applications we will often think of $\mathbf{A}$ as being $\mathbf{E}$, $\mathbf{B}$ or $\mathbf{J}$. The above is the generalization of the partial wave expansion of a *scalar* function, which we know works like this:

$$A(r, \theta, \phi) = \sum_{\ell m} A_{\ell m}(r) Y_{\ell m}(\theta, \phi)$$

$$A_{\ell m}(r) = \int d\Omega Y_{\ell m}^*(\theta, \phi) A(r, \theta, \phi)$$

$$\int d\Omega Y_{\ell' m'}^*(\theta, \phi) Y_{\ell m}(\theta, \phi) = \delta_{\ell', \ell} \delta_{m', m}$$

orthogonality

$$\sum_{\ell m} Y_{\ell m}^*(\theta, \phi) Y_{\ell m}(\theta', \phi') = \delta(\Omega - \Omega')$$

completeness

We are familiar with the use of this multipole expansion in electrostatic potential theory for the potential ϕ .

Some Useful Properties of the Basis Functions

$$\nabla \cdot \mathbf{X}_{\ell m} = 0$$

$$\hat{\mathbf{r}} \cdot \mathbf{X}_{\ell m} = 0$$

$$\nabla \cdot \mathbf{Y}_{\ell m} = (2/r) Y_{\ell m}$$

$$\hat{\mathbf{r}} \cdot \mathbf{Y}_{\ell m} = Y_{\ell m}$$

$$\nabla \cdot \mathbf{Z}_{\ell m} = -\ell(\ell+1) (1/r) Y_{\ell m}$$

$$\hat{\mathbf{r}} \cdot \mathbf{Z}_{\ell m} = 0$$

$$\nabla \times \mathbf{X}_{\ell m} = -[\ell(\ell+1)/r] \mathbf{Y}_{\ell m} - (1/r) \mathbf{Z}_{\ell m}$$

$$\hat{\mathbf{r}} \times \mathbf{X}_{\ell m} = -\mathbf{Z}_{\ell m}$$

$$\nabla \times \mathbf{Y}_{\ell m} = - (1/r) \mathbf{X}_{\ell m}$$

$$\hat{\mathbf{r}} \times \mathbf{Y}_{\ell m} = 0$$

$$\nabla \times \mathbf{Z}_{\ell m} = (1/r) \mathbf{X}_{\ell m}$$

$$\hat{\mathbf{r}} \times \mathbf{Z}_{\ell m} = \mathbf{X}_{\ell m}$$

It turns out that all operations you can think of to apply to the basis functions, or to scalar functions of r times the basis functions, always keep you within the ℓm manifold. The reason for this fact is summarized by the following exhaustion of all possible dimensionless vectors you can make out of the building blocks $\mathbf{r}$ and ∇ ,

$$\hat{\mathbf{r}} Y_{\ell m}$$

$$\mathbf{Y}$$

$$r \nabla Y_{\ell m}$$

$$\mathbf{Z}$$

$$\mathbf{r} \times \nabla Y_{\ell m}$$

$$\mathbf{X}$$

$$\hat{\mathbf{r}} (\mathbf{r} \cdot \nabla Y_{\ell m})$$

$$0$$

$$r^2 \nabla \times \nabla Y_{\ell m}$$

$$0$$

$$\hat{\mathbf{r}} r^2 \nabla \cdot \nabla Y_{\ell m}$$

$$\hat{\mathbf{r}} r^2 \nabla^2 Y_{\ell m} = -\hat{\mathbf{r}} L^2 Y_{\ell m} = -\ell(\ell+1) \hat{\mathbf{r}} Y_{\ell m} = -\ell(\ell+1) \mathbf{Y}$$

The last item is the only but crucial place where the properties of the $Y_{\ell m}$ are required, and no other functions will do! The $Y_{\ell m}$ work because they are coordinate-space representations of angular momentum states $|\ell m\rangle$ which diagonalize L^2 and L_3 , which is possible because the Hamiltonian is a rotational scalar.

From now on, we will write the expansion formula using Carleton's X,Y,Z form, to wit,

$$\mathbf{A} = \sum_{\ell m} [A_2 \mathbf{X} + A_\tau \mathbf{Y} + A_1 \mathbf{Z}]$$

in a shorthand notation where the three coefficients really have ℓm labels and are functions of r , and of course the basis vectors also have these labels. Note that 1 and 2 indicate the transverse vector coefficient functions of r , while τ is the radial vector coefficient function.

Diagonalizing Maxell's equations involves applying $\nabla \cdot$ and $\nabla \times$ applied to the above expansion for the E and B fields. Using the properties noted above, one can show these essential facts:

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \sum_{\ell m} [\nabla \cdot (A_2 \mathbf{X}) + \nabla \cdot (A_r \mathbf{Y}) + \nabla \cdot (A_1 \mathbf{Z})] \\ &= \sum_{\ell m} \{ 1/r^2 \partial_r (r^2 A_r) - A_1 \ell(\ell+1) (1/r) \} Y_{\ell m}\end{aligned}$$

The result is of course a scalar, and the A_2 coefficient does not appear in the result, indicating that no constraint is put on A_2 by a divergence condition one might find in Maxell's equations. Similarly we have

$$\begin{aligned}\nabla \times \mathbf{A} &= \sum_{\ell m} [\nabla \times (A_2 \mathbf{X}) + \nabla \times (A_r \mathbf{Y}) + \nabla \times (A_1 \mathbf{Z})] \\ &= \sum_{\ell m} \{ \underbrace{-(1/r) \partial_r (r A_2)}_1 \mathbf{Z}_{\ell m} - \underbrace{\ell(\ell+1)/r A_2}_{\tau} \mathbf{Y}_{\ell m} + \underbrace{[(1/r)(\partial_r (r A_1) - A_r/r)]}_{2} \mathbf{X}_{\ell m} \}\end{aligned}$$

This result is more complicated and all three coefficients are involved. We have shown the correct expansion labels for the resulting coefficient combinations.

At this point in the handout, Carleton does two warm-up exercises.

Electrostatics Exercise. Use $\nabla \cdot \mathbf{E} = 4\pi\rho$ and $\nabla \times \mathbf{E} = 0$ and see what happens. The result is that $E_2 = 0$ and we get two coupled first order DEs for E_2 and E_r which we can combine to get the Poisson radial equation for the quantity (rE^1) which we can interpret as the potential $-\phi$. This equation is driven by source term $4\pi\rho_{\ell m}$. We find in fact that $E^1_{\ell m} = -\phi_{\ell m}/r$ where the $\phi_{\ell m}$ are the *usual* scalar multipole coefficients used in electrostatics multipole expansion.

Magnetostatics Exercise. Use $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 4\pi/c \mathbf{J}$ and see what happens. The $\nabla \times \mathbf{B}$ equation results in three equations (39)-(41), and the $\nabla \cdot \mathbf{B}$ equation gives a fourth (42). By combining two of these, we obtain again Poisson's radial equation, but it is now for the quantity (rB^r) and the equation is driven by a J^2 term. Carleton provides in (44) the Green's Function solution for B^r . Via a separate path, he writes the conventional scalar multipole expansion for *magnetic* potential ϕ in terms of moments $\mu_{\ell m}$. Then $\mathbf{B} = -\nabla\phi$ is expanded in VSH giving equations (36)-(38). Comparing our Green's solution with (36) gives a traditional formula for the magnetic moment as an integral over the current distribution $\mathbf{J}$.

After these warm-up exercises, we go for the whole nine yards. On page 6 we write out the four Maxwell equations (with sources), and applying the above div and curl expansions, we end up with 8 little equations for the 6 VSH component functions. Now comes *the* key fact. These 8 equations can be grouped into two sets of 4 equations which I have called the α group and the β group. Each group involves only 3 of the 6 unknown functions, so we have really *two completely independent solution sets*.

TE = magnetic:	$\alpha: E_r = E_1 = B_2 = 0$	$\beta: B_r, B_1, E_2$ all active
TM = electric:	$\beta: B_r = B_1 = E_2 = 0$	$\alpha: E_r, E_1, B_2$ all active

The name TE arises because $E_r = 0$ in this solution set -- there is no radial E field. We know from Jackson that the conventional names magnetic and electric arise later on when we see what the radiation solutions look like. In general, you solve the TE and TM problems independently, and then of course you could superpose solutions together. Notice this fact: in each active set, you solve for a #2 field, and that is ALL of that kind of field. The other two fields in the set are of the OTHER type. So in TE the entire E field is simply E^2 , for example, and the other two fields are B type.

Think now about what we have done! We take the four very complicated Maxwell vector equations with nasty curls and vector components, and we reduce them in the partial wave expansion to 2 independent sets of 4 scalar equations for three field components. In each set, we can show that the #2 field component function must solve the radial wave equation driven by an appropriate current mixture, as shown in (59) and (60).

Full Maxwell Exercise: the cavity resonator. Imagine no sources and a conducting metal cavity. What E&M modes (resonances) can exist in there? The BC's are the usual $\hat{r} \cdot \mathbf{B} = 0$ and $\mathbf{r} \times \mathbf{E} = 0$ at $r = a$, which boil down to the extremely simple (61). Since no sources, solutions of the radial wave equation are spherical Bessel's like $j_\ell(kr)$. The TE and TM solutions are just multiples of $j_\ell(kr)$ as in (62,3). However, the BC's cause the solution values of k to be quantized, and these are then the cavity normal modes. The conditions for TM and TE are different, (65) and (66), so modes have different eigenfrequencies. This is a fairly complex problem to even think about, but the VSH method provides an almost instant solution! Of course the geometry is optimal for VSH work!

By the way, these same BC's apply to Jackson's scattering from a conducting sphere. In that problem, we have extra BC's in the full coordinate space of an incoming plane wave and outgoing spherical wave. I now realize that Jackson 16.139 is nothing more than the expansion of a plane wave traveling in the z direction onto the VSH basis functions. He uses circular polarizations. I recall at the time how completely obscure this seemed! You can see how Jackson prefers to use the combination $\nabla \times \mathbf{f}_\ell(r) \mathbf{X}_{\ell m}$ to incorporate the Y and Z components of the expansion, and he has even provided a little orthogonality for this combination in 16.132 which is unbelievably obtuse! Carleton has shown how this can all be understood in a systematic fashion. I could resolve this problem using the VSH from the start, project the plane wave onto the basis functions from scratch, but of course I would end up with the same 16.139 with three terms in place of Jackson's two.

Radiation from a localized source. Here we just take those two #2 component wave equations and do the Green's function solutions, as done in Jackson. The B^2 solution is shown in (69) (this is of course the particular solution, to which we can add solutions of the sourceless problem, like those cavity modes, in order to match some boundary conditions.)

Now on page 9 we come to the normal "statement" of multipole fields. For TM, we know that the B field has only the B_2 component, and that is what (70) says. Here we use the Hankel 1 function because we are implicitly matching the spherical outgoing boundary condition, since our topic is "radiation from a localized source". Jackson in 6.42 is slightly more general, not specifying the type of radial function. Away from the source, you get the E field from the B field in trivial fashion just from the usual Maxwell. By installing our specific Green's solution (69) for B_2 , we get a specific result for the radiation field in terms of newly minted coefficient a_E as shown. It is an integral over the curl of $\mathbf{J}$ in the source times other stuff as in (72), and this is the result that took me 5 pages of notes to convert into form (73) where you see that a TM mode is driven by charge ρ if it is present, hence the name "electric".

At this point, declaration of the form of the TE mode follows at once with (76), just looking at the different source in the radial equation. In this case, we don't have to do the elaborate conversion, we see that a_M is an integral of J as shown.

So fine, we have recovered Jackson's multipole expansion results in a much more organized approach, with much less mystery. I presume Jackson has held firm in his 2nd and 3rd editions.

Comment on Wave Equations. We know that components of E and B in free space solve the wave equation, let there be light! In the presence of sources ρ and J , these wave equations are not so nice. Here is a quick shot,

$$\begin{aligned} (\nabla^2 + k^2) E &= 4\pi\nabla\rho - 4\pi i k/c J & \text{and} & \quad ikB = \nabla \times E \\ (\nabla^2 + k^2) B &= -4\pi/c \nabla \times J & \text{and} & \quad ikE = -\nabla \times B + 4\pi/c J \end{aligned}$$

This is why we introduce the vector potential A .

Now in the VSH approach, we find that in each mode, the #2 component satisfies a *radial* wave equation. The B_2 solution for the TM is driven by the #2 VSH expansion coefficient of $\nabla \times J$, in analogy with the second line above, giving us result (72). However, after the transformation to (73) using continuity, we find that in fact ρ is really the main driver of the TM mode B , contrary to what appears above, where we don't even see ρ in the second line.

Similarly, the E_2 solution for TE is driven by the #2 coefficient of J , as shown in (76). This dimly reflects the first line above, but again we marvel at the absence of ρ in affecting E_2 directly. Of course in light of continuity, the two sources ρ and J are always connected and all these comments are not very productive I guess.

I don't think the vector basis functions satisfy any wave equations, though I saw someone claim this once. See other notes on the N and M functions.

These started out being notes on my little Mie Theory 4-page downloaded paper which happens to state all the coefficients in formulas that look like mine.

Review. We have seen how the functions X , Y and Z span the ℓm manifold. We have also seen how one solves an E&M problem separately in the two "modes" known as TM and TE. Each mode has its own private set of equations and operates independently. In either Jackson or Carleton's notes we have this basic idea:

$$\text{TM:} \quad \mathbf{B}_{\ell m} = a_E(\ell m) f_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi)$$

$$\mathbf{E}_{\ell m} = (i/k) \nabla \times \mathbf{B}_{\ell m}$$

$$\text{TE:} \quad \mathbf{E}_{\ell m} = a_M(\ell m) g_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi)$$

$$\mathbf{B}_{\ell m} = (-i/k) \nabla \times \mathbf{E}_{\ell m}$$

That is to say, for each ℓm , we can have the two independent solutions noted above.

The quantities $f_\ell(r)$ and $g_\ell(r)$ each represent a linear combination of spherical Bessel functions. The two coefficients of this linear combination are dependent on ℓ and must be determined from boundary conditions. Usually they are normalized so the sum of their squares is 1, but this is not necessary, since the $a_E(\ell m)$ type coefficient can pick up the slack. The asymptotic behaviors of $j_\ell(kr)$ and $h^{(1)}_\ell(kr)$ are especially useful in meeting these boundary conditions, the first being finite at $r=0$, the second having outgoing spherical wave behavior.

The normalization of the function $\mathbf{X}_{\ell m}(\theta, \phi)$ varies with author, but always has this form

$$\mathbf{X}_{\ell m}(\theta, \phi) = C(\ell, m) * \mathbf{r} \times \nabla Y_{\ell m}(\theta, \phi)$$

$$C(\ell, m) = \begin{array}{ll} +1 & \text{Carleton} \\ -1 & \text{Mie} \\ \frac{-i}{\sqrt{\ell(\ell+1)}} & \text{Jackson} \end{array}$$

When we write Z and Y , we always mean Carleton's normalization for these two basis functions. It would seem that the X vector basis function, one of the two transverse ones, is somehow more "fundamental" than the other two, since it provides the "main field" in each of the modes. The "other" field that goes with each mode is found from the $\nabla \times$ operation as shown, and we can write out:

$$\nabla \times [f_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi)] = -C(\ell, m) \{ (f'_\ell(r) + 1/r) \mathbf{Z}_{\ell m}(\theta, \phi) + \ell(\ell+1) f_\ell(r)/r \mathbf{Y}_{\ell m}(\theta, \phi) \}$$

Thus, as we well know, the "other field" in a given (TM or TE, transverse) mode has components in both the Z (other transverse) and the Y (radial) direction. We can rewrite the above vector expansion as follows:

$$\nabla \times [f_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi)] = f'_\ell(r) \hat{\mathbf{r}} \times \mathbf{X}_{\ell m}(\theta, \phi) - C(\ell, m) \ell(\ell+1) f_\ell(r)/r \hat{\mathbf{r}} \cdot \mathbf{Y}_{\ell m}(\theta, \phi)$$

where we have kept it general so that anybody's $\mathbf{X}$ can be used here. This appears as 16.143 in Jackson where his $C(\ell, m)$ is as shown above. To do this, we used the fact (from our appendix in other paper) that

$$\mathbf{Z} = -\hat{\mathbf{r}} \times \mathbf{X}_{\text{Carleton}} = -(1/C(\ell, m)) \hat{\mathbf{r}} \times \mathbf{X}$$

Now when we write the full expansion over all the modes, we get this result:

$$\mathbf{B} = \sum_{\ell m} [a_E(\ell m) f_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi) + (-i/k) \nabla \times \{ a_M(\ell m) g_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi) \}]$$

$$\mathbf{E} = \sum_{\ell m} [(i/k) \nabla \times \{ a_E(\ell m) f_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi) \} + a_M(\ell m) g_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi)]$$

and this appears as the "grand finale" 16.47 in Jackson, which I now see is not all that grand. It is just the statement of what we are saying all along in the VSH expansion. The game is to determine the two coefficients $a_E(\ell m)$ and $a_M(\ell m)$ as well as the spherical Bessel implied linear combination coefficients for each ℓm . Notice that in the form above, there is no immediate need to expand the secondary terms in either way shown above, but we know we can do it.

The Mie basis functions M and N. I call them Mie because I suspect they appeared in his 1908 paper in this form. In my little 4-page downloaded Mie Theory paper we see the following definition:

$$\mathbf{M}_{\ell m} = \nabla \times [f_\ell(r) Y_{\ell m}(\theta, \phi) \mathbf{r}] = \nabla \times [(f_\ell(r)/r) Y_{\ell m}(\theta, \phi)]$$

Warning: The $Y_{\ell m}(\theta, \phi)$ which appear in this Mie paper may have different normalization than we are used to, but we will absorb any difference into our linear combination of spherical Bessel coefficients, which are already implied when we write the generic $f_\ell(r)$

From the first form, we can quickly simplify as follows:

$$\nabla \times [f_\ell(r) Y_{\ell m}(\theta, \phi) \mathbf{r}] = \nabla [f_\ell(r) Y_{\ell m}(\theta, \phi)] \times \mathbf{r}$$

because the second term has $\nabla \times \mathbf{r}$ which is 0. In the next step, can see that the

$$\nabla [f_\ell(r) Y_{\ell m}(\theta, \phi)] = f_\ell(r) \nabla Y_{\ell m}(\theta, \phi) + f'_\ell(r) \hat{\mathbf{r}} Y_{\ell m}(\theta, \phi)$$

so the second term gives nothing since $\hat{\mathbf{r}} \times \mathbf{r} = 0$. So we end up with

$$\begin{aligned} \mathbf{M}_{\ell m}(r, \theta, \phi) &= \nabla \times [f_\ell(r) Y_{\ell m}(\theta, \phi) \mathbf{r}] = -f_\ell(r) \mathbf{r} \times \nabla Y_{\ell m}(\theta, \phi) \\ &= -f_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi) / C(\ell, m) \quad // \text{our old friend} \end{aligned}$$

Thus, we can write our exact same multipole fields as

$$\begin{aligned} \text{TM:} \quad \mathbf{B}_{\ell m} &= -a_E(\ell m) / C(\ell, m) \mathbf{M}_{\ell m}(r, \theta, \phi) \\ \mathbf{E}_{\ell m} &= (i/k) \nabla \times \mathbf{B} \end{aligned}$$

$$\begin{aligned}\text{TE:} \quad \mathbf{E}_{\ell m} &= -a_M(\ell m) / C(\ell, m) \mathbf{M}_{\ell m}(r, \theta, \phi) \\ \mathbf{B}_{\ell m} &= (-i/k) \nabla \times \mathbf{B}\end{aligned}$$

At this point, why not define

$$\mathbf{N}_{\ell m}(r, \theta, \phi) = (1/k) \nabla \times \mathbf{M}_{\ell m}(r, \theta, \phi) = (1/k) \nabla \times \nabla \times [f_\ell(r) Y_{\ell m}(\theta, \phi) \mathbf{r}]$$

Then we can be even more compact and say

$$\begin{aligned}\text{TM:} \quad \mathbf{B}_{\ell m} &= [-a_E(\ell m) / C(\ell, m)] \mathbf{M}_{\ell m E}(r, \theta, \phi) \\ \mathbf{E}_{\ell m} &= i [-a_E(\ell m) / C(\ell, m)] \mathbf{N}_{\ell m E}(r, \theta, \phi) \\ \text{TE:} \quad \mathbf{E}_{\ell m} &= [-a_M(\ell m) / C(\ell, m)] \mathbf{M}_{\ell m M}(r, \theta, \phi) \\ \mathbf{B}_{\ell m} &= -i [-a_M(\ell m) / C(\ell, m)] \mathbf{N}_{\ell m M}(r, \theta, \phi)\end{aligned}$$

Here I have added E (electric) and M (magnetic) labels on the M and N functions to make us remember that each one can have a different spherical Bessel function. That is, we might write

$$\begin{aligned}\mathbf{M}_{\ell m E}(r, \theta, \phi) &= \nabla \times [f_\ell(r) Y_{\ell m}(\theta, \phi) \mathbf{r}] \\ \mathbf{M}_{\ell m M}(r, \theta, \phi) &= \nabla \times [g_\ell(r) Y_{\ell m}(\theta, \phi) \mathbf{r}]\end{aligned}$$

Now let's rescale our coefficients in the obvious manner $c_E(\ell m) = -a_E(\ell m) / C(\ell, m)$ to get this final form

$$\begin{aligned}\text{TM:} \quad \mathbf{B}_{\ell m} &= c_E(\ell m) \mathbf{M}_{\ell m E}(r, \theta, \phi) = c_E \mathbf{M}_{\ell m E} \\ \mathbf{E}_{\ell m} &= i c_E(\ell m) \mathbf{N}_{\ell m E}(r, \theta, \phi) = i c_E \mathbf{N}_{\ell m E} \\ \text{TE:} \quad \mathbf{E}_{\ell m} &= c_M(\ell m) \mathbf{M}_{\ell m M}(r, \theta, \phi) = c_M \mathbf{M}_{\ell m M} \\ \mathbf{B}_{\ell m} &= -i c_M(\ell m) \mathbf{N}_{\ell m M}(r, \theta, \phi) = -i c_M \mathbf{N}_{\ell m M}\end{aligned}$$

where we are trying to achieve a goal of very compact notation. At this point we can write our full expansion as

$$\begin{aligned}\mathbf{B} &= \sum_{\ell m} [c_E \mathbf{M}_{\ell m E} - i c_E \mathbf{N}_{\ell m E}] \\ \mathbf{E} &= \sum_{\ell m} [i c_E \mathbf{N}_{\ell m E} + c_M \mathbf{M}_{\ell m M}]\end{aligned}$$

which I have to admit is pretty compact.

Now the paper I have has a different notation for things:

<u>them</u>		<u>me</u>
even e	=>	a special meaning not discussed yet! (electric)
odd o	=>	a special meaning not discussed yet! (magnetic, see below)
n	=	ℓ , the main index on the spherical harmonics

ℓ	=	m , the minor index
m	=	n , the index of refraction (they call this the "optical constant")
ka	=	x , known as "the size parameter"

The superscript on the N and M functions relates to the choice of Bessel function. The functions have label 3 (which would be a Hankel 1) and field has subscript s. The "internal" wave (inside the dielectric) has label 1 (which would be j_ℓ) and subscript t (transmitted), while the plane wave expression also has the internal 1 label and subscript i. The three labels are correct. You can see that the coefficients in (1.4) and (1.5) are defined in a peculiar manner, perhaps as Mie did things.

Status: So, at this point I am happy about the expansions (1.4) for the scattered wave, and (1.5) for the transmitted or internal wave. My big problem is (1.6) for the plane wave incident. I know from Jackson that the simplest forms for the plane wave are with circular polarizations, and we have

$$\begin{aligned} \mathbf{E}_\pm &= \sum_\ell (i)^\ell \sqrt{4\pi(2\ell+1)} \left[j_\ell(kr) \mathbf{X}_{\ell,\pm 1} \pm (1/k) \nabla \times \{ j_\ell(kr) \mathbf{X}_{\ell,\pm 1} \} \right] \\ \mathbf{B}_\pm &= \sum_\ell (i)^\ell \sqrt{4\pi(2\ell+1)} \left[\mp i j_\ell(kr) \mathbf{X}_{\ell,\pm 1} - (i/k) \nabla \times \{ j_\ell(kr) \mathbf{X}_{\ell,\pm 1} \} \right] \end{aligned}$$

For each partial wave ℓ , we know that the $m = \pm 1$ are involved. The formula (1.6) does not show this detail. What is most annoying is that the author simply makes no comment whatsoever about polarization, as if it did not even exist.

Aside: What would happen if we averaged over the two polarizations for the intensity? We could think about the Poynting vector time averaged which would be

$$\langle S_\pm \rangle = c/8\pi \mathbf{E}_\pm \times \mathbf{B}_\pm^*$$

Because of the *, there will be non-vanishing cross terms. Rewrite as

$$\begin{aligned} \mathbf{E}_\pm &= \sum_\ell (i)^\ell \sqrt{4\pi(2\ell+1)} \left[j_\ell(kr) \mathbf{X}_{\ell,\pm 1} \right. \\ &\quad \left. \pm (1/k) \{ -C(\ell,\pm 1) \{ (f'_\ell(r) + 1/r) \mathbf{Z}_{\ell\pm 1}(\theta, \phi) + \ell(\ell+1) f_\ell(r)/r \mathbf{Y}_{\ell\pm 1}(\theta, \phi) \} \} \right] \\ \mathbf{B}_\pm^* &= \sum_\ell (i)^\ell \sqrt{4\pi(2\ell+1)} \left[\mp i j_\ell(kr) \mathbf{X}_{\ell,\pm 1}^* \right. \\ &\quad \left. - (i/k) \{ -C(\ell,\pm 1) \{ (f'_\ell(r) + 1/r) \mathbf{Z}_{\ell\pm 1}^*(\theta, \phi) + \ell(\ell+1) f_\ell(r)/r \mathbf{Y}_{\ell\pm 1}^*(\theta, \phi) \} \} \right] \end{aligned}$$

It is the X-Z cross terms that survive as in Carleton (7), along with the diagonal terms. I see no trivial argument that says you can somehow replace $\mathbf{X}_{\ell,\pm 1}$ with $\mathbf{X}_{\ell,0}$ in any kind of average sense.

So I have now (momentarily) "lost the faith" in this paper due to the unsupported expansion (1.6). This makes me not understand the quoted Mie coefficients, although they do look very much like the ones I got. I did find that the coefficients in fact did not depend on the polarization state, something that seemed strange at the time. So for the moment, I will not regard this as a confirmation source for my coefficients, but will instead learn some generalities about Mie scattering!

Some new realizations. For given index and a parameters, there will be values of k for which the coefficients blow up -- **poles**! I did not think this could happen. Perhaps the numerators compensate so things don't go infinite, but these poles at least tell you where there will be lots of large field action! In

experiments with a laser, you let droplets evaporate so radius decreases, and then you can pass through resonances. No doubt these are the resonances of the dielectric sphere just as we had in the case of the conducting cavity! The Q values quoted are finite, even for lossless dielectrics, so the numerators must be providing cancellation. (?) Author states that you can generate fields near the object that are a million times stronger than the incident field, due to the resonance effects!

This paper is no doubt correct, but it is not the right place to learn this subject. I did manage to learn a few things from it, however, such as the N and M basis functions. Recall that there are whole BOOKS just on the subject of this kind of scattering!

I notice on the web discussion of s and p polarization states!

Fact: I cannot find a real Mie derivation on the web! I think it will be in my new book arriving soon. Otherwise I will have to go to Marriott and dig something up on this subject! // I did just that! I did find one little paper that had similar M and N notation, and as I was just giving up and storing PDF documents away, I looked in the krugl one and found how that plane wave expansion works!

Plane Wave Expansion Explained. See details in Appendix A, but this is the basic idea. Write down Jackson's exact result 16.139. Then take the appropriate linear combination of his results that corresponds to $\mathbf{E} = (\mathbf{E}_+ + \mathbf{E}_-)/2 = \hat{\mathbf{x}} e^{ikz}$. His plane wave formulas then read,

$$\begin{aligned} \mathbf{E} &= \sum_{\ell} (1/2)(i)^{\ell} \sqrt{4\pi(2\ell+1)} \left[j_{\ell}(kr) [\mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}] + (1/k) \nabla \times \{ j_{\ell}(kr) [\mathbf{X}_{\ell,+1} - \mathbf{X}_{\ell,-1}] \} \right] \\ \mathbf{B} &= \sum_{\ell} (1/2)(i)^{\ell} \sqrt{4\pi(2\ell+1)} \left[-i j_{\ell}(kr) [\mathbf{X}_{\ell,+1} - \mathbf{X}_{\ell,-1}] - (i/k) \nabla \times \{ j_{\ell}(kr) [\mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}] \} \right] \end{aligned}$$

Then make the following *definitions* for o = odd, and e = even:

$$\begin{aligned} \mathbf{M}_{o1\ell} &\equiv j_{\ell}(kr) \mathbf{r} \times \nabla \psi_{o1\ell} & \text{where} & & \psi_{o1\ell} &\equiv P_{\ell}^1(\cos\theta) \sin\phi \\ \mathbf{M}_{e1\ell} &\equiv j_{\ell}(kr) \mathbf{r} \times \nabla \psi_{e1\ell} & \text{where} & & \psi_{e1\ell} &\equiv P_{\ell}^1(\cos\theta) \cos\phi \end{aligned}$$

(note that $\sin\phi$ is "odd" in ϕ) with corresponding formulas for the N functions

$$\begin{aligned} \mathbf{N}_{o1\ell} &\equiv (1/k) \nabla \times \mathbf{M}_{o1\ell} = (1/k) \nabla \times [j_{\ell}(kr) \mathbf{r} \times \nabla \psi_{o1\ell}] \\ \mathbf{N}_{e1\ell} &\equiv (1/k) \nabla \times \mathbf{M}_{e1\ell} = (1/k) \nabla \times [j_{\ell}(kr) \mathbf{r} \times \nabla \psi_{e1\ell}] \end{aligned}$$

The vector form of M is the same as above, but we have something *other than* $Y_{\ell m}$ as the pilot function. It is in fact the sum of two $Y_{\ell m}$'s of opposite m values, allowing us to make this replacement in Jackson's formula,

$$\begin{aligned} [Y_{\ell,+1}(\theta,\phi) + Y_{\ell,-1}(\theta,\phi)] &= 2i \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}} \psi_{o1\ell} \\ [Y_{\ell,+1}(\theta,\phi) - Y_{\ell,-1}(\theta,\phi)] &= 2 \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}} \psi_{e1\ell} \end{aligned}$$

Then Jackson's plane wave formulas shown above boil down to this:

$$\mathbf{E}_p = \hat{\mathbf{x}} e^{ikz} = \sum_{\ell} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [\mathbf{M}_{01\ell}^{(1)} - i \mathbf{N}_{e1\ell}^{(1)}]$$

$$\mathbf{B}_p = \hat{\mathbf{y}} e^{ikz} = \sum_{\ell} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [-\mathbf{M}_{e1\ell}^{(1)} - i \mathbf{N}_{01\ell}^{(1)}]$$

In the B equation, the first term is a (- i) from the Jackson formula, but then we divide by i because the difference of Y's does not have the i that the sum did have, and we change o to e as well. For the *second* term, we get -i times the N function that matches the first term in the E equation. // I now have a general rule for doing B from E in any expansion: cross mult coeff's by -i and steal the nature.

This result agrees with (1.6) in all respects except one: in the derivation above, we clearly see that the m-index should have the value 1, as shown in the subscript labels above. But in (1.6), this subscript appears as the general m (which they happen to call ℓ). This is an error in (1.6)! In the Krugal PDF paper, the above formula appears exactly as I have stated it as equation 2.44. The superscript (1) implies that we are using the j_{ℓ} radial functions.

Recall that Jackson's "first term" in the E equation 16.139 is the "magnetic" or TE term, and this is still true when we combine the two helicity solutions. Thus, we can associate "magnetic" with "odd", and of course "electric" with "even". That is a nice label to have on the M and N functions. So if the first term is odd, we always expect the second term to be odd.

Aside on Krugel: I stumbled onto the contents and first three chapters on-line of the following book:

E. Krugel, *The Physics of Interstellar Dust*, 12/02

\$135 Amazon

He is the one who cleared up the above plane wave mystery. Krugel is at the Max Planck Institute (MPI) for Radio Astronomy in Bonn. I notice that the PDF documents cannot be printed, nor can anything be selected for cut and paste!

Attempt to set up the scattering problem in the N and M world.

First, let's look back at our earlier derivation where I wrote down the *four* boundary conditions. My two curl conditions implied matching of tangential field components. The $\mathbf{r} \times \mathbf{B}$ condition is problematical for a conductor because there really is a surface current. However, If you use H instead, I guess H does not see the surface current, the way D does not see surface charge. So I guess I accept Krugel's two $\mathbf{r} \times$ boundary conditions in the general case of arbitrary isotropic μ and ϵ , which is what he is talking about. Krugel then notes that each of these $\mathbf{r} \times$ boundary conditions implicate 2 equations if you use the $\hat{\phi}$ and $\hat{\theta}$ directions for tangentialness. He says you then get four equations for the four unknown coefficients and you can then solve.

When I did this problem, I was indeed finding redundancy with my boundary conditions. So I guess the answer is that the two $\mathbf{r} \times$ conditions are all you need! Here is Krugel's setup of the problem:

$$\mathbf{E}_p = \sum_{\ell} \mathbf{E}_{\ell} [\mathbf{M}_{01\ell}^{(1)} - i \mathbf{N}_{e1\ell}^{(1)}] \quad \mathbf{E}_{\ell} = i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} \quad // \text{ incident plane wave}$$

$$\mathbf{E}_s = \sum_{\ell} E_{\ell} [-b_{\ell} \mathbf{M}_{01\ell}^{(3)} + a_{\ell} \mathbf{N}_{01\ell}^{(3)}] \quad // \text{ incident plane wave, (3) means } h_{\ell}^{(1)}(kr)$$

$$\mathbf{E}_i = \sum_{\ell} E_{\ell} [c_{\ell} \mathbf{M}_{01\ell}^{(1)} - i d_{\ell} \mathbf{N}_{01\ell}^{(1)}] \quad // \text{ incident plane wave, (1) means } j_{\ell}(kr)$$

I presume the expressions for the $\mathbf{H}$ fields are the same as the above, except I expect to see the even and odd indices switched. This is because the "first term" in Jackson's B fields are always going to be the "other kind" from what you see in the E field.

What is the general rule for converting from an E equation to a B equation? I did it for the plane wave above:

- (1) negate the first term's sign, and replace odd with even
- (2) keep the second term's sign and replace even with odd.

So applying these rules to the above,

$$\mathbf{H}_p = \sum_{\ell} H_{\ell} [-\mathbf{M}_{01\ell}^{(1)} - i \mathbf{N}_{01\ell}^{(1)}] \quad H_{\ell} = i^{\ell} \frac{2\ell+1}{\ell(\ell+1)}$$

$$\mathbf{H}_s = \sum_{\ell} H_{\ell} [b_{\ell} \mathbf{M}_{01\ell}^{(3)} + a_{\ell} \mathbf{N}_{01\ell}^{(3)}]$$

$$\mathbf{H}_i = \sum_{\ell} H_{\ell} [-c_{\ell} \mathbf{M}_{01\ell}^{(1)} - i d_{\ell} \mathbf{N}_{01\ell}^{(1)}]$$

Now the boundary conditions are

$$H_{i\theta} = H_{s\theta} + H_{p\theta} \quad r \times \mathbf{H} = 0$$

$$H_{i\phi} = H_{s\phi} + H_{p\phi}$$

$$E_{i\theta} = E_{s\theta} + E_{p\theta} \quad r \times \mathbf{E} = 0$$

$$E_{i\phi} = E_{s\phi} + E_{p\phi}$$

This will be continued in another paper!

Appendix A. Trying to get a simple plane wave expansion in terms of the N and M functions

Attempt #1 (not successful)

Let's just try this as an exercise in the VSH X,Y,Z that we think we know about. We start with

$$\mathbf{E} = e^{ikz} \hat{\mathbf{x}}$$

Our full expansions are written,

$$\mathbf{B} = \sum_{\ell m} [a_E(\ell m) f_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi) + (-i/k) a_M(\ell m) \nabla \times \{ g_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi) \}]$$

$$\mathbf{E} = \sum_{\ell m} [(i/k) a_E(\ell m) \nabla \times \{ f_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi) \} + a_M(\ell m) g_\ell(r) \mathbf{X}_{\ell m}(\theta, \phi)]$$

but try to shorten it by writing,

$$\mathbf{B} = \sum_{\ell m} [a_E f \mathbf{X} + (-i/k) a_M \{ \nabla \times (g \mathbf{X}) \}]$$

$$\mathbf{E} = \sum_{\ell m} [(i/k) a_E \{ \nabla \times (f \mathbf{X}) \} + a_M g \mathbf{X}]$$

Use the fact that (again in compact notation just so we know where factors are sitting!)

$$\nabla \times (f \mathbf{X}) = -C [(f' + 1/r) \mathbf{Z} + \ell(\ell+1) f/r \mathbf{Y}]$$

and we then have

$$\mathbf{B} = \sum_{\ell m} [a_E f \mathbf{X} + (-i/k) a_M \{ -C [(g' + 1/r) \mathbf{Z} + \ell(\ell+1) g/r \mathbf{Y}] \}]$$

$$\mathbf{E} = \sum_{\ell m} [(i/k) a_E \{ -C [(f' + 1/r) \mathbf{Z} + \ell(\ell+1) f/r \mathbf{Y}] \} + a_M g \mathbf{X}]$$

which we then group by term as

$$\mathbf{B} = \sum_{\ell m} [\{ a_E f \} \mathbf{X} + \{ (i/k) C a_M \ell(\ell+1) g/r \} \mathbf{Y} + \{ (i/k) C a_M (g' + 1/r) \} \mathbf{Z}]$$

$$\mathbf{E} = \sum_{\ell m} [\{ a_M g \} \mathbf{X} - \{ (i/k) C a_E \ell(\ell+1) f/r \} \mathbf{Y} - \{ (i/k) C a_E (f' + 1/r) \} \mathbf{Z}]$$

In the lower equation, we swap the E and M labels, the f and g functions, AND the secondary terms have an extra minus sign.

Now let's try to compute the three projections as in (11) - (13). Start with

$$E_{\ell m}^2 = [1/\ell(\ell+1)] \int d\Omega \mathbf{X}_{\ell m}^* / C \cdot \mathbf{E} = [1/C \ell(\ell+1)] \int d\Omega \mathbf{X}_{\ell m}^* \cdot \mathbf{E}$$

where in the last form we mean Carleton's X. For the E field we shall use

$$\mathbf{E} = e^{ikz} \hat{\mathbf{x}} = \hat{\mathbf{x}} \sum_{\ell} (i)^{\ell} \sqrt{4\pi(2\ell+1)} j_{\ell}(kr) Y_{\ell,0}(\theta, \phi) \quad // \text{ Jackson 16.129}$$

Installing this gives,

$$E_{\ell m}^2 = [1/\ell(\ell+1)] \sum_{\ell} (i)^{\ell} \sqrt{4\pi(2\ell+1)} j_{\ell}(kr) \int d\Omega Y_{\ell,0}(\theta, \phi) [\mathbf{X}_{\ell m}^*(\theta, \phi) \cdot \hat{\mathbf{x}}]$$

At this point, we need to go off and do an overdue piece of work, which is to relate the Cartesian unit vectors to the X,Y,Z vectors. As shown in Appendix B we have

$$[\mathbf{X}_{\ell m}^*(\theta, \phi) \cdot \hat{\mathbf{x}}] = [-S_\phi \frac{\partial Y_{\ell m}^*}{\partial \theta} - (C_\theta C_\phi / r S_\theta) \frac{\partial Y_{\ell m}^*}{\partial \phi}]$$

which makes for a fairly unpleasant angular integral,

$$E_{\ell m}^2 = [1/\ell(\ell+1)] \sum_{\ell'} (i)^{\ell'} \sqrt{4\pi(2\ell'+1)} j_{\ell'}(kr) \int d\Omega Y_{\ell' 0}(\theta, \phi) [-S_\phi \frac{\partial Y_{\ell m}^*}{\partial \theta} - (C_\theta C_\phi / r S_\theta) \frac{\partial Y_{\ell m}^*}{\partial \phi}]$$

But now what do we do? Notice that we cannot conclude that $m=0$ due to the other ϕ factors present. This shows the wisdom of Jackson's approach using the raising and lowering operators on page 568. Although I cannot trivially do the integral shown, I think I can conclude that whatever it is, it is a function of ℓ, ℓ' and m . I suspect the integral will vanish unless $m = -1, 0$, or $+1$ based on my intuitive angular momentum addition notions. Also it will vanish unless $\ell' = \ell-1, \ell$ or $\ell+1$. But for the moment let's write:

$$E_{\ell m}^2 = [1/\ell(\ell+1)] \sum_{\ell'} (i)^{\ell'} \sqrt{4\pi(2\ell'+1)} j_{\ell'}(kr) K_{\ell m}^2 =$$

Presumably the other two components have similar results. Then we have to put these sums over ℓ' into our original sum over ℓm . The result is a mess and I have no direction really as to where I am going with this, so let's can it for now.

Attempt #2 (successful)

Let's here piggyback on Jackson's calculations! We know that:

$$\mathbf{E}_\pm = (\hat{\mathbf{x}} \pm i \hat{\mathbf{y}}) e^{ikz}$$

gives the result

$$\begin{aligned} \mathbf{E}_\pm &= \sum_{\ell} (i)^{\ell} \sqrt{4\pi(2\ell+1)} [j_{\ell}(kr) \mathbf{X}_{\ell \pm 1} \pm (1/k) \nabla \times \{ j_{\ell}(kr) \mathbf{X}_{\ell \pm 1} \}] \\ \mathbf{B}_\pm &= \sum_{\ell} (i)^{\ell} \sqrt{4\pi(2\ell+1)} [\mp i j_{\ell}(kr) \mathbf{X}_{\ell \pm 1} - (i/k) \nabla \times \{ j_{\ell}(kr) \mathbf{X}_{\ell \pm 1} \}] \end{aligned}$$

So let's use this

$$\begin{aligned} \mathbf{E} &= (\mathbf{E}_+ + \mathbf{E}_-)/2 = \hat{\mathbf{x}} e^{ikz} \\ \mathbf{B} &= (\mathbf{B}_+ + \mathbf{B}_-)/2 = \hat{\mathbf{y}} e^{ikz} \quad // \text{ see 16.130 and fact that } \mathbf{B} = \hat{\mathbf{z}} \times \mathbf{E} \end{aligned}$$

Then our results will be

$$\begin{aligned} \mathbf{E} &= \sum_{\ell} (1/2)(i)^{\ell} \sqrt{4\pi(2\ell+1)} [j_{\ell}(kr) [\mathbf{X}_{\ell+1} + \mathbf{X}_{\ell-1}] + (1/k) \nabla \times \{ j_{\ell}(kr) [\mathbf{X}_{\ell+1} - \mathbf{X}_{\ell-1}] \}] \\ \mathbf{B} &= \sum_{\ell} (1/2)(i)^{\ell} \sqrt{4\pi(2\ell+1)} [-i j_{\ell}(kr) [\mathbf{X}_{\ell+1} - \mathbf{X}_{\ell-1}] - (i/k) \nabla \times \{ j_{\ell}(kr) [\mathbf{X}_{\ell+1} + \mathbf{X}_{\ell-1}] \}] \end{aligned}$$

I have confirmed this second equation several times, every sign and i factor is important!

Now I am having trouble relating this form to (1.6) in my little Mie web paper.

BUT WAIT!!!! Think about what these things look like:

$$[\mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}] = C(\ell,m) * \mathbf{r} \times \nabla [Y_{\ell,+1}(\theta,\phi) + Y_{\ell,-1}(\theta,\phi)]$$

Look at Jackson 3.53 and 3.54

$$Y_{\ell,+1}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}} P_{\ell}^1(\cos\theta) e^{i\phi}$$

$$Y_{\ell,-1}(\theta,\phi) = (-1)^1 Y_{\ell,+1}^*(\theta,\phi) = - \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}} P_{\ell}^1(\cos\theta) e^{-i\phi}$$

Therefore,

$$[Y_{\ell,+1}(\theta,\phi) + Y_{\ell,-1}(\theta,\phi)] = \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}} P_{\ell}^1(\cos\theta) \{ e^{i\phi} - e^{-i\phi} \} = 2i \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}} P_{\ell}^1(\cos\theta) \sin\phi$$

If we leave off the constant, my krugel PDF claims that

$$\psi_{01\ell} = P_{\ell}^1(\cos\theta) \sin\phi$$

And then the "odd" M function is the curl of this thing! That is why it is "odd", perhaps. That is to say

$$\mathbf{M}_{01\ell} = j_{\ell}(kr) \mathbf{r} \times \nabla \psi_{01\ell}$$

If we keep track of the constants

$$\begin{aligned} j_{\ell}(kr) [\mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}] &= j_{\ell}(kr) C(\ell,m) * \mathbf{r} \times \nabla [Y_{\ell,+1}(\theta,\phi) + Y_{\ell,-1}(\theta,\phi)] \\ &= j_{\ell}(kr) C(\ell,m) 2i \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}} \mathbf{r} \times \nabla \{ P_{\ell}^1(\cos\theta) \sin\phi \} \\ &= \frac{-i}{\sqrt{\ell(\ell+1)}} 2i \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}} j_{\ell}(kr) \mathbf{r} \times \nabla \psi_{01\ell} \\ &= \frac{-i}{\sqrt{\ell(\ell+1)}} 2i \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}} j_{\ell}(kr) \mathbf{r} \times \nabla \psi_{01\ell} \\ &= 2 \sqrt{\frac{2\ell+1}{4\pi}} \frac{1}{\ell(\ell+1)} \mathbf{M}_{01\ell} \end{aligned}$$

Then the first term in Jackson's sum becomes:

$$\begin{aligned} \mathbf{E} &= \sum_{\ell} (1/2)(i)^{\ell} \sqrt{4\pi(2\ell+1)} \quad [\quad j_{\ell}(kr) [\mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}] \quad] \\ &= \sum_{\ell} (i)^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [\mathbf{M}_{0\ell} + \dots \end{aligned}$$

Now recall that the N function is defined as

$$\mathbf{N} = (1/k) \nabla \times \mathbf{M} = (1/k) \nabla \times \{ j_{\ell}(kr) \mathbf{r} \times \nabla \psi \}$$

You can see that the second term has the difference of the two Y functions

$$\begin{aligned} j_{\ell}(kr) [\mathbf{X}_{\ell,+1} - \mathbf{X}_{\ell,-1}] &= j_{\ell}(kr) C(\ell, m) * \mathbf{r} \times \nabla [Y_{\ell,+1}(\theta, \phi) - Y_{\ell,-1}(\theta, \phi)] \\ &= j_{\ell}(kr) C(\ell, m) * \mathbf{r} \times \nabla \left\{ 2 \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}} P_{\ell}^1(\cos\theta) \cos\phi \right\} \\ &= (2/i) \sqrt{\frac{2\ell+1}{4\pi}} \frac{1}{\ell(\ell+1)} \mathbf{M}_{\ell\ell} \quad \mathbf{M}_{\ell\ell} = j_{\ell}(kr) \mathbf{r} \times \nabla \psi_{\ell\ell} \quad \psi_{\ell\ell} = P_{\ell}^1(\cos\theta) \cos\phi \end{aligned}$$

The second term in the Jackson sum is then

$$\begin{aligned} + (1/k) \nabla \times \{ j_{\ell}(kr) [\mathbf{X}_{\ell,+1} - \mathbf{X}_{\ell,-1}] \} &= (1/k) \nabla \times \left\{ (2/i) \sqrt{\frac{2\ell+1}{4\pi}} \frac{1}{\ell(\ell+1)} \mathbf{M}_{\ell\ell} \right\} \\ &= 2 \sqrt{\frac{2\ell+1}{4\pi}} \frac{1}{\ell(\ell+1)} (-i \mathbf{N}_{\ell\ell}) \end{aligned}$$

which has the same form as our M result above, so the final result is this:

$$\mathbf{E} = \hat{\mathbf{x}} e^{ikz} = \sum_{\ell} (i)^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [\mathbf{M}_{0\ell} - i \mathbf{N}_{\ell\ell}]$$

And so we have *precisely* produced equation (1.6) now at 5 of midnight 2/14/03. Thus, we may conclude that they are considering fixed polarization in the $\hat{\mathbf{x}}$ direction! The big mystery is finally resolved.

Now what about the corresponding B equation?

Comments. You get the definite feeling that we are combining a spin 1 "thing" (polarization vector, say) with ℓm spatial functions, so we should be doing some kind of $j = \ell \otimes 1$ somewhere. The underlying group theory is being masked in our efforts so far, but I think I have found a source that discusses this subject. I think the idea might be this concerning the plane wave expansion: If you pre-combine into values of j , then you need only have a sum over j . I know this is a vague comment, but it might explain why the polarization is not appearing in the plane wave expansion I keep looking at.

I did a Marriott library trip today on this subject, was reminded that you cannot park before 6 PM. I found a little paper which I copied a bit of, and it repeats the funny plane wave expansion in terms of N and M functions. However, this paper gives a lot more detail on what these functions look like

Appendix A. Relating Cartesian unit vectors to X,Y,Z

$$\mathbf{X}_{\ell m} = \mathbf{r} \times \nabla Y_{\ell m}$$

$$\mathbf{Y}_{\ell m} = \hat{\mathbf{r}} Y_{\ell m}$$

$$\mathbf{Z}_{\ell m} = \mathbf{r} \nabla Y_{\ell m}$$

It is possible to do this, but the results are not very nice. For example

$$\mathbf{r} \times \nabla Y_{\ell m} = \hat{\mathbf{x}} [-S_{\phi} Y_{\theta} - (C_{\theta} C_{\phi} / r S_{\theta}) Y_{\phi}] + \hat{\mathbf{y}} [C_{\phi} Y_{\theta} - (C_{\theta} S_{\phi} / r S_{\theta}) Y_{\phi}] + \hat{\mathbf{z}} [1/r Y_{\phi}]$$

$$\hat{\mathbf{r}} Y_{\ell m} = \hat{\mathbf{x}} [S_{\theta} C_{\phi} Y] + \hat{\mathbf{y}} [S_{\theta} S_{\phi} Y] + \hat{\mathbf{z}} [C_{\theta} Y]$$

$$\mathbf{r} \nabla Y_{\ell m} = \hat{\mathbf{x}} [C_{\theta} C_{\phi} Y_{\theta} - S_{\phi} / S_{\theta} Y_{\phi}] + \hat{\mathbf{y}} [C_{\theta} C_{\phi} Y_{\theta} + C_{\phi} / S_{\theta} Y_{\phi}] + \hat{\mathbf{z}} [-S_{\theta} Y_{\theta}]$$

where Y_{θ} means the θ derivative of Y , etc. As noted elsewhere, this must somehow be the same as a certain rotation acting on our unit vectors.

1. About the M function

In the previous paper on this topic, I made this identification for the "first term" multipole fields,

$$\mathbf{M}_{\ell m}(r, \theta, \phi) = \nabla \times [f_{\ell}(r) Y_{\ell m} \mathbf{r}] = -f_{\ell}(r) \mathbf{r} \times \nabla Y_{\ell m} = -f_{\ell}(r) \mathbf{X}_{\ell m} / C(\ell, m)$$

where

$$\mathbf{X}_{\ell m}(\theta, \phi) = C(\ell, m) * \mathbf{r} \times \nabla Y_{\ell m}(\theta, \phi)$$

$C(\ell, m) =$	$+1$	Carleton
	-1	Mie (not sure what this means now)
	$\frac{-i}{\sqrt{\ell(\ell+1)}}$	Jackson

Sometimes, such as in Krugel, we found that $Y_{\ell m}$ was replaced with a linear combination of $Y_{\ell m}$ within the same ℓ group.

Let's now prove a few theorems concerning the M functions

Theorem 1: If $\mathbf{M} \equiv \nabla \times (\mathbf{c} \psi)$ where $\mathbf{c}$ is either a constant vector or $\mathbf{r}$, then $\mathbf{M} = -\mathbf{c} \times \nabla \psi$.

Proof: $\nabla \times (\mathbf{c} \psi) = \nabla \psi \times \mathbf{c} + \psi \nabla \times \mathbf{c} = \nabla \psi \times \mathbf{c} = -\mathbf{c} \times \nabla \psi$

Theorem 2: Let $\mathbf{M} \equiv \nabla \times (\mathbf{c} \psi)$. If $\mathbf{c}$ is either a constant vector or $\mathbf{r}$, and if ψ satisfies the wave equation $(\nabla^2 + k^2) \psi = 0$, then so does $\mathbf{M}$: $(\nabla^2 + k^2) \mathbf{M} = 0$

Proof: If we apply ∇^2 onto $\mathbf{M}$, it moves through onto the $\mathbf{c} \psi$, and then we can apply our identity

$$\nabla^2 (\mathbf{c} \psi) = \mathbf{c} \nabla^2 \psi + \psi \nabla^2 \mathbf{c} + 2 \nabla \mathbf{c} \cdot \nabla \psi$$

where we think of a component of $\mathbf{c}$ at a time here, but write it as a vector. Now if $\mathbf{c}$ is a constant, the last two terms vanish and we get our result,

$$\nabla^2 \mathbf{M} \equiv \nabla \times \nabla^2 (\mathbf{c} \psi) = \nabla \times (\mathbf{c} \nabla^2 \psi) = -k^2 \nabla \times (\mathbf{c} \psi) = -k^2 \mathbf{M}$$

It turns out that this still works when $\mathbf{c} = \mathbf{r}$ even though $\mathbf{r}$ is not a constant vector. Consider,

$$\nabla^2 (\mathbf{r} \psi) = \mathbf{r} \nabla^2 \psi + \psi \nabla^2 \mathbf{r} + 2 \nabla \mathbf{r} \cdot \nabla \psi = -k^2 (\mathbf{r} \psi) + 0 + 2 \nabla \psi$$

where we have used the fact that $\nabla^2 \mathbf{r} = 0$ since $\partial_j \partial_i r_i = \partial_j \delta_{j,i} = 0$. Obviously since $\mathbf{r}$ is linear, it cannot have a Laplacian. In the second term, we note that $\nabla \mathbf{r} = \mathbf{I}$, the identity matrix, which we might write in the form $\overset{\leftrightarrow}{\mathbf{1}}$ and then we would say that $\overset{\leftrightarrow}{\mathbf{1}} \bullet \nabla \psi = \nabla \psi$ (or do as components!) . So at this point we have

$$\nabla^2 \mathbf{M} \equiv \nabla \times \nabla^2 (\mathbf{r} \psi) = \nabla \times [-k^2 (\mathbf{r} \psi) + 2 \nabla \psi] = -k^2 \mathbf{M} + 2 \nabla \times \nabla \psi = -k^2 \mathbf{M} \quad \text{QED}$$

Theorem 3: Let $\mathbf{M} \equiv \nabla \times (\mathbf{r} \psi) = -\mathbf{r} \times \nabla \psi$. If ψ satisfies the wave-like equation $\nabla^2 \psi = f(r)\psi$, then $\mathbf{M}$ satisfies the same equation, $\nabla^2 \mathbf{M} = f(r) \mathbf{M}$. { This theorem is not true if we replace $\mathbf{r}$ with constant $\mathbf{c}$. }

Proof: $\nabla^2 \mathbf{M} = \nabla \times \nabla^2 (\mathbf{r} \psi) = \nabla \times (\mathbf{r} \nabla^2 \psi) = \nabla \times [(f(r) \psi) \mathbf{r}] = \nabla(f(r) \psi) \times \mathbf{r} + (f(r) \psi) \nabla \times \mathbf{r}$
 $= \nabla(f(r) \psi) \times \mathbf{r} = [f(r) \nabla \psi + \psi \nabla f(r)] \times \mathbf{r} = -\mathbf{r} \times f(r) \nabla \psi + 0 = -f(r) \mathbf{r} \times \nabla \psi = f(r) \mathbf{M}$ by Theorem 1.

Example of Theorem 3. Consider $\mathbf{X}_{\ell m} = \mathbf{r} \times \nabla Y_{\ell m}$ so that $\psi = -Y_{\ell m}$. We know that $\nabla^2 Y_{\ell m} = -\ell(\ell+1)/r^2 Y_{\ell m}$ so that $f(r) = -\ell(\ell+1)/r^2$. Thus we know that $\nabla^2 \mathbf{X}_{\ell m} = -\ell(\ell+1)/r^2 \mathbf{X}_{\ell m}$. We proved this fact in gory detail in our "vector harmonics.doc" paper. We might think of this as "the wave equation within a partial wave", loosely speaking.

Summary of what we have so far:

- Define $\mathbf{M} \equiv \nabla \times (\mathbf{r} \psi)$. Then:
- $\mathbf{M} = -\mathbf{r} \times \nabla \psi$, which is one of our usual VSH basis functions $\mathbf{X}$, up to a constant
- $\nabla \bullet \mathbf{M} = 0$
- If ψ satisfies a wave or wave-like equation, then so does $\mathbf{M}$.

Theorem 4. Suppose we define $\mathbf{M} \equiv f_\ell(r) \mathbf{M}_0 = f_\ell(r) \nabla \times (\mathbf{r} \psi) = -f_\ell(r) \mathbf{r} \times \nabla \psi$, where $\mathbf{M}_0 = -\mathbf{r} \times \nabla \psi$ and where:

- (1) $\psi = \psi(\theta, \phi)$;
- (2) ψ satisfies the wave-like equation $\nabla^2 \psi = g(r)\psi$ where $g(r) = -\ell(\ell+1)/r^2$;
- (3) $f_\ell(r)$ is a linear combination of spherical Bessel functions such as $j_\ell(kr)$, where k is the constant appearing in the Bessel equation.

$$\text{Then } (\nabla^2 + k^2) \mathbf{M} = 0.$$

Corollary: If ψ is any linear combination of spherical harmonics $Y_{\ell m}$ within the ℓ manifold, theorem 4 is true.

Proof of Theorem 4: From Theorem 2 we know that $\nabla^2 \mathbf{M}_0 = g(r) \mathbf{M}_0$. So consider,

$$\begin{aligned} \nabla^2 \mathbf{M} &= \nabla^2 [f_\ell(r) \mathbf{M}_0] = [\nabla^2 f_\ell(r)] \mathbf{M}_0 + f_\ell(r) \nabla^2 \mathbf{M}_0 + 2 \nabla f_\ell(r) \bullet \nabla \mathbf{M}_0 \\ &= \{ 1/r^2 \partial_r [r^2 \partial_r f_\ell(r)] + g(r) f_\ell(r) \} \mathbf{M}_0 + 2 f'_\ell(r) \partial_r \mathbf{M}_0 \end{aligned}$$

Now comes a somewhat unobvious fact that I got caught on. Consider:

$$\begin{aligned}
 -\mathbf{M}_\theta &= \mathbf{r} \times \nabla \psi = \mathbf{r} \times \left(\frac{1}{r} \psi_\theta \hat{\theta} + \frac{1}{r} \frac{1}{S_\theta} \psi_\theta \hat{\phi} \right) \\
 &= \psi_\theta \hat{\phi} - \frac{1}{S_\theta} \psi_\theta \hat{\theta}.
 \end{aligned}$$

So $\mathbf{M}_\theta$ is not even a function of r , and therefore $\partial_r \mathbf{M}_\theta = 0$. Thus we get:

$$\nabla^2 \mathbf{M} = \left\{ \frac{1}{r^2} \partial_r \left[r^2 \partial_r f_\ell(r) \right] - \ell(\ell+1)/r^2 f_\ell(r) \right\} \mathbf{M}_\theta = \left\{ -k^2 f_\ell(r) \right\} \mathbf{M}_\theta = -k^2 \mathbf{M} \quad \text{QED}$$

Since $f_\ell(r)$ is a Bessel function as noted, we get the simplification of $\{...\}$ shown on the right above.

Comment: The above theorem shows an interesting result. We know that $f_\ell(r)\psi_\ell$ satisfies the **scalar** wave equation (where $f_\ell(r)$ is a spherical Bessel of parameter k , and ψ_ℓ is a linear combination of spherical harmonics within ℓ). If we are interested in solving the **vector** wave equation, then $\mathbf{M} = -f_\ell(r) \mathbf{r} \times \nabla \psi_\ell$ works. It has the extra nice property that $\nabla \cdot \mathbf{M} = 0$.

Observation: The multipole field "main terms" for $\mathbf{E}$ and $\mathbf{B}$ are of exactly this form. Usually we have ψ_ℓ as a straight $Y_{\ell m}$, but it is OK to instead use something like $\psi_\ell = C [Y_{\ell m} \pm Y_{\ell, -m}]$ if that is convenient. For the expansion of an $\hat{\mathbf{x}}$ polarized plane wave, this does turn out to be convenient, and $m = 1$. We are also satisfying $\nabla \cdot \mathbf{E}, \mathbf{B} = 0$ with this solution.

2. About the N function

Now let's look at $\mathbf{N}$ defined by:

$$\mathbf{N} = \nabla \times \mathbf{M}$$

where $\mathbf{M}$ satisfies the wave equation. Then $\mathbf{N}$ must also satisfy the wave equation, since

$$\nabla^2 \mathbf{N} = \nabla^2 \nabla \times \mathbf{M} = \nabla \times (\nabla^2 \mathbf{M}) = \nabla \times (-k^2 \mathbf{M}) = -k^2 \nabla \times \mathbf{M} = -k^2 \mathbf{N}$$

Notice also that $\nabla \cdot \mathbf{N} = 0$.

Usually, a multiplicative constant is used out in front,

$$\mathbf{N} = (1/k) \nabla \times \mathbf{M}$$

since this removes a factor of $1/k$ in the "second terms" in the multipole expansions for $\mathbf{E}$ and $\mathbf{B}$. We have just shown, then, that these "second terms" satisfy both the divergence condition and wave equation, so they represent a second solution to the wave equation mentioned earlier. So, we can now extend our earlier comment:

Comment: We know that $f_\ell(r)\psi_\ell$ satisfies the **scalar** wave equation (where $f_\ell(r)$ is a spherical Bessel of parameter k , and ψ_ℓ is a linear combination of spherical harmonics within ℓ). If we are interested in solving the **vector** wave equation, then we know of two solutions:

$$(1) \mathbf{M} = -f_\ell(r) \mathbf{r} \times \nabla \psi_\ell, \text{ and we also know that } \nabla \cdot \mathbf{M} = 0.$$

$$(2) \mathbf{N} = (1/k) \nabla \times \mathbf{M} = -(1/k) \nabla \times [f_\ell(r) \mathbf{r} \times \nabla \psi_\ell], \text{ and } \nabla \cdot \mathbf{N} = 0.$$

These are exactly the first and second terms we see in Jackson's multipole expansion formulas, apart from overall constants.

We can think of $\mathbf{M}$ as $\mathbf{M} = \text{constant} \times f_\ell(r) * \mathbf{X}$ where $\mathbf{X}$ is one of Carleton's three VSH's. We know that we can explode $\mathbf{N}$ into a linear combination of the $\mathbf{Y}$ and $\mathbf{Z}$ VSH basis vectors. Only $\mathbf{X}$ of these three basis vectors satisfies the wave equation! However, the $\mathbf{N}$ linear combination of $\mathbf{Y}$ and $\mathbf{Z}$ also solves it!

One other fact. With $\mathbf{N}$ scaled by the $(1/k)$ factor as shown

$$\nabla \times \mathbf{N} = k \mathbf{M}$$

So we really have this symmetrical situation:

$$\mathbf{N} = (1/k) \nabla \times \mathbf{M}$$

$$\mathbf{M} = (1/k) \nabla \times \mathbf{N}$$

Also we know that (from inspection of $\mathbf{M}$)

$$\mathbf{r} \cdot \mathbf{M} = 0 \quad \text{so } \mathbf{M} \text{ is the "transverse" solution}$$

Of course we know this since $\mathbf{M} \sim \mathbf{X}$ and we know that $\mathbf{X}$ and $\mathbf{Z}$ are transverse, $\mathbf{Y}$ radial. Note that, since $\mathbf{N}$ is a mixture of $\mathbf{Y}$ and $\mathbf{Z}$, it has both transverse and radial pieces.

DUBIOUS

→ See "Mmm m+n..."

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The spherical harmonics can be generalized to vector spherical harmonics by looking for a scalar function ψ and a constant vector $\mathbf{c}$ such that

$$\begin{aligned} \mathbf{M} &\equiv \nabla \times (\mathbf{c}\psi) = \psi(\nabla \times \mathbf{c}) + (\nabla\psi) \times \mathbf{c} \\ &= (\nabla\psi) \times \mathbf{c} = -\mathbf{c} \times \nabla\psi \Rightarrow \vec{\nabla} \times \vec{M} = -\vec{c} \nabla^2 \psi \end{aligned} \quad (1)$$

SO

$$\nabla \cdot \mathbf{M} = 0. \quad \text{true of any curl.} \quad (2)$$

Now use the vector identities

$$\begin{aligned} \nabla^2 \mathbf{M} &= \nabla(\nabla \cdot \mathbf{M}) - \nabla \times (\nabla \times \mathbf{M}) \\ &= \nabla \times (\nabla \times \mathbf{M}) = \nabla \times (\mathbf{c} \nabla^2 \psi) \\ &= \nabla \times (\nabla \times (\mathbf{c} \psi)) = \nabla \times (\mathbf{c} \nabla^2 \psi) \end{aligned} \quad \begin{aligned} \nabla^2 \vec{M} &= -\vec{\nabla} \times (\vec{\nabla} \times \vec{M}) \\ &= -\vec{\nabla} \times (-\vec{c} \nabla^2 \psi) \\ &= \vec{\nabla} \times (\vec{c} \nabla^2 \psi) \end{aligned} \quad (3)$$

$$k^2 \mathbf{M} \equiv k^2 \nabla \times (\mathbf{c}\psi) = \nabla \times (\mathbf{c} \nabla^2 \psi), \quad (4)$$

SO

$$\nabla^2 \mathbf{M} + k^2 \mathbf{M} = \nabla \times [\mathbf{c}(\nabla^2 \psi + k^2 \psi)], \quad (5)$$

and $\mathbf{M}$ satisfies the vector Helmholtz differential equation if ψ satisfies the scalar Helmholtz differential equation

$$\nabla^2 \psi + k^2 \psi = 0. \quad (6)$$

Construct another vector function

$$\mathbf{N} \equiv \frac{\nabla \times \mathbf{M}}{k}, \quad \text{OK} \quad (7)$$

which also satisfies the vector Helmholtz differential equations since

$$\nabla^2 \mathbf{N} \equiv \frac{1}{k} \nabla^2 (\nabla \times \mathbf{M}) \equiv \frac{1}{k} \nabla \times (\nabla^2 \mathbf{M}) \quad (8)$$

$$= \frac{1}{k} \nabla \times (-k^2 \mathbf{M}) = -k \nabla \times \mathbf{M} = -k^2 \mathbf{N},$$

which gives

$$\nabla^2 \mathbf{N} + k^2 \mathbf{N} = 0. \quad (9)$$

We have the additional identity

$$\begin{aligned} \nabla \times \mathbf{N} &= \frac{1}{k} \nabla \times (\nabla \times \mathbf{M}) = \frac{1}{k} \nabla (\nabla \cdot \mathbf{M}) - \frac{1}{k} \nabla^2 \mathbf{M} \\ &= \frac{1}{k} \nabla \times (-\vec{c} \nabla^2 \psi) \\ &= \frac{1}{k} \nabla \times (+\vec{c} k^2 \psi) \\ &= \frac{1}{k} \nabla^2 \mathbf{M} - \frac{1}{k} \nabla^2 \mathbf{M} = \frac{\nabla^2 \mathbf{M}}{k} = k \mathbf{M}. \quad (10) \end{aligned}$$

In this formalism, ψ is called the generating function and $\vec{c}$ is called the pilot vector. The choice of generating function is determined by the symmetry of the scalar equation, i.e., it is chosen to solve the desired scalar differential equation. If $\mathbf{M}$ is taken as

$$\mathbf{M} = \nabla \times (\mathbf{r} \psi), \quad (11)$$

where $\mathbf{r}$ is the radius vector, then $\mathbf{M}$ is a solution to the vector wave equation in spherical coordinates. If we want vector solutions which are tangential to the radius vector, ~~or~~

$$\mathbf{M} \cdot \mathbf{r} = \mathbf{r} \cdot (\nabla \psi \times \vec{c}) = (\nabla \psi) (\vec{c} \times \mathbf{r}) = 0, \quad (12)$$

so

$$\vec{c} \times \mathbf{r} = 0 \quad (13)$$

and we may take

$$\vec{c} = \mathbf{r} \quad (14)$$

(Arfken 1985, pp. 707-711; Bohren and Huffman 1983, p. 88).

A number of conventions are in use. Hill (1954) defines

$$\begin{aligned} \mathbf{V}_l^m &\equiv -\sqrt{\frac{l+1}{2l+1}} Y_l^m \hat{\mathbf{r}} + \frac{1}{\sqrt{(l+1)(2l+1)}} \frac{\partial Y_l^m}{\partial \theta} \hat{\theta} \\ &\quad + i M \sqrt{(l+1)(2l+1)} \sin \theta Y_l^m \hat{\phi} \end{aligned} \quad (15)$$

$$\mathbf{W}_l^m = \sqrt{\frac{l}{2l+1}} Y_l^m \hat{\mathbf{r}} + \frac{1}{\sqrt{l(2l+1)}} \frac{\partial Y_l^m}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{iM}{\sqrt{l(2l+1)} \sin \theta} Y_l^m \hat{\boldsymbol{\phi}} \quad (16)$$

$$\mathbf{X}_l^m = -\frac{M}{\sqrt{l(l+1)} \sin \theta} Y_l^m \hat{\boldsymbol{\theta}} - \frac{i}{\sqrt{l(l+1)}} \frac{\partial Y_l^m}{\partial \theta} \hat{\boldsymbol{\phi}}. \quad (17)$$

Morse and Feshbach (1953) define vector harmonics called **B**, **C**, and **P** using rather complicated expressions.

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NOTES

There are many ideas here that are new to me, so I have tried to number each of the steps to keep them from blurring into each other. I had guidance from:

<http://farside.ph.utexas.edu/~rfitzp/teaching/jk1/lectures/node83.html>

1. Radiation resistance R_{rad} of a Hertzian Dipole antenna. From Jackson, we know the power and pattern of a "Hertzian Dipole" antenna, a center-fed antenna which is $d/2$ in length on each side, and where $d \ll \lambda$. That power is, from 9.29,

$$P = I_o^2 (kd)^2 / 12c = I_o^2 / 2 [2 \cdot 4 \pi^2 / 12 c * (d/\lambda)^2] = I_o^2 / 2 * R_{\text{rad}}$$

Here, I_o is a peak current, so $I_o^2 / 2 = I_{\text{rms}}^2$ and R_{rad} must then be the radiation resistance of this antenna, of course here in cgs units. Secondly, we know the power dipole radiation pattern is $\sin^2\theta$.

2. Converting R_{rad} to mks units.

$$1 \text{ Coulomb} = c(\text{cgs})/10 \text{ esu} \quad \text{so} \quad 1 \text{ amp} = c(\text{cgs})/10 \text{ statamp}$$

$$1 \text{ volt} = 1/300 \text{ statvolt}$$

$$1 \text{ ohm} = 1 \text{ volt} / 1 \text{ amp} = 1/300 \text{ statvolt} / [c(\text{cgs})/10 \text{ statamp}] = 1/[30c(\text{cgs})] \text{ statohms}$$

$$\Rightarrow R(\text{ohms}) = 30 c(\text{cgs}) R(\text{statohms})$$

Thus, for the Hertzian antenna we have

$$R_{\text{rad}} (\text{statohms}) = [2 \pi^2 / 3 c * (d/\lambda)^2]$$

$$R_{\text{rad}} (\text{ohms}) = [20 \pi^2 * (d/\lambda)^2] = 197 (d/\lambda)^2$$

3. Model for a receiving antenna. When any antenna is configured for reception, it is modeled as an induced voltage in series with R_{rad} , and your best bet for power into a load is to have $R_{\text{load}} = R_{\text{rad}}$. If the antenna emf source generates V_o peak voltage, half of that will then be on the load, so the load receives power of

$$P_{\text{load}} = (1/2) (V_o/2)^2 / R_{\text{rad}} = (1/8) V_o^2 / R_{\text{rad}}$$

where the first factor of $1/2$ comes because we want to talk rms power, not peak power. In any matched antenna, therefore, fully **half the received power is re-radiated** into R_{rad} !

4. P_{load} For the Hertzian Dipole antenna. We know that $V_o = E_o (d/2)$ [or twice this for peak to peak], just from definition of the field, where E_o is the peak electric field amplitude. Thus, from the previous paragraph,

$$P_{load} = (1/8) V_o^2 / R_{rad} = (1/32) E_o^2 d^2 / R_{rad}$$

On the other hand, from 2 paragraphs ago we know that for this antenna, in mks units,

$$R_{rad} \text{ (ohms)} = 20 \pi^2 * (d/\lambda)^2$$

Therefore, the **power delivered to the matched load of a short dipole antenna** is

$$P_{load} = 1/[32*20*\pi^2] * E_o^2 * \lambda^2$$

which is a fundamental result, saying power received is proportional to λ^2 and to E_o^2 . We suspect that the E_o^2 proportionality will hold for any antenna, not just a short dipole.

5. Effective area concept. For any antenna, we can *imagine* an "effective antenna of area A" that scoops up 100% of the Poynting flux it receives and converts that 100% to P_{load} . We would then say

$$P_{load} = A * S$$

6. Poynting flux in mks units and Z_o . In cgs units, for a plane wave in free space we know that the time averaged S is given by

$$S = (c/8\pi) \mathbf{E} \times \mathbf{H}^* \quad \Rightarrow \quad S = (c/8\pi) E_o^2$$

Converting this to mks as on page 619 Jackson requires

$$c \rightarrow 1/\sqrt{\mu_o \epsilon_o} \quad E_o \rightarrow \sqrt{4\pi \epsilon_o} E_o$$

so we would expect to get

$$S = (1/8\pi \sqrt{\mu_o \epsilon_o}) 4\pi \epsilon_o E_o^2 = (1/\sqrt{\mu_o \epsilon_o}) E_o^2/2 = [E_o^2/2] / Z_o \quad \Rightarrow \quad S = [E_o^2/2] / Z_o$$

where $Z_o = 376.7 = 120\pi \text{ ohms} = 1/(\epsilon_o c)$.

7. Area of a Hertzian Dipole antenna. Combining the previous two items, we find that

$$P_{load} = A * S = A * [E_o^2/2] / Z_o$$

being the definition of this effective area A. But for the matched Hertzian Dipole we know that

$$P_{\text{load}} = 1/[32 \cdot 20 \cdot \pi^2] \cdot E_0^2 \cdot \lambda^2$$

from which we may find the effective area

$$A \cdot [E_0^2/2] / Z_0 = 1/[32 \cdot 20 \cdot \pi^2] \cdot E_0^2 \cdot \lambda^2$$

$$A = 1/[32 \cdot 20 \cdot \pi^2] \cdot \lambda^2 \cdot 2 Z_0 = 1/[32 \cdot 20 \cdot \pi^2] \cdot \lambda^2 \cdot 240 \pi = 12/[32 \cdot \pi] \cdot \lambda^2 = (3/8\pi) \lambda^2$$

$$\Rightarrow A(\text{short dipole}) = (3/8\pi) \lambda^2$$

8. **Gain of the Hertzian dipole antenna.** Now the Hertzian dipole is not an isotropic radiator, but has a $\sin^2\theta$ dependence for $dP/d\Omega$. Let's compare a dipole radiator to an isotropic one:

$$dP/d\Omega = D \sin^2\theta \quad \Rightarrow \quad P = D \cdot 8\pi/3$$

$$dP/d\Omega = I \quad \Rightarrow \quad P = I \cdot 4\pi$$

If we want both these radiators to radiate the same total power, we must have

$$D \cdot 8\pi/3 = I \cdot 4\pi \quad \Rightarrow \quad D \cdot 2/3 = I$$

Now if we consider the dipole in its peak radiating direction, we have $dP/d\Omega = D$, so the dipole antenna therefore has a GAIN over the same-total-power isotropic antenna of $G = 3/2$.

9. **The hypothetical isotropic antenna.** Imagine a transmitter sending a signal to our Hertzian antenna from a point in the peak of the beam of our receiving antenna. We have $P_{\text{load}} = A \cdot S$ in our receiver load. Now, suppose we replace our *receiving* antenna with a hypothetical isotropic one. From reciprocity, we know that this antenna for the same incoming S is going to have *less* P_{load} by exactly the power gain factor of the Hertz antenna. So we would write

$$\begin{aligned} P_{\text{load}} &= A \cdot S & // \text{ Hertz antenna} \\ P'_{\text{load}} &= A' \cdot S & // \text{ isotropic antenna} \\ P_{\text{load}} / P'_{\text{load}} &= G = \text{gain} \end{aligned}$$

Therefore,

$$A/A' = P_{\text{load}} / P'_{\text{load}} = G$$

So you imagine that you start with an isotropic receiving antenna of area A_0 and the antenna's gain makes the effective area larger by $A = G \cdot A_0$.

10. Effective area A_0 of the hypothetical isotropic antenna. So, at this point we computed the A for a Hertzian short dipole to be $A = (3/8\pi)\lambda^2$, and we know that the gain for this antenna is $G = 3/2$. If we replaced this antenna with a hypothetical isotropic receiving antenna, that hypothetical antenna would have a smaller effective area A_0 and would have a smaller power delivered to the receiving antenna load. We would have

$$A_0 = (3/8\pi)\lambda^2/G = (3/8\pi)\lambda^2 * 2/3 = \lambda^2/4\pi = \pi \chi^2 \quad \chi = \lambda/2\pi$$

11. Interaction between two antennas. Imagine now a transmitting antenna driven by P_{xmit} all of which is delivered into the transmitting radiation resistance. The flux from this antenna at distance r is going to be G x the flux from an isotropic radiator of the same power, so

$$S = G_t * P_{xmit} / (4\pi r^2)$$

The power delivered to the load in a receiving antenna will be

$$\begin{aligned} P_{load} &= A_r * S = G_r A_0 * \{ G_t * P_{xmit} / (4\pi r^2) \} \\ &= G_r G_t P_{xmit} (\lambda^2/4\pi) / (4\pi r^2) = G_r G_t [\lambda/4\pi r]^2 P_{xmit} = [G_r \lambda^2/4\pi] [G_t \lambda^2/4\pi] P_{xmit}/r^2 \lambda^2 \\ &= [A_r A_t / \lambda^2] P_{xmit} / r^2 \end{aligned}$$

This is the Friis Transmission Formula, symmetric in both directions of course.

12. Conclusions about Area

Therefore we arrive at these conclusions:

(1) The effective area A_0 of a hypothetical (and non-existent) isotropic antenna is a $\pi \chi^2$ which is the area of a circle of radius χ . The meaning of this area is that, if you actually had such a receiving antenna, and put it into the flux S of some transmitted plane wave, and if you matched this antenna to its load, the power delivered to the load would be $S * A_0$.

(2) Any real antenna has an effective area $A = G A_0 = G \pi \chi^2$, where G is the directivity or gain factor in presumably the best direction, which is assumed to be the way it would be used. Again, the power delivered to a matched load would be $S * A$.

Consider a point charge q moving at some slow constant velocity $\mathbf{v}$. What is $\mathbf{J}$ and what is ρ ?

$$\rho(\mathbf{r}) = q \delta(\mathbf{r} - \mathbf{a}) \quad // \text{point charge located at position } \mathbf{a}$$

We know this is right because we integrate over a surrounding volume to get q . Next, let's guess for $\mathbf{J}$:

$$\mathbf{J}(\mathbf{r}) = q \mathbf{v} \delta(\mathbf{r} - \mathbf{a})$$

To verify this conjecture, consider ($\mathbf{v}$ is a constant vector)

$$\nabla \cdot \mathbf{J}(\mathbf{r}) = q (\mathbf{v} \cdot \nabla) \delta(\mathbf{r} - \mathbf{a})$$

On the other hand, we can see that

$$\partial_t \rho(\mathbf{r}) = q \partial_t \delta(\mathbf{r} - \mathbf{a}) = q (\partial_t \mathbf{a}) \cdot \nabla_a \delta(\mathbf{r} - \mathbf{a}) = q \mathbf{v} \cdot (-\nabla_r) \delta(\mathbf{r} - \mathbf{a}) = -q (\mathbf{v} \cdot \nabla) \delta(\mathbf{r} - \mathbf{a})$$

Since this satisfies $\nabla \cdot \mathbf{J} = -\partial_t \rho(\mathbf{r})$, we know it must be the right form for $\mathbf{J}$. Remember that continuity just says that if there is a net outflow of current from a volume, the enclosed charge must decrease.

Example in the z direction

Let $\mathbf{v} = v \hat{\mathbf{z}}$. Then $\mathbf{J}(\mathbf{r}) = q \mathbf{v} \delta(\mathbf{r} - \mathbf{a}) \hat{\mathbf{z}}$. Then $\nabla \cdot \mathbf{J}(\mathbf{r}) = \partial_z J_z = q v \partial_z \delta(\mathbf{r} - \mathbf{a})$, where ∂_z acts only on the z factor of the delta function. Now imagine an aligned box containing the charge. Integrate this divergence

over the box and we get $q v \int_0^h dz \partial_z \delta(z - a_z) = \text{only the parts} = q v [\delta(h - a_z) - \delta(a_z)]$. This is only non-

zero when the charge is passing through the upper or lower face of the box. On the other hand, if we integrate the current density over the box, we get $Q(a_z) = q \theta(a_z) \theta(h - a_z)$. That is, it is non-zero only if a_z is in the box. If we apply a time derivative to this Q , we use $\partial_x \theta(x) = \delta(x)$ to get

$$\begin{aligned} \partial_t Q(a_z) &= q \partial_t [\theta(a_z) \theta(h - a_z)] = q \partial a_z / \partial t \partial_{a_z} [\theta(a_z) \theta(h - a_z)] && // \text{chain rule} \\ &= q v \partial_{a_z} [\theta(a_z) \theta(h - a_z)] && // \text{recognize } v \\ &= q v \{ \theta(a_z) [-\delta(h - a_z)] + \delta(a_z) \theta(h - a_z) \} && // \text{product rule} \\ &= q v \{ \theta(h) [-\delta(h - a_z)] + \delta(a_z) \theta(h - 0) \} && // \text{use deltas} \\ &= q v \{ [-\delta(h - a_z)] + \delta(a_z) \} \\ &= -q v [\delta(h - a_z) - \delta(a_z)] \end{aligned}$$

So things do work out. It just seems a little odd to say that $\nabla \cdot \mathbf{J}(\mathbf{r}) = q v \partial_z \delta(\mathbf{r} - \mathbf{a})$ in this case, and generally

$$\rho(\mathbf{r}) = q \delta(\mathbf{r} - \mathbf{a}) \quad \mathbf{J}(\mathbf{r}) = q \mathbf{v} \delta(\mathbf{r} - \mathbf{a}) \quad \nabla \cdot \mathbf{J}(\mathbf{r}) = -\partial \rho / \partial t = q v \nabla \delta(\mathbf{r} - \mathbf{a})$$

Application to a mechanical dipole radiator

Imagine charges q and $-q$ separated by distance $a \cos(\omega t)$. In other words, $+q$ is located at

$$q \text{ is at: } \mathbf{a}(t) = (d/2) \exp(-i\omega t) \hat{\mathbf{z}} \quad -q \text{ is at: } -\mathbf{a}(t) = -(d/2) \exp(-i\omega t) \hat{\mathbf{z}}$$

$$\rho(\mathbf{r}) = (+q) \delta(\mathbf{r} - \mathbf{a}) + (-q) \delta(\mathbf{r} + \mathbf{a}) = q [\delta(\mathbf{r} - \mathbf{a}) - \delta(\mathbf{r} + \mathbf{a})]$$

The velocity of the $+q$ charge is $\partial_t \mathbf{a}(t) = -i\omega \mathbf{a}(t)$, while that of $-q$ is the opposite. So we have

The current density is

$$\mathbf{J}(\mathbf{r}) = (+q)(-i\omega \mathbf{a}(t))\delta(\mathbf{r} - \mathbf{a}) + (-q)(+i\omega \mathbf{a}(t))\delta(\mathbf{r} + \mathbf{a}) = -i\omega q \mathbf{a}(t) [\delta(\mathbf{r} - \mathbf{a}) + \delta(\mathbf{r} + \mathbf{a})]$$

Now use Jackson (9.13) to compute the $n=0$ term of the vector potential! Note that the two currents are in the same direction! We get:

$$\mathbf{A}(\mathbf{r}, t) = \frac{e^{ikr}}{cr} \int \mathbf{J} d^3r = -ik \frac{e^{ikr}}{r} [2q \mathbf{a}(t)] = -ik \frac{e^{ikr}}{r} [2q (d/2) \exp(-i\omega t) \hat{\mathbf{z}}] = -ik \frac{e^{ikr}}{r} \{qd\hat{\mathbf{z}}\} \exp(-i\omega t)$$

The thing $\{qd\hat{\mathbf{z}}\}$ is usually called $\mathbf{p}$, the **dipole moment**,

$$\mathbf{p} = q d \hat{\mathbf{z}} \quad \mathbf{A}(\mathbf{r}) = -ik \frac{e^{ikr}}{r} \mathbf{p} \quad // \text{ agreeing with (9.16)}$$

We could also compute $\mathbf{p}$ by integrating :

$$\mathbf{p} = \int \rho \mathbf{r} d^3r = \int q [\delta(\mathbf{r} - \mathbf{a}) - \delta(\mathbf{r} + \mathbf{a})] \mathbf{r} d^3r = q \mathbf{a} - (q(-\mathbf{a})) = 2q \mathbf{a} = q d \hat{\mathbf{z}}$$

Note that things are defined with and without the time phase in various places.

Jackson

p109

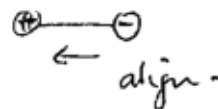
$$\vec{P} = \chi_e \vec{E}$$

↑ "electric susceptibility"

$$\vec{D} = \epsilon \vec{E}$$

$$\epsilon = 1 + 4\pi\chi_e$$

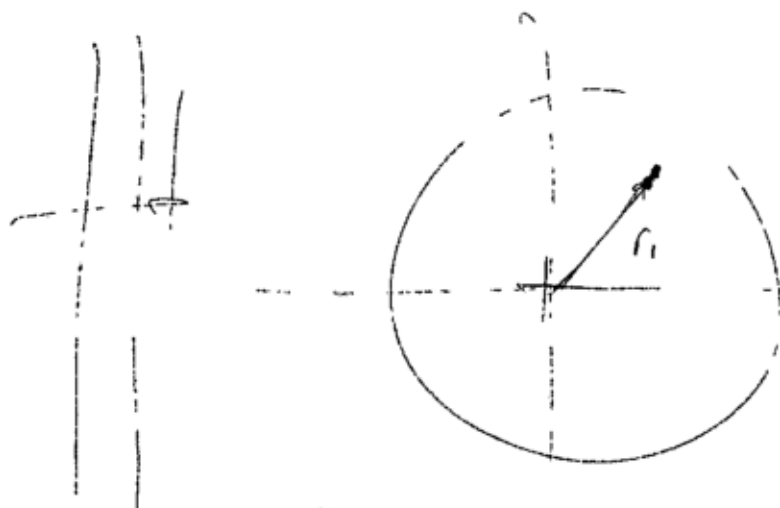
↑ "dielectric constant"



$$k = \frac{\omega}{c} \sqrt{\mu\epsilon} = \frac{\omega}{[c/\sqrt{\mu\epsilon}]} \quad \text{so} \quad n = \frac{c}{[c/\sqrt{\mu\epsilon}]}$$

$$n = \sqrt{\mu\epsilon} \quad \text{so} \quad n = \frac{c}{[c/\sqrt{\mu\epsilon}]}$$

$$\text{So, } n \approx \sqrt{\epsilon} = \sqrt{1 + 4\pi\chi_e}$$



$$\frac{e^{i(kz - \omega t)}}{z} = e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad \vec{k} = k\hat{z}$$

Put origin at sphere center

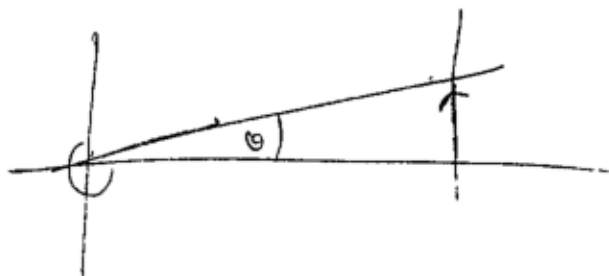
$$\Rightarrow \vec{E}_{in}(\vec{r}_1) = e^{i(\vec{k} \cdot \vec{r}_1 - \omega t)} \quad \checkmark$$

$$f(0) = \frac{-\epsilon_0}{e^2/m\omega^2} \chi \cdot \left(\frac{4}{3}\pi a^3\right). \quad n = \sqrt{1+4\pi\chi}$$

$$n^2 - 1 = 4\pi\chi.$$

$$\text{So } f(0) = \frac{-\epsilon_0}{(e^2/mc^2)} \left[\frac{n^2-1}{4\pi}\right] \cdot \frac{4\pi}{3} a^3 \equiv N_{\text{eff}}.$$

$$\text{Then: } \Delta k = |\hat{k} - k\hat{z}|.$$



Answer is

$$|\vec{E}| \propto N_{\text{eff}} \left[\frac{\sin k\Delta h}{(k\Delta h)^3} - \frac{\cos k\Delta h}{(k\Delta h)^2} \right].$$

$$\Delta k = |\hat{r} - \hat{z}|^2 = 1 + 1 - 2\cos\hat{r} \cdot \hat{z} = 2(1 - \cos(\hat{r} \cdot \hat{z}))$$

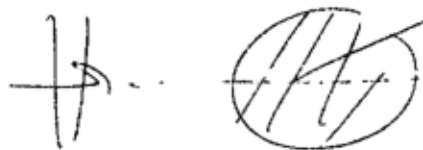
$$\hat{r} \cdot \hat{z} = \cos\theta \Rightarrow \Delta k = 2(1 - \cos\theta).$$

$$\text{So } \Delta k = 2(1 - \cos\theta) \quad -1) \Rightarrow$$

$$= 4\sin^2\frac{\theta}{2} \approx 4\left(\frac{\theta}{2}\right)^2 = \theta^2$$

So messy result:

Small θ .



$$[\] = \frac{\sin(R\theta^2)}{(R\theta^2)^3} - \frac{\cos(R\theta^2)}{(R\theta^2)^2}$$

1. What is the energy density in a plane wave?

In statics, page 21 Jackson shows that $|\mathbf{E}|^2/8\pi$ is the energy density of an E field. The $||$ is there only because E is a vector, not because something is complex-valued. The understanding here is that $\mathbf{E}$ is a real-valued vector. When we do plane waves, we usually say $\mathbf{E} = E_0 \exp(ikz) \hat{\mathbf{z}}$. We really mean the real part of this thing which is $E_0 \cos(kz) \hat{\mathbf{z}}$. The peak energy density of such a plane wave stored only in the E field would then be $E_0^2/8\pi$. The average however is the average of cosine squared which is 1/2, so $\langle U_E \rangle = E_0^2/16\pi$. But in a plane wave, the B field has the same energy density, we get $\langle U \rangle = E_0^2/8\pi$ as the total average energy density.

Notice that the average energy density of a plane wave is independent of frequency ω . Higher frequency plane waves have a fewer number of photons that each have a higher energy quantum.

2. What is the momentum of a plane wave?

(a) Photon has energy $h\nu$ and momentum p such that $E^2 - p^2 = p_\mu p^\mu = m^2 c^2 = 0$ so $p = h\nu/c = \hbar\beta$. This would then be $p = \hbar\omega/2\pi c = \hbar k$. That sounds familiar. But E&M classical theory does not know about $\hbar$, so question remains unanswered.

(b) Use the fact that, for photons, $p = E/c$ where E is energy. Then average energy density U in a volume is $E_0^2/8\pi$ in free space, so: $d\text{Energy} = (E_0^2/8\pi) dV$ and $dp = (E_0^2/8\pi c) dV$. So the average momentum density is $P = (E_0^2/8\pi c)$. For a unit amplitude plane wave, we have $P = (1/8\pi c)$. The number of photons per unit volume is $n = (E/dV)/E_1 = (E_0^2/8\pi) / \hbar\omega = (E_0^2/8\pi)/\hbar\omega = n$. The number of photons in a volume $dz dA$ is $ndz dA$, they all bounce off our mirror in time Δt , so force $F = \Delta p/\Delta t$. We have $\Delta p = 2np dz dA$, and $\Delta t = dz/c$ (the time for this volume to hit the wall completely), so $F = 2np c dz dA$ or Pressure = $2np c = (E_0^2/4\pi)$. So we are in the end able to show these things about momentum without reference to a photon, except we need the idea that $p = E/c$ and then $P = U/c$. I am afraid this can only come from quantum theory. Notice that the momentum of an individual photon $\hbar k$ seems to be irrelevant so our momentum calculation. The momentum of a plane wave is not proportional to k . In a plane wave, as we increase k , we increase the energy of a photon, but at the same time we decrease the number of photons, so the momentum p or P for a given value of E_0 remains unchanged.

Summary for a plane wave of the form $\mathbf{E} = E_0 \exp(ikz) \hat{\mathbf{z}}$:

average energy density = $(E_0^2/8\pi) = U$

average momentum density = $(E_0^2/8\pi c) = P = U/c$

average energy flow per unit area per unit time = $\Delta U/\Delta t = (E_0^2/8\pi) dz / dz/c = (E_0^2/8\pi) * c = U * c$

This is also called "energy flow" or "power/area" and symbol S of Poynting is usually used.

Some people refer to power/area as intensity and use symbol I .

average momentum flow per unit area per unit time = $\Delta P/\Delta t = (E_0^2/8\pi c)dz / dz/c = (E_0^2/8\pi) = U = P \cdot c$
 This is called "**momentum flow**" but there is no term like "power" to describe this thing.

average pressure on a mirror = $F/dA = (\Delta p/\Delta t)/dA = 2 \cdot PdV/(dA dz/c) = 2 \cdot (P \cdot c) = 2 \cdot U = (E_0^2/4\pi)$

Comment: If you have an absorber instead of a mirror, radiation pressure is half the above.

3. Problematic bogus calculation of Pressure

Average energy density in a plane wave is $E_0^2/8\pi$, see above. Imagine plane wave bouncing off a mirror. Push the mirror dz into the field, stored field energy decreases by $dz E_0^2/8\pi$ per unit area just because there is less field present. And this reduction occurs in both the incoming and the outgoing beams? Or should you first compute the total coherent E and B fields to get the true stored energy in the beam space? If I do this, I find that $E = 2E_0 \sin(kz)\sin(\omega t) = B$. Then I get $\langle E^2 \rangle = (2E_0)^2 \cdot 1/2 = 2 E_0^2$ and the same for $\langle B^2 \rangle$. Then I would claim that the average energy density in the volume to the left of the mirror is the sum of these which would be $4 E_0^2$, then we add the 8π to get $\langle u \rangle = 4 E_0^2 / 8\pi = E_0^2 / 2\pi$. If this is right, then we would claim that after the mirror has moved dz , the field has lost stored energy $(E_0^2 / 2\pi) dz$ per unit area. Now, the pushing hand does work on the mirror which does work on the field, so you would think the field energy would *increase*, not decrease! This simple argument gives twice the right answer for pressure with the wrong sign, since pressure = $F/dA = dU/(dzdA)$! Something is not kosher with this argument.

Idea: while we are pushing on the mirror for time dt and it is moving at speed v , the reflected beam is Doppler up-shifted. The frequency of the reflected beam is upshifted in frequency by amount $1/(1 - 2v/c) \sim 1 + 2v/c$ during the time of movement. This ought to have something to do with the problem!

After spending about 2 full hours on this paradox, I have no solution to it. So I don't know how to compute radiation pressure using a change-in-energy-stored argument. I sent an email to Jim, I bet he will have the obvious answer that I am unable to see today.

4. What about Poynting's Vector?

Start off with $S = k E \times B$ where k is to be determined. This is for real fields E and B. If we write $S = k E \times B$ in esu units, then $|S| = k |E|^2$ instantaneous. Then $|S|_{\text{peak}} = k E_0^2$ and $\langle |S| \rangle = k E_0^2/2$. But from the section above this must be $U \cdot c = (E_0^2/8\pi) \cdot c = (E_0^2/2) \cdot (c/4\pi)$, so $k = c/4\pi$. Therefore

$$S = (c/4\pi) E \times B \quad \text{instantaneous energy flow} \quad \Rightarrow \quad |S|_{\text{peak}} = (c/4\pi) E_0^2 = (E_0^2/4\pi) \cdot c$$

$$\langle S \rangle = (c/8\pi) E \times B = (E_0^2/8\pi) \cdot c$$

How do we adapt this for complex E and B? Try $S' = j E \times B^*$ where j is to be determined. For plane wave in esu units we have $|S'| = j E_0^2$ because the phasors cancel. Clearly S' so defined cannot refer to an instantaneous value at some z because there is no function of z left. Consider:

$$|S'| = j E_0^2 = (j/k) k E_0^2 = (j/k) |S|_{\text{peak}} = (2j/k) \langle |S| \rangle$$

If we select $j = k$, then S' refers to a peak value, and if we pick $j = k/2$, then S' is an average value. I think Jackson on page 205 wants to be thinking average flow, so he would say

$$S' = (c/8\pi) \mathbf{E} \times \mathbf{B}^* \quad \text{average energy flow} \quad // \text{ as we see in 7.14}$$

5. How do we come to associated $\mathbf{p}$ with ∇ , and in what context(s) ?

In wave theory, consider a plane wave $\exp(ikx)$. The operator $(\hbar/i)\nabla$ yields $\hbar k$ which we know is the momentum of a photon in the plane wave (see 2(a) above). So we would associate $\mathbf{p} = (\hbar/i)\nabla$. So this association, which is the same in the Saxon QM book, requires quantum theory.

On page 542 of Jackson, we seem him use $\mathbf{p} = (1/i)\nabla$ as a shorthand, so he can use $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ as another shorthand. For him, these are operators closely related to the quantum operators, but they are have the $\hbar$ scaled out. If we apply Jackson's $\mathbf{p}$ operator to $\exp(ikx)$, we get just k , the momentum of a photon in units of $\hbar$. I should think of these things as mere shorthands in Jackson. But if we talk about individual photons, they have that interpretation.

6. What is the potential of a plane wave, both ϕ and $\mathbf{A}$?

Let $\hat{\mathbf{e}}_t$ be an arbitrary unit vector transverse to $\mathbf{k}$ of the plane wave. Then try this form:

$$\mathbf{A} = (1/ik) \exp(i\mathbf{k} \cdot \mathbf{r}) \hat{\mathbf{e}}_t$$

Then

$$\mathbf{B} = \nabla \times \mathbf{A} = (1/ik) \{ \nabla [\exp(i\mathbf{k} \cdot \mathbf{r})] \times \hat{\mathbf{e}}_t \} + \exp(i\mathbf{k} \cdot \mathbf{r}) \nabla \times \hat{\mathbf{e}}_t.$$

The second term is 0 since $\hat{\mathbf{e}}_t$ is a fixed vector. Then $\nabla [\exp(i\mathbf{k} \cdot \mathbf{r})] = i\mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{r})$ so we have

$$\mathbf{B} = \nabla \times \mathbf{A} = (1/ik) \{ i\mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{r}) \times \hat{\mathbf{e}}_t \} = \exp(i\mathbf{k} \cdot \mathbf{r}) \hat{\mathbf{k}} \times \hat{\mathbf{e}}_t$$

So this looks like a reasonable $\mathbf{B}$ field for a plane wave in the $\mathbf{k}$ direction. Given this $\mathbf{B}$ and this $\mathbf{k}$, we then know that

$$\mathbf{E} = \exp(i\mathbf{k} \cdot \mathbf{r}) \hat{\mathbf{e}}_t \quad \hat{\mathbf{e}}_t \times (\hat{\mathbf{k}} \times \hat{\mathbf{e}}_t) = \hat{\mathbf{k}}$$

because then $\mathbf{E} \times \mathbf{B}$ is in the $\mathbf{k}$ direction. Now we know that

$$\mathbf{E} = -\nabla\phi - (1/c)\partial_t \mathbf{A} = -\nabla\phi - (1/c)(-i\omega)(1/ik) \exp(i\mathbf{k} \cdot \mathbf{r}) \hat{\mathbf{e}}_t = -\nabla\phi + \exp(i\mathbf{k} \cdot \mathbf{r}) \hat{\mathbf{e}}_t = -\nabla\phi + \mathbf{E}$$

Thus, if we take $\phi = \text{constant}$ in space, we get the right answer.

Summary: If we start with these potentials:

$$(A, \phi) = (e^{ik \cdot r} \hat{\epsilon}_t / ik, \phi(t))$$

we obtain these plane-wave fields:

$$\mathbf{E} = e^{ik \cdot r} \hat{\epsilon}_t \quad \mathbf{B} = e^{ik \cdot r} \hat{\mathbf{k}} \times \hat{\epsilon}_t$$

where $\mathbf{E}$ is polarized in the $\hat{\epsilon}_t$ direction. Notice that

$$\nabla \cdot \mathbf{A} = (1/ik) \hat{\epsilon}_t \cdot \nabla (e^{ik \cdot r}) = e^{ik \cdot r} (\hat{\mathbf{k}} \cdot \hat{\epsilon}_t)$$

$$1/c \partial \phi / \partial t = f'(t)/c$$

Since these don't add up to 0, we are not in a Lorentz gauge here (see 6.36). Nor are we in the Coulomb gauge. This is one of an infinite possible sets of potentials that give these fields.

7. In non-relativistic classical mechanics, how do $\mathbf{E}$ and $\mathbf{B}$ fields interact with a particle?

(a) The $\mathbf{E}$ and $\mathbf{B}$ fields put a force on a particle which is $\mathbf{F} = q\mathbf{E} + (v/c) \times \mathbf{B}$. The first term is normal in electrostatics. The second term represents an $\mathbf{E}$ field that appears in a moving frame of reference. I think this force law comes entirely from Maxwell's four equations.

(b) A static charged particle makes an $\mathbf{E}$ field which is $\mathbf{E} = -\nabla\phi$ where ϕ is the potential made by the particle, and we find ϕ by solving the Poisson equation.

(c) If the particle makes a current $\mathbf{J}$ because it is moving, then we get an $\mathbf{A}$ that is a certain integral of $\mathbf{J}$, and then $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E}$ then comes from a Maxwell equation.

(d) In general, we have $\mathbf{J}^\mu$ driving an ODE for $\mathbf{A}^\mu$ which encapsulates (b) and (c) above.

However, we know that the fields affect the charges and the charges affect the fields, so not obvious what general solutions exist. We often talk about a particle in some "external fields" which are imposed from the outside. We also talk about particles moving and making fields, for example, some "fixed currents" in an antenna. We also talk about particles without fields and fields without particles.

8. In non-relativistic quantum mechanics, how do $\mathbf{E}$ and $\mathbf{B}$ fields interact with a particle?

I think this is going to have something to do with $\mathbf{A} \cdot \mathbf{J}$ and maybe $\phi \rho$.

For electric field, we should be able to add a Hamiltonian term $-eEz$ for field in z direction. Maybe add something like $-evxB$ for magnetic field. I will find some books where this is done.

9. In relativistic classical mechanics, how do E and B fields interact with a particle?

Jackson talks about some of this in his book. At least how fast moving particles generate fields. I forget how the force equation changes (if it does at all) for a fast-moving particle

10. In relativistic quantum mechanics, how do E and B fields interact with a particle?

This is perhaps Dirac electron theory with fields not yet quantized.

11. In relativistic quantum mechanics and quantum field theory, how do E and B fields interact with a particle?

The general idea is you write a Lagrangian and add the interaction term $e A^\mu J_\mu$ which couples the photon field to the electric current, represented by the electron Dirac spinor business.

The Average Auroral Zone Electric Field

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Four hundred and seventy-eight hours of electric field data obtained by the flight of 32 balloons between $L = 5.4$ and $L = 8.2$ have been averaged to determine mean properties of the auroral zone electric field. When it is mapped into the equatorial plane, the average $\mathbf{E} \times \mathbf{B}$ flow has a sunward component at all local times and produces a convection pattern consistent with that expected for flow around a rotating obstacle. The average auroral zone electric field strength increases by a factor of about 2 as Kp increases from 0 to 6, but the larger-scale convection pattern is not greatly affected by the level of magnetic activity. The average electric field variation with Kp is small compared to that caused by local turbulent fluctuations, and so local field strengths can be large during quiet periods or small during magnetic activity. The average electric field magnitude is enhanced by a factor of about 2 when the interplanetary magnetic field has a southward component; this finding provides further evidence in favor of the generation of the auroral zone electric field by magnetic field reconnection. The y component of the interplanetary magnetic field causes the northern auroral zone electric field strength to be larger than average near local dawn, when $B_y > 0$, and larger than average near local dusk, when $B_y < 0$. These behaviors are understood in terms of a model in which the two-cell convection pattern is rotated about the sun-earth line by an amount that depends on the sign and magnitude of B_y . This model and the data suggest either that important parallel potential drops exist along auroral zone magnetic field lines or that simple conjugacy between the two hemispheres does not exist along such field lines.

Auroral ionospheric and magnetospheric electric fields have been measured by a variety of techniques (see Mozer [1973] for a review of the various types of measurements), and much information on certain aspects of their morphology has been obtained [Mozer, 1971b; Cauffman and Gurnett, 1972; Doupnik et al., 1972; Haerendel, 1972; McIlwain, 1972]. However, relatively few attempts have been made to determine the diurnal, latitude, magnetic activity, and other dependencies of the average auroral zone electric field, probably because sufficiently large bodies of directly interpretable electric field data have not yet been obtained. Since one of the larger bodies of such data has resulted from the flight of 32 balloons between $L = 5.4$ and $L = 8.2$ [Mozer and Serlin, 1969; Mozer and Manka, 1971], the 478 hours of data thereby collected have been analyzed to determine the dependence of the average auroral zone electric field on these parameters. Whereas this body of data has been sufficient to elucidate certain general relationships that will be discussed below, it is important to note that further detailed analyses will probably require thousands of hours of data in order to average better over local turbulent fluctuations of these fields.

DATA

Table 1 summarizes the locations and quantities of data collected from each of the seven sites in which balloon measurements of auroral zone ionospheric electric fields have been made. One-hour averages of the data from each of these flights are presented in a nonrotating frame of reference in Figure 1. Because of the scientific interest in nighttime measurements, data collection has been greatest at night. The largest sample of data obtained by any local time is the 31 hours acquired at 0200 LT, and the smallest amount is the 6 hours of data obtained at 1900 LT.

Figure 1 shows both a spread in the data at any local

time and a general trend in the average field as a function of local time. Whereas the data are characterized as much by their spread as by their mean diurnal pattern, this turbulence will generally be eliminated from further discussion in the present paper by averaging over the data collected at any given local time. Properties of this turbulent behavior have been described elsewhere [Mozer, 1971a].

AVERAGE LARGE-SCALE MAGNETOSPHERIC ELECTRIC FIELD

In Figure 2 the hourly averages of the data of Figure 1 are plotted with error bars that represent the standard deviations of the means. The solid curves of this figure are smooth fits to the experimental data drawn with the constraint that the 24-hour average of the westward field component be zero in order to satisfy $\nabla \times \mathbf{E} = 0$. In agreement with expectation the electric field patterns of Figure 2 imply a sunward $\mathbf{E} \times \mathbf{B}$ convection, since the dominant field component is southward near dawn and northward near dusk. These data are also in excellent agreement with average flows deduced by barium cloud measurements [Haerendel, 1972], ground magnetometer observations [Akasofu, 1965], and radar incoherent backscatter [Doupnik et al., 1972]. It is further noted that the southward field component is typically a factor of about 3 larger than the westward component, in agreement with all of the above techniques.

To deduce large-scale magnetospheric convection patterns from the average ionospheric data of Figure 2, it is convenient to map the ionospheric electric fields to the magnetic equatorial plane. On the assumption that the variation of the potential drop along neighboring magnetic field lines is small compared with the difference in potential between the lines, the ratios of the ionospheric to the equatorial electric field components are given by geometric factors that depend on the magnetic field configuration [Mozer, 1970] and are plotted in Figure 3 for an empirical magnetic field model [Fairfield, 1968] having magnetic field lines that enter the ionosphere at three latitudes. Because the equatorial

TABLE 1. Summary of Data

Site	L Value	Number of Flights	Hours of Data
Churchill	8.2	12	160
Yellowknife	8.2	3	62
Uranium City	7.1	4	63
Great Whale	6.6	1	12
Thompson	6.4	8	126
College	5.4	3	41
McMurray	5.4	1	14

magnetic field strength varies rapidly with radius near local midnight in the field model (Figure 3, top), the ratio of the ionospheric to the equatorial electric field components, which are given in the bottom plots of Figure 3, depend critically on the latitude of the ionospheric end of the magnetic field line. Thus if the equatorial electric field strength is roughly independent of position near midnight over a region of a few earth radii near 10 R_E , the ionospheric field should vary rapidly with latitude, especially near midnight.

To determine whether such a rapid variation of ionospheric electric field strength with latitude is present in the experimental data, the curves of Figure 1 have been averaged over two latitude intervals and plotted in Figure 4. The higher-latitude data of this figure come from the first four sites listed in Table 1, and data during periods of extreme magnetic activity have been deleted from these averages. The differences between the higher- and the lower-latitude data are generally small near midnight and are not statistically significant at other times (near local noon there are only two lower-latitude measurements). Thus either the magnetic field model overestimates the radial gradient of the equatorial magnetic field at local midnight near 10 R_E , or there is a rapid radial increase of equatorial electric field strength that nearly compensates for the large magnetic field gradient. There is independent evidence that the former interpretation is more likely to be correct [Sugiura, 1972].

For the above reasons the 67° curves of Figure 3 have been used to map the mean electric fields of Figure 2 into

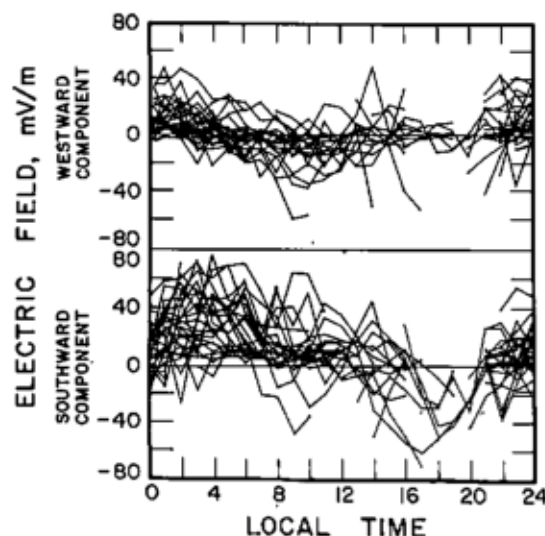


Fig. 1. One-hour averages of electric field components measured in a nonrotating frame of reference on 32 balloons flown in the auroral zone.

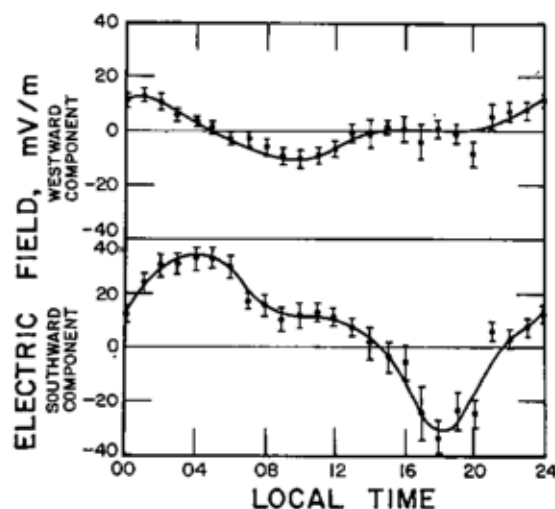


Fig. 2. Hourly averages of the data presented in Figure 1. The error bars are standard deviations of the means, and the solid curves are empirical fits to the data satisfying the constraint imposed by $\nabla \times \mathbf{E} = 0$ that the 24-h average westward field be zero.

the equatorial plane. The fields obtained by this mapping and given by the vectors in Figure 5 might be in error by factors of the order of 2 near midnight, owing to uncertainties in the magnetic field geometry. Nevertheless, certain general features of the deduced equatorial electric field pattern must still be valid. The field direction is generally from dawn to dusk, corresponding to sunward convection, and the field magnitude is typically about 1 mV/m. The field is small near 2200 and 1400 LT, corresponding to times when the average ionospheric field is nearly zero.

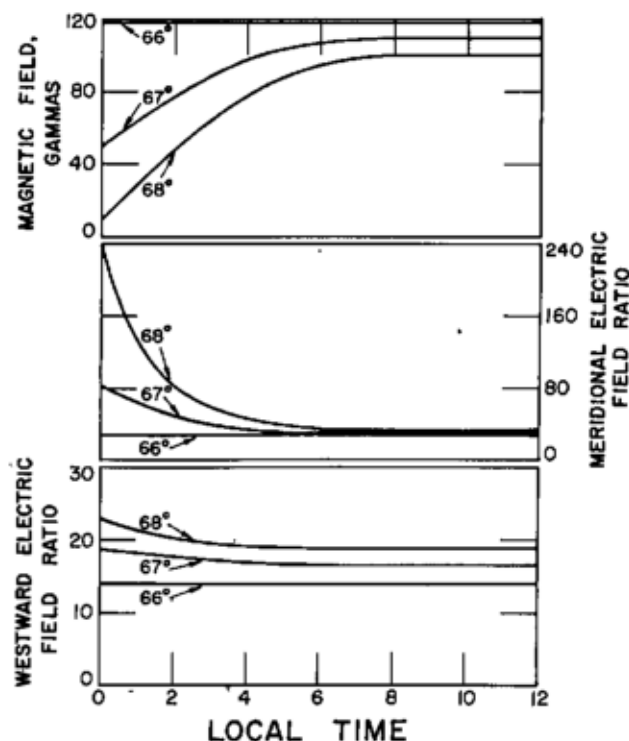


Fig. 3. (Top) Equatorial magnetic field strengths as a function of local time on field lines that intersect the ionosphere at three different latitudes. (Middle and Bottom) The ratios of ionospheric to equatorial electric field components on the same three magnetic field lines.

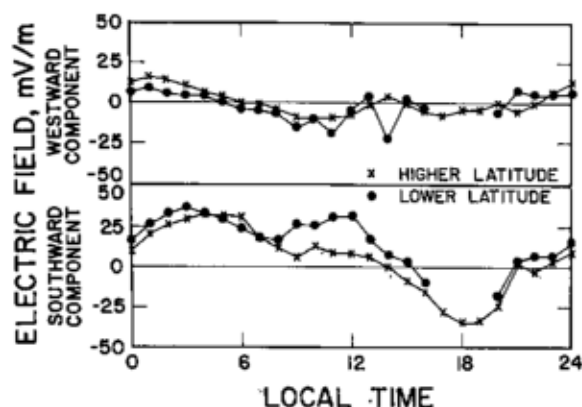


Fig. 4. Hourly averages of the electric field components measured at the higher- and lower-latitude sites.

Equipotentials of the electric field pattern are presented in Figure 5 as deduced from the measured vectors and the assumptions that the electric field is uniform across the tail at a tailward distance of $15 R_E$, that the plasmapause is an electric equipotential, and that the magnetopause is not. Since the experimental data alone do not provide enough information for drawing equipotential patterns everywhere, assumptions other than these will produce different patterns. However, the experimental data do place important constraints on such patterns. The average total potential difference inside $8 R_E$ cannot greatly exceed 30 kV. There are nulls in the flow near 1400 and 2200 LT.

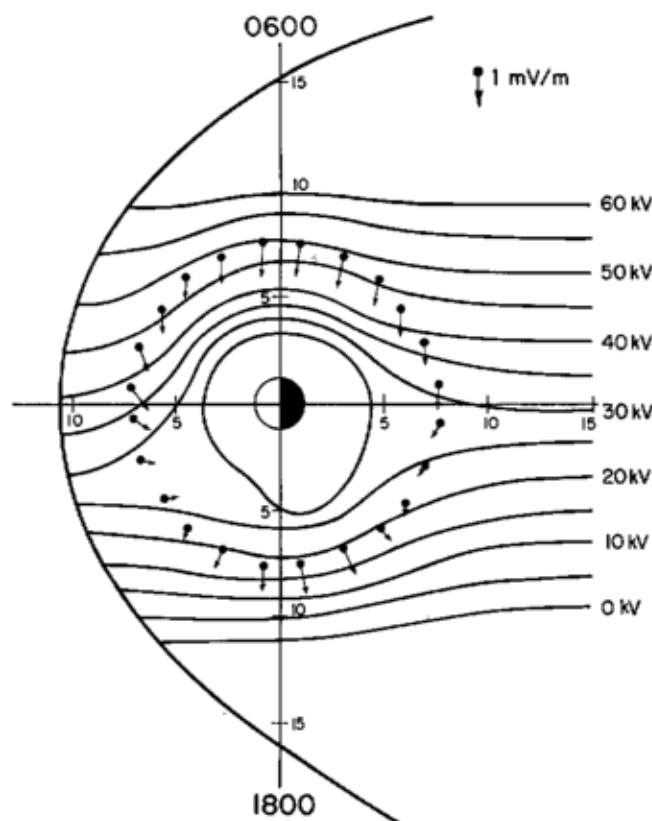


Fig. 5. Hourly averaged electric field vectors plotted in the equatorial plane of a nonrotating frame of reference, as viewed from above the north pole. The solid curves are electric equipotentials deduced from the experimental data on the assumption that the electric field is uniform across the tail at a tailward distance of $15 R_E$ from the earth.

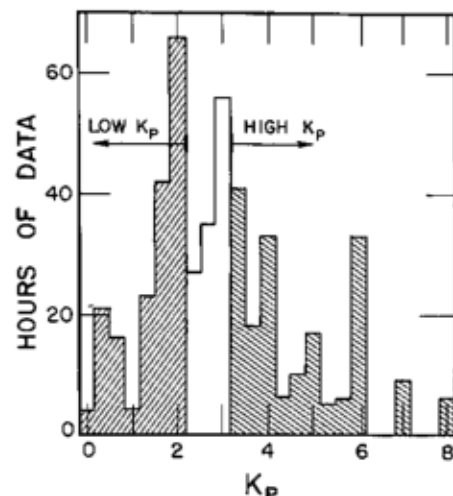


Fig. 6. Histogram of the amount of electric field data obtained as a function of K_p , shaded areas defining regions called low and high K_p .

The fact that the fields near dawn and dusk are larger than average limits the potential drop across an assumed non-equipotential plasmapause to about half of that existing if the electric field were uniform everywhere.

The experimental data and the above assumptions combine to suggest an equatorial potential drop of about 60 kV and an $E \times B$ potential flow in agreement with earlier theories (see review by *Axford* (1969)) and with expectations based on flow around a rotating impervious object. The equipotential plots of Figure 5 also generally agree with average equatorial field patterns deduced from plasma measurements [*McIlwain*, 1972, and revision, private communication, 1973].

VARIATION OF THE AVERAGE ELECTRIC FIELD WITH MAGNETIC ACTIVITY

Figure 6 presents a histogram of the number of hours of electric field data collected as a function of the planetary K index K_p . The data have been divided into high- and low- K_p intervals as illustrated in Figure 6 such that roughly 40% of the data lies in each interval. The hourly averages of the electric field components for these high- and low- K_p intervals are plotted in Figure 7. Within the statistical accuracy of this figure the electric field is roughly twice as large during periods of enhanced K_p as it is when K_p is small, but the large-scale convection electric field pattern is not strongly dependent on the level of magnetic activity.

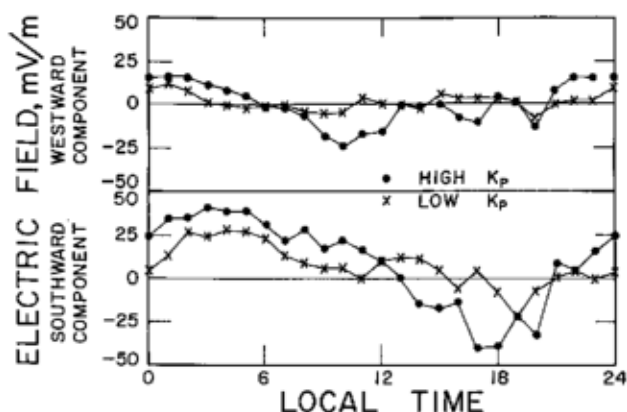


Fig. 7. Hourly averages of electric field components measured during periods of high and low K_p , as defined in Figure 6.

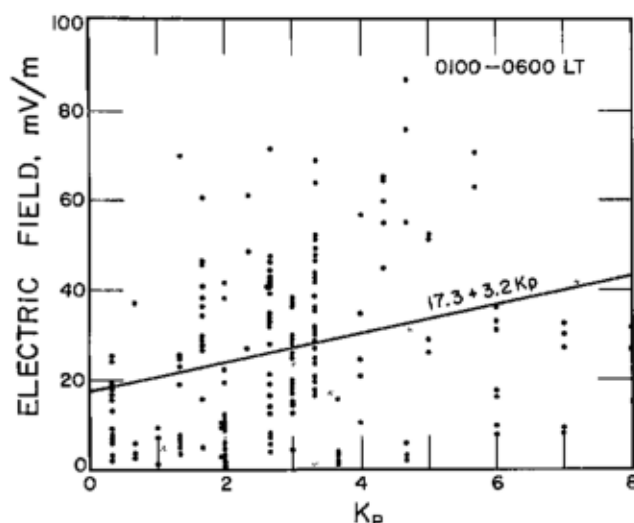


Fig. 8. Scatter diagram of 1-h averages of all electric field magnitudes measured between 0100 and 0600 LT, as a function of K_p . The solid curve is a linear least squares fit to the data.

The dependence of electric field strength on magnetic activity is further elucidated by the scatter plot of 1-h averages of the electric field strength versus K_p presented in Figure 8. Data from the local time interval of 0100-0600 have been selected for this study both because a large amount of data exists and because the average field strength

varies by less than $\pm 10\%$ with local time over this interval. Although there is a small dependence of field strength on K_p , as illustrated by the linear least squares fit to the data, the major variations of the field are not explained in this way. In conclusion the average electric field strength varies by a factor of roughly 2 with magnetic activity, but this variation is less than that caused by local turbulent variations.

DEPENDENCE OF THE AVERAGE ELECTRIC FIELD ON THE INTERPLANETARY MAGNETIC FIELD

One-hour averages of the electric field data have been divided into two subsets depending on the sign of the northward component of the interplanetary magnetic field, B_z , and the data in each subset have been averaged to produce the diurnal plots of Figures 9 and 10. The time delays Δt of these figures are the intervals between measurement of the interplanetary magnetic field and the ionospheric electric field. If attention is focused on the local time interval of 0000-1000 because differences between the two curves are statistically most significant during these times, then for any time delay between -4 and $+4$ hours it is observed that the average auroral zone electric field is roughly twice as large when the interplanetary magnetic field has a southward component as it is when the magnetic field is northward. This result is consistent with earlier electric field measurements [Mozer, 1971b; Haerendel, 1972] and offers strong evidence that the convection electric field is

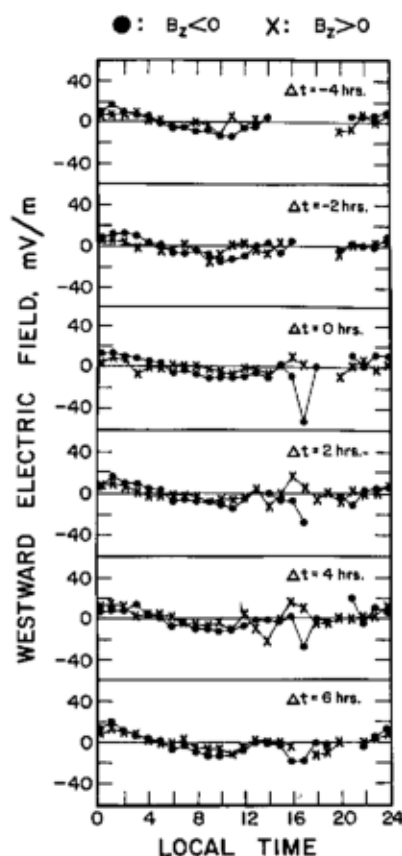


Fig. 9. Hourly averages of the westward electric field measured when the northward component of the interplanetary magnetic field was positive and when it was negative. The quantity Δt refers to the time delay between the measurement of the interplanetary magnetic field and that of the ionospheric electric field.

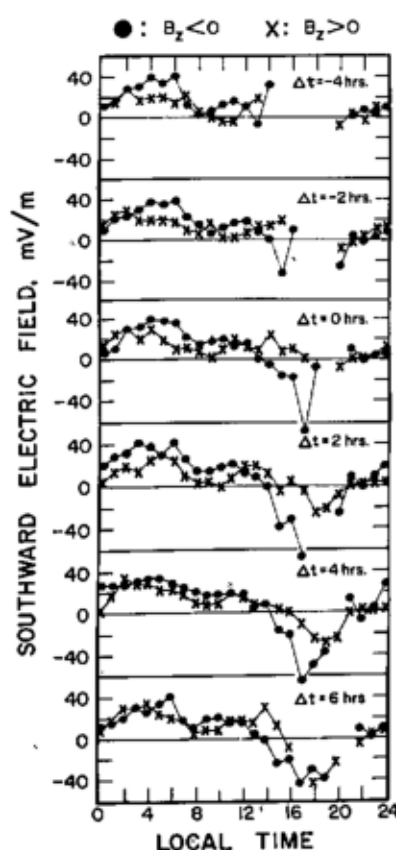


Fig. 10. Hourly averages of the southward electric field measured when the northward component of the interplanetary magnetic field was positive and when it was negative. The quantity Δt refers to the time delay between the measurement of the interplanetary magnetic field and that of the ionospheric electric field.

produced by magnetic field reconnection [Dungey, 1961]. It is also noted that the shape of the convection pattern is approximately independent of B_z and is given by the curves of Figure 5 even though the average magnitude of the convection is related to the sign of B_z . That the shapes of the curves of Figures 9 and 10 are not strongly dependent on the assumed time delay is probably accounted for by the statistical uncertainties in the curves and by the observation that the correlation time scale of the interplanetary magnetic field is apparently several hours.

The dependence of the average auroral zone electric field on the y component of the interplanetary magnetic field is given in Figures 11 and 12. For any time delay less than about 8 hours the southward component of electric field is largest in the postmidnight time interval when $B_y > 0$. The curves of Figure 12 also suggest with less statistical significance that the magnitude of the southward field component is largest near dusk when $B_y < 0$. These results may be understood in terms of a reconnection model (W. Gonzalez and F. S. Mozer, unpublished manuscript, 1973) in which the line of reconnection depends on the direction of the interplanetary magnetic field. For a purely southward magnetic field the reconnection line is in the equatorial

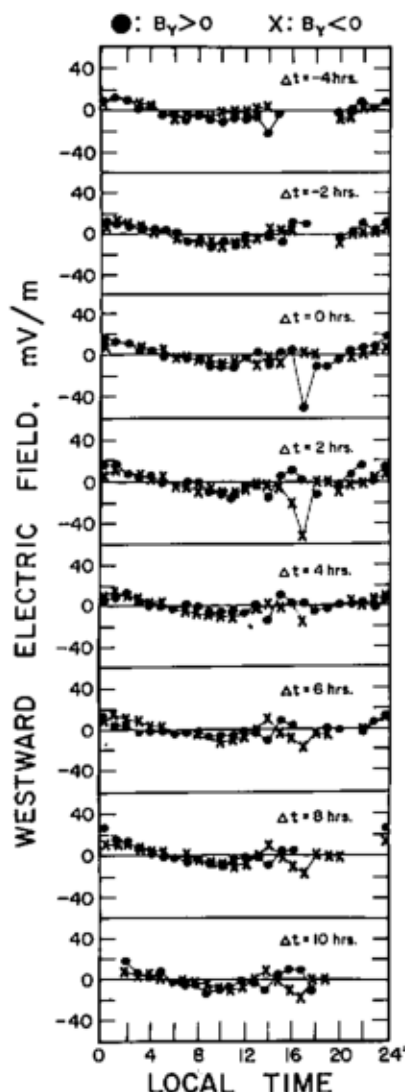


Fig. 11. Hourly averages of the westward electric field measured when the y component of the interplanetary magnetic field was either positive or negative.

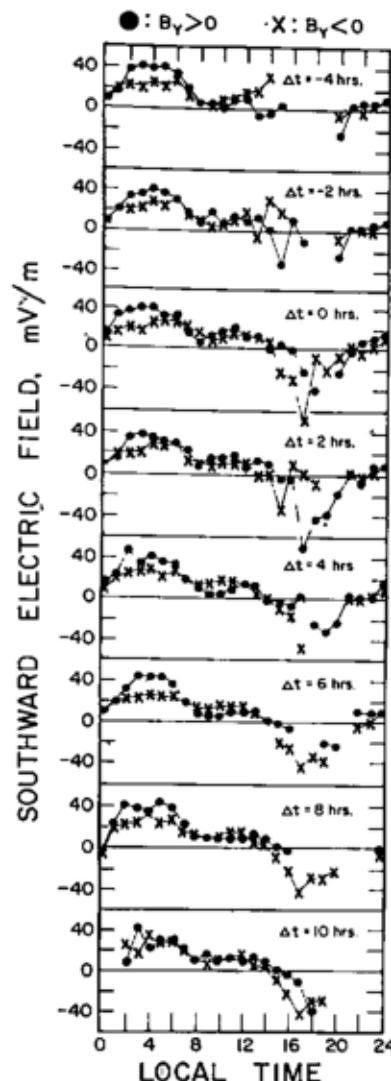


Fig. 12. Hourly averages of the southward electric field measured when the y component of the interplanetary magnetic field was either positive or negative.

plane; and the reconnection line rotates out of this plane as the interplanetary field rotates about the sun-earth line. Since the postreconnection flow is initially perpendicular to the reconnection line, the two-cell magnetospheric convection pattern tends to rotate about the sun-earth line as the interplanetary magnetic field direction rotates. This rotation causes a larger return flow at a fixed northern hemisphere sub-polar cap latitude at dawn, when $B_y > 0$, and at dusk, when $B_y < 0$. This rotation of the two-cell convection pattern has been observed in polar cap electric field data [Heppner, 1972; Mozer et al., 1974] as well as in the southward electric field component of Figure 12. It is also consistent with ground observations of magnetic field variations in the polar cap [Svalgaard, 1968; Mansurov, 1969; Berthelier and Guerin, 1972]. That the sign B_z has little effect on the westward field component of Figure 11 is understood from the facts that the major effect is expected near local dawn and dusk and that these are times when the average westward electric field is zero.

Since the hypothesized rotation of the two-cell convection pattern occurring when $B_y > 0$ causes an enhanced electric field near dawn at a fixed latitude in the northern hemisphere and a reduced field at the same time and latitude in

the southern hemisphere, the ionospheric electric fields in the dawn-dusk magnetic meridional plane at equal northern and southern hemisphere auroral zone latitudes are not equal. Figure 12 suggests that the magnitude of this difference is a reasonable fraction of the larger field strength. This result may be due to a lack of conjugacy at auroral zone latitudes near dawn and dusk or may be due to parallel potential drops that are comparable to the ~60-kV average potential drop across the magnetosphere. There is considerable experimental [Mozer and Fahleson, 1970; Kelley et al., 1971a, b; Mozer et al., 1973] and theoretical [Kindel and Kennel, 1971; Coroniti and Kennel, 1972] evidence that parallel potential drops of tens of kilovolts exist in the magnetosphere. Such results are consistent with many laboratory experiments [Hamberger and Jancarik, 1970] and plasma theoretical calculations [Buneman, 1958].

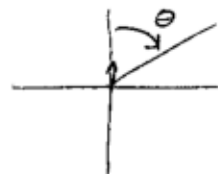
Acknowledgments. This work was supported by the Atmospheric Sciences Section of the National Science Foundation under grants GA 1317, GA 11259, GA 28307, GA 17328, and GA 33112X, by the Office of Naval Research under contracts N00014-67-A0112-0068 and N00014-69-A-02000-01015, and by NASA under grant NAS 9-9502.

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DIPOLE FIELD

Note: I spent a whole half day thinking about these things; I do not want to ever think about them again.

potential:

$$V = p \cos(\theta) / r^2$$

$$V = \frac{p \cos \theta}{r^2} = \frac{\vec{p} \cdot \vec{r}}{r^3}$$

field:

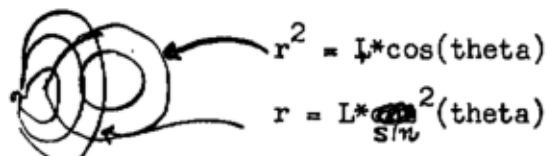
$$E_r = 2p \cos(\theta) / r^3$$

$$E_r = \frac{2p \cos \theta}{r^3}$$

$$E_\theta = p \sin(\theta) / r^3$$

$$E_\theta = \frac{p \sin \theta}{r^3}$$

equipotential lines:



$$r^2 = L \cos(\theta)$$

$$r^2 = L \cos \theta$$

field lines:

$$r = L \sin^2(\theta)$$

$$r = L \sin^2(\theta)$$

Note: the potential V is not a harmonic function in two dimensions, so complex methods are useless.

$$|B(r)| = \left(4 - 3 \frac{r}{L}\right)^{1/2} \frac{p}{r^3}$$

$$|B(\theta)| = \frac{(3 \cos^2 \theta + 1)^{1/2} p}{L^3 \sin^6 \theta}$$

$$L = \csc^2(\theta) = \sec^2(\text{LAT})$$

Invariant Phase Space:

$$\int \frac{d^3 k}{2E} = \int d^4 k \delta(k^2 - m^2) \Theta(k_0)$$

B+D p34, II.

Cabrera counts flux quanta to find a Dirac monopole

Take a superconducting loop, place it in an ultralow magnetic-field device, and monitor the current with a Superconducting Quantum Interference Device for many months. That was the experiment that Blas Cabrera of Stanford University did in his search for a moving magnetic monopole. On Valentine's Day he found a single event consistent with one Dirac unit of magnetic charge. When rumors of the observation spread, the organizers of the Third Workshop on Grand Unification phoned Cabrera in Blacksburg, Virginia (where he was giving a colloquium), and asked him to address the Workshop the next morning (16 April). After a long, wee-hours-of-the-morning drive to Chapel Hill, N. C., Cabrera electrified the workshop participants with his careful, low-key account of his possible monopole discovery.

One noted theorist remarked, "We came to scoff and stayed to praise. It was a very impressive glitch. One shouldn't be convinced by one event. But it's about as impressive as one event can be."

By using a superconducting ring, the Cabrera experiment can detect a moving magnetic charge solely from the long-range electromagnetic interactions between the magnetic charge and the macroscopic quantum state of the

ring. Such a detector is insensitive to the monopole velocity, mass, electric charge or magnetic dipole moment.

A similar approach was used in 1970 by Philippe Eberhard, Ronald Ross and Luis Alvarez (Berkeley) and Robert Watt (SLAC). They circulated moon dust (collected by the Apollo 11 astronauts, Neal Armstrong, Edwin Aldrin and Michael Collins) through a superconducting solenoid and looked for the emf generated. Cabrera's paper was published in the 17 May issue of *Physical Review Letters*.

The existence of magnetic monopoles was proposed by Paul Dirac in 1931 in an attempt to explain the existence of the smallest electric charge, e . Dirac's quantization condition said that the strength of a single magnetic pole, g , would be restricted by the relation

$$eg = \hbar c/2$$

In 1948 Dirac showed that if a magnetic charge exists, the integrity of quantum mechanics, and in particular the quantization of angular momentum, is violated unless there is a smallest electric charge.

In remarks prepared for the Monopole Meeting held in Trieste last December, Dirac said, "I am inclined now to believe that monopoles do not exist. So many years have gone by without

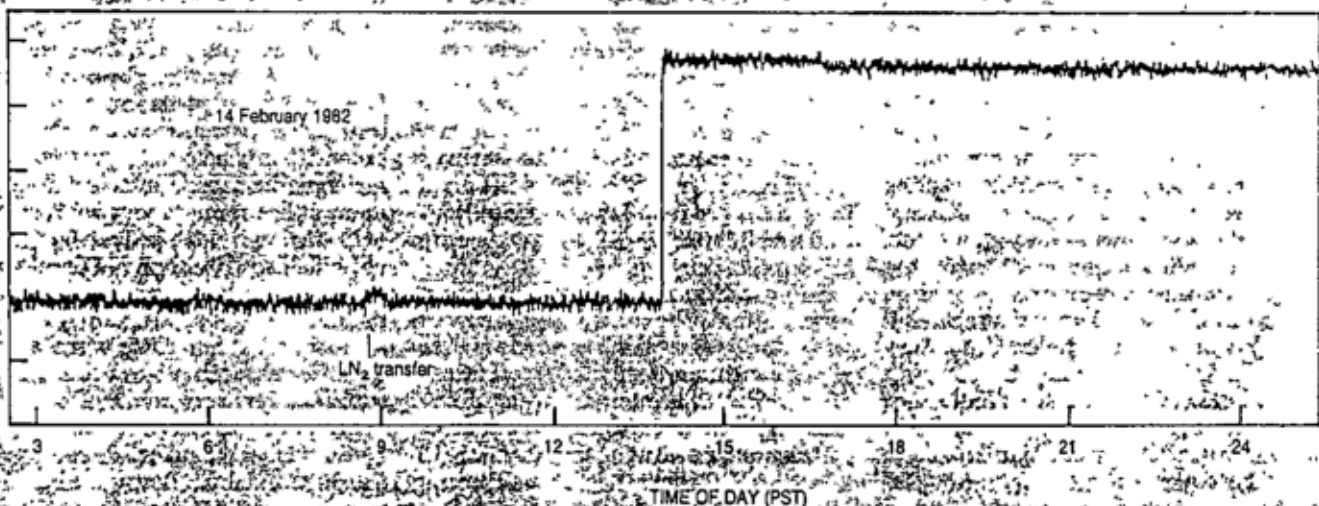
any encouragement from the experimental side."

But practically all previous searches for monopoles—in cosmic rays, at particle accelerators, and in matter—would not have detected an extremely massive monopole. Currently popular grand unified theories predict the existence of monopoles, which are indeed extremely massive— 10^{-8} grams or 10^{16} GeV.

In 1975 P. Buford Price, Edward Shirk (Berkeley), Zack Osborne and Lawrence Pinsky (University of Houston) reported finding in cosmic rays a monopole with twice the Dirac strength and a mass at least 200 proton masses (*PHYSICS TODAY*, October 1975, page 17). The Price group later withdrew its claim to have found a monopole.

Cabrera's way to detect a monopole is to use the quantized flux in a superconductor. The flux quantum $\phi_0 = \hbar c/2e$ gives a direct measurement of magnetic charge.

Cabrera's apparatus is a four-turn, 5-cm-diameter loop made of niobium, positioned with its axis vertical; the loop is connected by twisted pair leads to the superconducting input coil of a SQUID magnetometer. If a single Dirac charge passes through the loop, one would expect an $8\phi_0$ change in the flux through the superconducting circuit,



Candidate monopole event found by Blas Cabrera of Stanford University. If a single Dirac magnetic charge passes through the detection loop, one would expect an $8\phi_0$ change in the flux through the superconducting circuit.

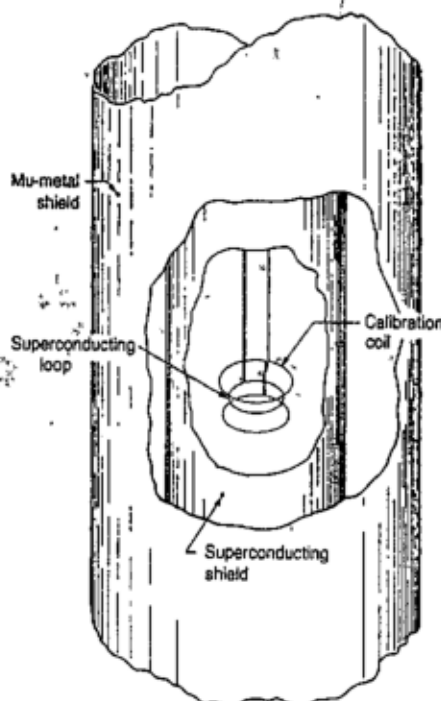
consisting of the detection loop and the SQUID input coil (a factor of 2 from $4\pi g = 2\phi_0$ and of 4 from the number of turns in the pick-up loop).

The SQUID and loop are mounted inside a 20-cm-diameter, 1-meter-long cylindrical superconducting shield closed at the bottom. Surrounding the superconducting shield is a single mu-metal cylinder that reduces the Earth's field to a few milligauss. The combined shielding provides 180 db isolation from external magnetic-field changes and an ambient field of 5×10^{-8} gauss. A flux quantum in this size cylinder would produce a field of about 3×10^{-9} gauss. So, Cabrera told us, the cylinder has a few unpaired vortices. Cabrera had used a similar ultralow field device for his PhD research (which he earned in 1974 under William Fairbank's direction).

[Blas Cabrera is a third-generation physicist. His father, Nicolas, is a solid-state physicist who spent 23 years at the University of Virginia (where his son earned his BS), and is now at the Autonomous University of Madrid. The father of Nicolas, also named Blas, specialized in magnetism and was an organizer of the 1930 Solvay Congress, along with Niels Bohr, Marie Curie and Albert Einstein. He is sometimes credited with founding modern experimental physics in Spain.]

To make the superconducting shield, Cabrera takes an expandable lead foil, 60 microns thick, and cools it to liquid-helium temperature while it is accordion-pleated into a long, thin ribbon. When the foil container becomes superconducting, it has trapped the ambient field. Then Cabrera uses a mechanical plunger to expand the foil into a cylinder, still keeping it at liquid-helium temperature. During the expansion, the magnetic field that occupied the volume of cylinder before expansion is expelled from the volume by induced supercurrents on the surface, leaving a lower magnetic field inside the foil container. Although some trapped field remains inside the shield because of the folded configuration, the inside of the cylinder has a field a factor of 10-100 lower than the field outside.

Then Cabrera takes a second folded shield ribbon, placed inside a cooling tube, which can be thought of as a vacuum jacket with a long, glass tube. The tube allows him to control the temperature of the inner shield while the outer shield is kept superconducting. He lets warm helium gas flow through the glass tube, then slowly reduces the flow so that the inner shield is cooled through its transition temperature. Because the inner shield is in the lower magnetic field provided by the previously expanded shield, a still lower field is trapped. He continues this bootstrap process, so that after



Monopole-search apparatus. The loop is connected to the input coil of a SQUID magnetometer. The combined shielding provides 180-db isolation from external field changes and an ambient field of 5×10^{-8} gauss.

three or four expansions, the field inside the apparatus is down to 5×10^{-8} gauss. Then Cabrera lowers his apparatus into the ultralow field region through an airlock. (Stanford has four such ultralow field devices operating; two of them are used for the gyro relativity experiment.) The Dewar is maintained continuously at liquid-helium temperature.

The four-turn loop is a flip coil that can be rotated 180° in the plane of the coil (to allow calibration of the absolute magnetic field). Cabrera had been using the facility for an experiment to measure $\hbar/m_e$. In his monopole search, he left the coil in a fixed position with its axis coincident to the shield axis.

By mid-May, Cabrera had been monitoring the dc current in the superconducting loop continuously for over 200 days and had seen only one candidate event—recorded at 1:53 PM on Sunday, 14 February, when the lab was unoccupied. Cabrera found the event an hour and a half later, when he read his strip chart recorder.

In his *Phys. Rev. Letter*, Cabrera reports that the event is consistent with the passage of a single Dirac charge within a combined uncertainty of $\pm 5\%$. It is the largest event of any kind in the record. He found a total of 27 events exceeding a threshold of $0.2 \phi_0$, which remain after excluding known disturbances such as transfers of liquid helium and nitrogen. He defines an event as a sharp offset with

well-defined stable levels for at least one hour before and after. Only six events were recorded during the 70% of the running time when the lab was unoccupied.

Sources of error. Discussing possible sources of spurious detector response, Cabrera says line-voltage fluctuations caused by two power outages and accompanying transients did not cause detectable offsets. Radio-frequency interference from the motor brushes of a heat gun failed to produce any offsets when operated close to the detector. The critical current of the loop is not reached for currents 1000 times greater. No seismic disturbance occurred on 14 February.

Could an energetic cosmic ray hitting the wire cause a piece of it to go normal? Cabrera says that such rays could deposit as much as 1 GeV/cm in traversing the wire, raising the local wire temperature by about 0.01 K. But a 5-K change would be needed to reach the critical temperature.

Cabrera has intentionally generated mechanically induced offsets by hitting the detector with a screwdriver handle, for example. Out of 25 attempts, two produced offsets up to $6\phi_0$, but the signal was not clean enough to mimic a monopole event, Cabrera told us. There was always an overshoot and settling-in period lasting several hours.

The only reasonable occurrence that would mimic a monopole passage, Cabrera feels, is a spontaneous internal stress-release mechanism. If the turns in the loop shifted with respect to each other, the inductance would change and produce a current change. A 1% shift in inductance could mimic the event.

Cabrera says his signal-to-noise level approaches 15:1 for the passage of a single Dirac charge. The 1970 experiment of Alvarez and his collaborators was not sensitive to a single Dirac charge in a single pass. So they circulated their moon-rock samples several times through the superconducting coil to increase the signal-to-noise ratio. Cabrera told us that the advantage of this type of experiment, compared to other monopole searches, is that a particle with any velocity or mass carrying a magnetic charge affects the experiment in the same way. The coupling is only through long-range electromagnetic fields. Although his present system, with its low bandwidth, is unable to discern the expected monopole velocity (about $10^{-3} c$), some SQUID systems built for research have been made with sufficient bandwidth.

Improved experiment. Cabrera, Michael Taber, Susan Felch, Robert Gardner and John Bourg are building a new detector with three mutually orthogonal loops wound on a spherical Pyrex bulb. Unlike the flip coil, the three-

loop arrangement is highly stable mechanically. For hypothetical monopoles passing through the loops, about 70% of the trajectories will intersect at least two of the loops, providing coincidence information. The total area is ten times that of the flip coil. An additional factor of five in effective detecting area is gained because Cabrera also will be able to detect near-miss trajectories that do not pass through because they would cause a field change within the ultralow-field shield. Cabrera expects that the group will be taking data with the new equipment beginning this month.

Cabrera stresses that the event he has reported is "not yet a discovery. It's an interesting event. We're working hard with new apparatus to see definitively if there are such particles. It's tough to be very extravagant with only one data point. But it's also difficult to make it go away. We're caught on a knife edge."

Theory. In 1974 Gerard 't Hooft (University of Utrecht) and A. M. Polyakov (Landau Institute for Theoretical Physics, Moscow) independently showed that if you have a non-Abelian gauge group that is semisimple and that spontaneously undergoes a breakdown of scale, such theories have solutions corresponding to Dirac magnetic monopoles. They further showed that the mass of the monopole would be the characteristic scale of symmetry breaking divided by the fine-structure constant. That same year Howard Georgi, Helen Quinn and Steven Weinberg (all then at Harvard) showed that for a large class of grand unified theories, the unification scale—the region where strong, weak and electromagnetic couplings become equal—is about 10^{14} GeV. All the grand unified theory predictions for monopole mass are in the ballpark of a hundred times the unification scale, that is 10^{16} GeV.

If any monopoles exist now, presumably they would have been produced in the very early Universe, about 10^{-35} sec after the Big Bang, at a time when the unified interactions break apart into strong, weak and electromagnetic.

Cabrera cites an observational upper bound on the mass density of monopoles to be given by the local "missing mass." This limit is in the range 0.03–0.05 solar masses/cubic parsec. One can assume that because the monopoles are so massive their velocities would be no greater than about 300 km/sec (as Alvarez says, just sauntering by an atom so that little or no ionization takes place). Then Cabrera estimates that the number of monopoles passing through the Earth's surface would be 4×10^{-10} cm⁻² sec⁻¹ ster⁻¹. Such a flux would yield 1.5 events per year through his loop.

In 1969 Eugene Parker (University of

Chicago) pointed out that if monopoles are distributed throughout the galaxy, they would leach energy out of the galactic magnetic field and soon destroy the field. Some argue that monopoles are concentrated locally, but it's difficult to produce a mechanism for this concentration. At the GUT workshop in April, Glashow mentioned an idea developed by him, Savvas Dimopoulos, Edward Purcell (Harvard) and Frank Wilczek (University of California, Santa Barbara). They speculate that the monopole flux arriving on Earth originates in the Sun. Alvarez has called attention to measurements of the Sun's magnetic field for the last five "quiet periods" that once led John Wilcox (Stanford) to write a 1972 paper called, "Why does the Sun sometimes

look like a magnetic monopole?" A few months ago Alvarez calculated, assuming those measurements are correct, that the monopoles in the Sun must have masses greater than 10^{12} GeV to keep them together by gravitational attraction in spite of their mutual repulsion.

Perhaps monopoles collect at the center of the Earth, too, Alvarez speculates. You couldn't hope to put a monopole on a table and experiment with it. As Paul Frampton (University of North Carolina) points out, "Because of its enormous mass, the thing would be unimpressed by a table. Earth's gravitational pull could be greater than the electromagnetic force between a monopole and a typical atom. So the monopole would go straight through."—GEL

Hunting neutron—antineutron oscillation

When Murray Gell-Mann and Abraham Pais predicted in 1955 that the K^0 meson should exhibit an oscillating probability for metamorphosis into its antiparticle, the $\bar{K}^0$, they cited the neutron as a counterexample. The conservation of baryon number, they pointed out, would prevent the neutron-antineutron analog of neutral-kaon oscillation. But nowadays, with grand unified theories of the elementary particles very much in favor, all bets based on baryon-number conservation are off. A prodigious experimental effort (PHYSICS TODAY, January 1980, page 17) attests to the widespread expectation that proton decay will be seen with a lifetime of about 10^{31} years.

But the same SU(5) grand unified theory that predicts this finite proton lifetime does not (in its simplest form) permit neutron oscillation. The SU(5) unification of quarks and leptons in a single gauge-theoretic framework, proposed by Howard Georgi and Sheldon Glashow (both at Harvard) in 1974, replaces baryon conservation by the conservation of $B - L$, the difference between baryon and lepton number, thus forbidding $n \leftrightarrow \bar{n}$, the neutron-oscillation transition.

There are, however, rival unification theories. Robert Marshak (Virginia Polytechnic Institute) and Rabindra Mohapatra (City College of New York) have recently put forward¹ a "partial (as distinguished from grand) unification theory" based on a left-right symmetric electroweak scheme which, when joined to the usual color-SU(3) group of quantum chromodynamics, yields a color-SU(4) group structure mathematically similar to an earlier proposal of Jogesh Pati (University of Maryland) and Abdus Salam (International Centre for Theoretical Physics, Trieste, and Imperial College, London).

In contrast to the SU(5) grand unification, the Marshak-Mohapatra theory predicts neutron oscillation on a time scale that may well be accessible to experiment. Proton decay, on the other hand, is not permitted in the Marshak-Mohapatra partial unification scheme.

One could complicate the spontaneous-symmetry-breaking mechanism in the SU(5) theory to get neutron oscillations at an observable level. But such departures from "minimal SU(5)" would deprive the theory of much of its simple elegance, Georgi contends. "I would be greatly surprised if we find neutron oscillation," he told us. "But if we do, it will be most instructive."

Aesthetic prejudices notwithstanding, Lay-nam Chang (Virginia Polytechnic Institute) and Ngee-pong Chang (City College of New York) have proposed² a modification of SU(5) that would permit both neutron oscillation and proton decay at observable levels by extending the theory's minimal symmetry-breaking mechanism to include Higgs bosons with masses around 10^4 or 10^5 GeV.

Because neutron oscillation plays such a central role in choosing among competing unification theories, a number of experimental attempts to look for this exotic phenomenon are now in various stages of planning, and one group has recently announced preliminary results. At the International Conference on Baryon Nonconservation, held in Bombay in January, Milla Baldo-Ceolin (University of Padua) reported the first results obtained by a CERN, Laue-Langevin, Padua, Rutherford, Sussex collaboration using the cold-neutron facilities of the Laue-Langevin Institute at Grenoble. Their first null finding—that the characteristic neutron oscillation time is greater than

UNITS

Details are in a separate document, here we just want to get the summary ideas.

There are three equations of interest

$$F = k_1 * qQ/r^2 \quad (1) \quad / \text{ force between two charges (electrostatics)}$$

$$dF/dx = k_2 * 2il/D \quad (2) \quad / \text{ force between two parallel wires (magnetism)}$$

$$k_1 = c^2 * k_2 \quad (3) \quad / \text{ see other doc for how this is shown.}$$

$$P = V * I \quad (4) \quad / \text{ power = potential * current}$$

The fact that constants k_1 and k_2 must be connected is clear since the charge appears in both equations (1) and (2). If we double the unit of charge in both equations, we have to reduce *both* k_1 and k_2 by 4, so they are proportional. The fact that the constant in (3) is c^2 comes from the place where electricity and magnetism meet -- waves in space.

Now in the beginning there was only electrostatics, before 1800 say, and the cgs-esu system was used where $k_1=1$. This defined a reasonable working unit of charge, the esu = statcoulomb. No one cared about equation (2) because it was not known yet.

After 1800, batteries and currents in wires appeared, and (4) appeared on the scene just from work with DC currents.

Soon, in the 1825 time frame, equation (2) appeared, and people who worked with currents and wires and magnetic fields and such used the cgs-emu system wherein $k_2=1$. This made $k_1 = c^2$ so the unit of charge here, the abcoulomb, was ridiculously large for electrostatic work, but fine for magnetic work. This system of course has an abvolt as well, which turns out to be quite small.

Since batteries typically put out 1.5 e8 abvolts, it was convenient to rescale and define a "volt" such that batteries then put out 1.5 volts, so $\text{volt/abvolt} = 10^8$. Current was still in abcoulombs/sec = abamperes.

Next, people starting liking the mks system with its joules and watts and meters and kg, all being closer to the human scale of things, so more appropriate for building things (engineering). So, instead of using a mixed system of volts and abamperes, a new system was developed called "the practical units".

In this new system, planners wanted to use the normal mks units and the volt. In equation (4) above, since P was watts and V was volts, a current unit was implied which was 1/10 of an abampere. This is simply because $\text{joule/erg} = 1e7$ and $\text{volt/abvolt} = 1e8$, so dividing gives 1/10. This new unit is the ampere, and the corresponding charge is the coulomb. Given this charge unit, and comparing that unit say to the esu, it was then easy to figure out what the right values are for k_1 and k_2 for this new system. It is easy to show with unit conversions that $k_2 = 10^{-7}$, written as $\mu_0/4\pi$ to make $H = \mu_0 B$ in free space:

$$dF(\text{dyne}) \sim 1 * i(\text{abamp})^2 \Rightarrow \text{nt/dyne} = 1/k_2 * (\text{amp}/\text{abamp})^2 \quad 1e5 = 1/k_2 * 1e-2$$

$$dF(nt) \sim k_2 i(\text{amp})^2$$

This made $k_1 = 10^{-7} \text{ c}^2(\text{mks}) = 9 \times 10^9$ written as $1/(4\pi\epsilon_0)$ so that $D = \epsilon_0 E$ in free space. The fact then that $k_1/k_2 = \text{sqrt}(1/(\mu_0\epsilon_0)) = c^2$ is then pretty much a no-brainer.

So this is the SI or mksa or practical system. We end up with volts, amps, watts, Coulombs, Teslas volts/meter, Webers and all that stuff. Everyone is happy, except that B and E have different units which is ugly since they are both really part of the same $F^{\mu\nu}$, and we have these two constants floating around everywhere to annoy us. A theorist would rather have c floating around in equations which at least means something and can be set to 1 by rescaling time, compared to c split into two pieces μ_0 and ϵ_0 each of which mean nothing at all.

One other comment. It is true that E is strong and B weak in normal life as v/c , and this is most clearly seen in the cgs-esu units where the magnetic equation has $k_2 = 1/c^2$ quashing magnetic effects. The reason we end up with E and B in different units in mksa is that this makes them similar in size for human life. The natural units (ie, same for E and B) make E much larger than B for humans. However, if we did things with large velocities like $c/2$ in our normal lives, E and B would be similar in size. This would be as if our natural time scale was nsec instead of sec. Imagine pieces of our equipment that move $c/2$ relative to each other, we would have similar E and B fields floating around in the natural units.

1. About Units in General

Define versus Use. Some equations *define* a new unit, other equations just *use* the unit in some manner in the solution of a problem. For example, $E=mv^2/2$ and $E=mgh$ both use an energy unit on the left, but neither is the defining equation for an erg. Perhaps $E = FxD$ is the defining equation here. Although $E=mv^2/2$ is true for what it means, since we are not defining the erg by this equation, we would never say $1 \text{ erg} = 1/2 \text{ gm-cm/sec}^2$ the way we might say $1 \text{ foot} = 12 \text{ inches}$. But we *would* use the general notion that $\text{erg} = \text{gm-cm/sec}^2$ in terms of "dimensions" to check work we do, see next item. A third type of equation is used to *rescale* a unit, such as $L(\text{in}) = 12 L(\text{ft})$ and for such an equation, it is always correct to say that $1 \text{ ft} = 12 \text{ in}$. See below.

Two ways of handling units in an equation. Consider:

$$\begin{array}{ll} F(\text{nt}) = m(\text{kg}) a(\text{m/sec}^2) & F = 36 \text{ kg} * 10 \text{ m/sec}^2 = 360 \text{ nt} \\ & F = m \text{ kg} * a \text{ m/sec}^2 = m * a \text{ nt} \end{array}$$

In the equation on the left, $m(\text{kg})$ means "mass measured in kilograms". In this form, $m(\text{kg})$ is just a number, it has no units! We have already handled the units with the parentheses. The two numbers on the right when multiplied give the number on the left. When we solve a problem, however, we often write things as done on the top right. The method shown on the bottom right seems confusing to me -- writing units like this seems better suited to doing numerical calculations and making sure you get no errors. With numbers, you will cancel and combine units as required and that acts as a check on your work. But from a theoretical point of view, I find the notation on the top left to be much clearer and unambiguous. Sometimes equations have a different form if one uses different units, so $F = ma$ would not be "enough" if that were the case for that equation, which it is not. But for the Coulomb force law, it is the case!

Defining a Unit by Rescaling. Consider:

$$1 \text{ foot} = 12 \text{ inches} \qquad L(\text{inch}) = 12 L(\text{foot})$$

The thing on the left is the *definition* of a foot, if the inch is already defined. Notice how the labels are reversed in the right equation.

Converting Equations to Different Units. The above is an example of a rescaling of some unit. Here is my general notation for doing this:

$$\begin{array}{ll} L(\text{inch}) = L(\text{foot}) * \{ \text{foot/inch} \} & // \{ \text{"right over left"} \} \\ \text{or} & \\ 1 \text{ foot} = 1 \text{ inch} * \{ \text{foot/inch} \} & // \{ \text{"left over right"} \} \end{array}$$

The symbol $\{ \text{foot/inch} \}$ is meant as a normal dimensions fraction. You put $\text{foot} = 12 \text{ inch}$ on the top, cancel the inch, and end up with $\{ \dots \} = \text{the number } 12$. $\{ \text{foot/inch} \}$ does **not** mean "feet per inch",

because if it did, that would give 1/12 which is not what is meant! When a dimension is on the bottom, you have to reverse the rule. For example:

$$\begin{aligned} v(\text{m/sec}) &= v(\text{mile/hour}) * \{\text{miles/m}\} * \{\text{sec/hour}\} \\ &= v(\text{mile/hour}) * \{\text{mile/ft}\} * \{\text{ft/in}\} * \{\text{in/cm}\} * \{\text{cm/m}\} * \{\text{sec/hour}\} \\ &= v(\text{mile/hour}) * \{5280\} * \{12\} * \{2.54\} * \{1/100\} * \{1/3600\} = .45 v(\text{m/hour}) \end{aligned}$$

This provides a 100% error-free clean way to convert any quantity from any units to any other units! We will use this method many times in what follows.

2. Units used in Electricity and Magnetism; comments applicable to all Systems

The Two Basic Equations for Defining Charge and Related Quantities. The first of these is associated with Mr. Coulomb's work done around 1788, now known as his law. The second is associated with Mr. Ampere's work done around 1827 (and other work of Biot and Savart and Oersted around the same time). Following Jackson's appendix, we install constants k_1 and k_2 into these equations:

$$F = k_1 * qQ/r^2 \quad (1) \quad / \text{ force between two charges}$$

$$dF/dx = k_2 * 2iI/D \quad (2) \quad / \text{ force between two parallel wires}$$

The second equation is really an integration of a more primitive equation which is this:

$$dF = k_2 * id\mathbf{x} \times (I d\mathbf{X} \times \mathbf{r})/r^3 \quad (3)$$

so the factor of 2 appearing in (2) just comes from the integration to get that result.

Sometimes (1) is used to define a unit of charge, and sometimes (2) is used. Different choices of k_1 and k_2 yield the different "systems of units" used in E&M.

How are k_1 and k_2 related to Maxwell's Equations? The first order of business is to say how we are going to define the electric field. Every system does this in the same way *with no constant* as follows:

$$F = qE \quad \text{from which we conclude that:} \quad E = k_1 * Q/r^2$$

That is to say, there is no constant between F and qE . The associated Maxwell equation must have this form as we show in the words that follow:

$$\text{div} \mathbf{E} = 4\pi k_1 \rho \quad \int \mathbf{E} \cdot d\mathbf{A} = \int \text{div} \mathbf{E} dV = 4\pi k_1 \int \rho dV$$

When we integrate the $\text{div} \mathbf{E}$ equation (Gauss's Theorem) over a sphere around a point charge, doing $\int \mathbf{E} \cdot d\mathbf{A}$ makes 4π from the angle integral, so there has to be a 4π put into the $\text{div} \mathbf{E}$ equation RHS to cancel it. The main point is that the k_1 appearing in (1) also appears in the $\text{div} \mathbf{E}$ equation.

In similar fashion we can rewrite (3) as

$$d\mathbf{F} = (1/\alpha) \, \mathbf{id}\mathbf{x} \times d\mathbf{B} \quad d\mathbf{B} = \alpha \, k_2^* (\mathbf{Id}\mathbf{x} \times \mathbf{r})/r^3 \quad // \text{ 1st is Biot-Savart} \quad (4)$$

We should think of the first equation in the manner of $\mathbf{F} = q\mathbf{E}$. For electrics, we use "q" as a test charge to sense the E field. For magnetics, we use "idx" as a test current to sense the B field. When equation (3) is split into two parts this way, allowing a definition of the B field, we have the chance to add a new constant which we might have called k_4 , but Jackson calls it α so we will call it α as well. Since it cancels when the first two equations are glued back together to give (3), it can be anything. [In the Gaussian cgs system it is set to c, in mks to 1, for example.]

Of course when we later learn that E and B are part of the same tensor $F^{\mu\nu}$, we must choose k_1 , k_2 and α so E and B had the same dimensions, and this is one major benefit of the cgs-Gaussian system. This is of little concern in the practical units.

So the right equation defines the B field produced by a practical current, and the left shows the force of that field on a test current. But the left equation now stated this way *also* applies to the force on a test current in the presence of a fixed-magnet field B. We now know this is due to the same current idea, but it was not known in 1810 (say), so this first formula correlated fixed-magnet B with electro-magnet B and implicitly said the two B's were the same thing. The internal currents idea of M then quickly followed.

As noted, the first equation is like $\mathbf{F} = q\mathbf{E}$, but because it is more complicated in nature and correlated previously separated ideas, it gets a name associated with it, in honor of Biot and Savart who, in 1820, discovered the law as the force on a wire in the presence of a fixed-magnet. Oersted in 1819 had found that a current deflected a compass, but did not measure or analyze the nature of that force.

The associated Maxwell equation is this (Ampere's Law)

$$\text{curl } \mathbf{B} = 4\pi \alpha \, k_2 \, \mathbf{J} \quad \int \mathbf{B} \cdot d\mathbf{l} = \int \text{curl } \mathbf{B} \cdot d\mathbf{A} = 4\pi \alpha \, k_2 \int \mathbf{J} \cdot d\mathbf{A}$$

To see the connection here, the simplest case is the straight wire. Integrate dB from (4)-right over the wire and you get $B = k_2 \, 2 \, I/r$. Compute this same B from Stokes theorem applied to the curl equation shown above, you get $2\pi r B$ on the left from the line integral, and you get $4\pi \, k_2 \, I$ on the right, so $B = k_2 \, I/r$ again. This shows why we had to put the 4π and k_2 into the curl B equation.

So, in this section we have related our constants to the two Maxwell equations that have sources (ρ and $\mathbf{J}$). The divB equation is uninteresting since it is just 0. We still have to worry about the 4th Maxwell equation about curl E, and we have to worry about the "other term" in the curl B equation, known as the "displacement current". We do these things right now.

A simple problem regarding the term $\text{curl } \mathbf{B} = k_4 \, \partial \mathbf{E} / \partial t$.

Let's assume a constant k_4 in this equation in current-free space and see if we can connect it with k_1 or k_2 . To do this, let's study a simple problem. Imagine a tiny ring (radius a) moving toward a point charge at speed v, all flat-on and symmetric. If I apply the above bolded curl rule to this problem,

$$\int \mathbf{B} \cdot d\mathbf{l} = \int \text{curl } \mathbf{B} \cdot d\mathbf{A} = +k_4 \, d/dt \left(\int \mathbf{E} \cdot d\mathbf{A} \right)$$

Using this formula, we can compute the B field in the tiny ring to be:

$$B = k_1 \, k_4 \, Q \, a \, v / r^3$$

In this problem, the ring is so small, that E is constant on the ring, so electric flux is just $\pi a^2 E = \pi a^2 k_1 Q/r^2$. Putting $k_4 d/dt$ on this makes $-k_4 2 v/r^3$, while the left side gives $2\pi a B$. The $2\pi a$ cancels on both sides, giving the above result.

Now I treat the same problem by sitting on the ring and watching Q approach at speed v , I can identify Qv with $I dx$ in the Biot-Savart B formula above, and I get that

$$B = \alpha k_2 Q a v/r^3$$

where $\sin\theta$ from the cross product is a/r . Comparing these two results, we find that,

$$k_4 = \alpha k_2/k_1.$$

Notice that this is just what Jackson has put in his (A.8).

A simple problem regarding curl $E = -k_3 \partial B/\partial t$ (Faraday's Law)

The simple problem needed here is to put a square loop at the edge of a region of constant B field, say between the poles of a magnet, and have the loop move to the right, say, at speed v . The equation of interest here is this

$$\int E \cdot dl = \int \text{curl} E \cdot dA = -k_3 d/dt (\int B \cdot dA)$$

In our simple problem, the change of flux on the right is Byv where y is the height of the loop, so the RHS is then $k_3 Byv$ in magnitude. The electrons in the moving loop wire feel a force $dF = (1/\alpha) i dx \times dB$ as noted above. Putting $J = qv\delta(r-a)$ and integrating in all 3D over a tiny "wire", $i dx = qv$, and we then get that the force is $F = (1/\alpha) qvB$. We interpret this as being due to an electric field $E' = F/q = (1/\alpha)vB$ in the rest frame of the wire, and the line integral then gives $(1/\alpha) yvB$. Thus, $k_3 = 1/\alpha$.

Jackson on page 172 deals with this issue in a more general way, and here is a summary of his method. You write the usual Stokes equation with the above curl rule with constant k_3 (he calls it k there). You then consider the situation of a current loop moving in the presence of a lab B field of some sort. The electric field in equation 6.4 is denoted E' to remind us that it is the field in a frame moving with the wire, because a contour integration is done in that frame. That is a key point. We are interested in the field that pushes a current in the wire, and that field must be the field that is in the rest frame of the wire. So I think (6.4) is evaluated in the lab, but you use the field E' as just stated. Then in (6.5) he computes the flux change in the lab and comes up with the famous two terms -- I proved the second term on a separate sheet. This lets you write (6.6) and then interpret E as the field in the lab. We have then proven the field transform law (6.8) for non-relativistic velocity v . We still have our k_3 in this formula. Now comes the magic trick. In the lab, an electron in the wire is in fact a current of the form $q v$. We know from Biot-Savart that such a current feels a force from the B field, and that force is $(1/\alpha) q v \times B$. So Jackson's conclusion is that $k_3 = 1/\alpha$. (In this section he uses k for k_3 , but uses k_3 in his appendix.)

Maxwell's Equations and the Connection to Speed of Light. So far then, based on the discussions above, we have these Maxwell's Equations:

$$\text{div} \mathbf{E} = 4\pi k_1 \rho$$

$$\text{curl} \mathbf{E} = - (1/\alpha) \partial \mathbf{B} / \partial t \quad (\text{Faraday})$$

$$k_1 = c^2 \cdot k_2$$

$$\text{div} \mathbf{B} = 0$$

$$\text{curl} \mathbf{B} = 4\pi \alpha k_2 \mathbf{J} + (\alpha k_2/k_1) \partial_t \mathbf{E}$$

It took some doing just to get to this point! Now if you do $\nabla \times \nabla \times \mathbf{E}$ and get the free-space wave equation, you conclude that $k_1/k_2 = c^2$ in your system of units whatever it is -- we have not picked one yet. This is because plane waves must travel the speed of light. If you start with $\nabla \times \nabla \times \mathbf{B}$, you reach the same conclusion. Either way you pick up both curl equations and you therefore conclude that $k_1/k_2 = c^2$.

Do the constants k_1 and k_2 and α have dimensions?

If we look back at our original two defining equations in which k_1 and k_2 first appear, and if we imagine that each of our constants have some kind of dimensions, then the dimensions in these two equations are only consistent IF this is true: (k_1/k_2) must have the dimensions of m^2/sec^2 (say in mksa). We know that in fact $k_1/k_2 = c^2$ so this all makes perfect sense. This does NOT tell us what we have to do for k_1 and k_2 separately, however. Similarly, constant α may or may not have dimensions, since we can make it be anything we want. Since α is a rescaling of the B-field, giving it dimensions will simply change the dimensions of the B field.

There are really only two reasonable choices to make about dimensioning k_1 and k_2 . One choice is to give each of these units its "natural" dimensions such that the dimensions cancel on both sides of the two basic equations.

$$F(\text{force}) = k_1(\text{force} \cdot \text{L}^2/\text{charge}^2) * q(\text{charge})Q(\text{charge})/r(\text{L})^2 \quad (1)$$

$$dF(\text{force})/dx(\text{L}) = k_2(\text{force} \cdot \text{time}^2/\text{charge}^2) * 2i(\text{charge}/\text{time})I(\text{charge}/\text{time})/D(\text{L}) \quad (2)$$

In this case, we cannot make any connection between the charge unit and the other units. This is a good approach to take if neither k_1 nor k_2 is going to be unity.

The other reasonable choice is to make $k_1 = 1$ with no dimensions, or to make $k_2 = 1$ with no dimensions. If this is done with k_1 , then the equation is regarded as making a connection between the unit of charge and the base units of force and length. You get $\text{force} = \text{charge}^2/\text{L}^2$. If this is done instead with k_2 , then you get $\text{force} = \text{charge}^2/\text{time}^2$ which is different!

The mksa system takes the first approach noted above. The esu systems sets $k_1 = 1$ and this defines the esu unit of charge such that $\text{dyne} = \text{esu}^2/\text{cm}^2$. The emu systems set $k_2=1$ so $\text{dyne} = \text{emu}^2/\text{sec}^2$

What about D and H?

As Jackson points out on page 617, this allows 6 more constants to appear in your world:

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \lambda \mathbf{P} = \epsilon' \mathbf{E}$$

$$\mathbf{H} \equiv (1/\mu_0) \mathbf{B} - \lambda' \mathbf{M} = (1/\mu') \mathbf{B}$$

I have used ϵ' and μ' because, although Jackson says everyone uses μ and ϵ , I see that Bleaney on page 19 have $\mathbf{D} = \epsilon \epsilon_0 \mathbf{E}$, so I would then say $\epsilon' = \epsilon \epsilon_0$ for Bleaney, whereas Portis on p 112 uses just $\epsilon' = \epsilon$.

Since we are defining D and H by these equations, we are really free to set all 4 constants arbitrarily. Of course P and M are zero in free space. In this limit, ϵ becomes ϵ_0 and μ becomes μ_0 and this is the

motivation for these notations, as if these numbers were properties of free space. Jackson says that λ and λ' are always both 1 or 4π .

It is desirable to have the $\text{div} \mathbf{D}$ and the $\text{curl} \mathbf{H}$ equations "look nice".

From above we have $\text{div} \mathbf{E} = 4\pi k_1 \rho$, so in free space we would then have $\text{div} \mathbf{D} = (4\pi \epsilon_0 * k_1) \rho$. The choice of k_1 and ϵ_0 is always coordinated so that the factor in parens is either 4π or 1, for "unrationalized" or "rationalized". In all systems shown on page 618, these are the only two possibilities for $\text{div} \mathbf{D}$.

The other interesting equation is $\text{curl} \mathbf{B} = (4\pi \alpha k_2) \mathbf{J} + (\alpha k_2/k_1) \partial_t \mathbf{E}$. In free space we would have this result: $\text{curl} \mathbf{H} = (4\pi \alpha k_2/\mu_0) \mathbf{J} + (\alpha k_2/(k_1 \mu_0 \epsilon_0)) \partial_t \mathbf{D}$. Again, k_2 and μ_0 are chosen so the first factor (in front of the $\mathbf{J}$) is either α or $4\pi\alpha$. In Jackson's page 618 list, there are two choices for α (1 and $1/c$), so we end up with four different forms for the $\text{curl} \mathbf{H}$ equation in different systems!

Jackson does not comment on the χ type factors one sees in $\mathbf{P} = \chi \mathbf{E}$ and $\mathbf{M} = \chi \mathbf{H}$. There is some variation in how these things are done as well. Bleaney says $\mathbf{P} = \chi \epsilon_0 \mathbf{E}$ but $\mathbf{M} = \chi \mathbf{H}$, so for each author you have to go look at exactly how they do things! I suspect that geophysicists are interested in $\mathbf{H}$ and μ and use units different from transformer designers.

Why batteries are about a volt

We are jumping ahead here a bit, but we want to show what the "volt" is a useful unit size. Using the Bohr radius and the electric charge, we have

$$|e| = 1.6 \times 10^{-19} \text{ C} \quad r_B = 5.3 \times 10^{-11} \text{ m}$$

$$|e|/r_B = 3 \times 10^{-9} \text{ C/m}$$

$$V(\text{volts}) = k_1^{\text{mks-C}} Q(\text{C})/r(\text{m}) = 9 \times 10^9 * 3 \times 10^{-9} = 27 \text{ volts as a rough scale}$$

Hydrogen levels are $E = -e^2/(2r_B n^2)$ so that $V = -13.5 \text{ volts}/n^2$. In a larger atom there is shielding so the general form is the same, but the radius is larger, so set $n = 4$ say and we get outer electron of something like Cu being bound by on the order of 1 eV, meaning the valence electron is at a potential of - 1 volt. In a battery, you might pull an electron off Cu to get Cu^{++} , and you might put it onto Zn^{++} to get Zn. Each of these actions will involve a different energy on the order of 1 eV, so the net result will be on the order of an eV, so the battery will put out something on the order of 1 volt. The ionization and affinity values are affected by the fact that the ions which result are not free, but are in a solution, but this does not change the scale of things. In fact, the energy obtained by putting Cu^{++} in solution probably offsets its ionization potential. (see chemistry books for more detail)

The potential unit determines the current unit.

In any system, we know that power $P = IV$. Thus, the product of the current and potential units must be the energy/sec unit in whatever system you are in. In mks we get watts = amperes * volts. In the other systems we get erg/sec = abamperes*abvolts = statamperes*statvolts.

3. The mksa System (= SI system = rationalized-mks).

We do this one first because most lookup information is in these units, which are sometimes called the "practical" units. Also, the world has adopted SI now and SI is "taking over". One writes equation (2) above as:

$$dF(NT)/dx(m) = k_2(NT\text{-}sec^2 / C^2) * 2i(C/sec)I(C/sec)/D(m) \quad k_2(NT\text{-}sec^2 / C^2) = 10^{-7} = \mu_0/4\pi$$

and one then *defines* the size of quantity C (the **Coulomb** unit of charge) by setting $k_2(NT\text{-}sec^2 / C^2) = 10^{-7}$. In this unit, an experimenter will find that $|e| = 1.6e-19$ C. WHY was k_2 selected to be this strange power? Because then a C/sec = ampere will be a reasonable "practical unit" of current, and will have a simple connection (1/10) with the formerly used abampere unit of the emu system. More on this historical subject later. In fact the **ampere** is really defined as per above, then Coulomb = ampere-sec.

The constant k_2 is usually written as $\mu_0/4\pi$. The 4π removes 4π from the Maxwell equation, and the μ_0 is so called because it ends up (via a long path) in the $H = \mu_0 B$ relation in free space. In the mksa system, the constant k_2 is assigned its natural dimensions:

$$k_2(NT\text{-}sec^2 / C^2) = 10^{-7} = \mu_0/4\pi \quad k_2 = 10^{-7} NT\text{-}sec^2/C^2 \quad (= 10^{-7} \text{ henry/meter})$$

The appearance of "**henry**" just comes from $V = L \partial I$, and is the way $\mu_0/4\pi$ is usually expressed. Now, because k_2 is assigned these "natural" units, we cannot make any connection between the Coulomb and any of our base units. The units just cancel out in the equation, as was noted earlier.

Notice that even with an "amp" of current, the force between two "practical wires" a meter apart is tiny, about 2/10,000,000 of 1 nt, and think of 1 lb = 4.45 nt, so nt really is an everyday unit. So in the *magnetic* world, a Coulomb/sec is not a very "big" thing. If you put the wires 1 cm apart and put 100 amps in each wire, you gain a factor of 10^6 , so force is then $2/10 = 1/5$ nt ~ 1 lb. This is easily observable by any dolt.

So in this system, charge is *defined* by the *magnetic* equation. The *electric* equation will then read,

$$F(NT) = k_1(NT\text{-}m^2/C^2) * q(C)Q(C)/r(m)^2 \quad = 9 \times 10^9 * q(C)Q(C)/r(m)^2$$

since we know from the above work that $k_1(NT\text{-}m^2/C^2) = k_2 * c(m/sec)^2 = 1e-7 * (3e8)^2 = 9 \times 10^9$. This constant k_1 is usually expressed as $k_1 = 1/(4\pi\epsilon_0)$ so we can have $D = \epsilon_0 E$ in free space. As with k_2 , the constant k_1 is given its "natural" dimensions:

$$k_1(NT\text{-}m^2 / C^2) = 9 \times 10^9 = 1/(4\pi\epsilon_0) \quad k_1 = 9 \times 10^9 NT\text{-}m^2/C^2 \quad (= 9 \times 10^9 \text{ meter/farad})$$

where the last comes from $Q=CV$ defining the **farad**. Notice that $k_1/k_2 = (3*10^8 \text{ m/sec})^2$.

The electrostatic potential is a derived concept, not a definition. We have $dW = Fdx$ so $dW/q = F/q dx$ which is then $E dx$. This work per charge is called the potential and is this $dV = E dx = (F/q)dx$. Thus, our definition of the volt is this:

$$1 \text{ volt} \equiv 1 NT\text{-}m/C$$

Now for the particular situation of a point charge, we can integrate E from r to infinity to get

$$V(\text{volts}) = 9 \times 10^9 Q(C)/r(m) \quad / = k_1 Q/r$$

where we have used the point charge for our definition. Now $E = q/r^2$ from its definition, =volts/m here.

In *electrostatics*, a Coulomb is a very large charge! If you put two of them 1 meter apart, the force between them would be 9 billion newtons, enough to explode any physical holding object to shreds in a real hurry! Another way to say this (see next paragraph) is that an aluminum sphere of radius 1 meter holding 1 Coulomb of charge would have a potential of 9 billion volts!

Side question: why don't the electrons just fly off a charged sphere into the vacuum? According to Purcell page 50, an average electron at the surface "feels" half the field. Perhaps we have 10,000 volts on a practical sphere, so the field just outside is 10,000 volts/meter. But this is a mere 10^{-6} volts/Angstrom. If you pull one extreme surface electron 10 Å above the surface, you "gain" about 10^{-5} eV of energy. But you would expect the electron to be bound in the metal by something on the order of 1 eV (see battery discussion earlier). Thus, you would need perhaps 1 billion volts on your sphere (~ Coulomb) before electrons would fly off spontaneously. If there is a conductor outside the sphere, then the "work function" to remove an electron goes away, and you get resistive flow. This same thing happens if the medium outside the sphere ionizes into a plasma due to the E field (spark, lightning bolt, etc).

This system is called "rationalized mks" only because we write $k_1 = 9 \times 10^9$ as $1/(4\pi\epsilon_0)$. Since $4\pi k_1$ appears in the Maxwell equation, this gets rid of the 4π floating around.

What about α ? Jackson says $\alpha=1$. The place to look for this is in Faraday's law. Bleaney page 256 shows clearly that $\alpha=1$ for that book, and the same for Portis on p 382. Bleaney claims to use the "rationalized mks" where Portis uses the "SI units system" which incorporates rationalized mks. Another term used is **mksa** where a stands for ampere.

The magnetic field formula above is $d\mathbf{F} = (1/\alpha) i d\mathbf{x} \times \mathbf{B}$ which in mksa is $d\mathbf{F} = i d\mathbf{x} \times \mathbf{B}$. Thus, B has units which are derived from other quantities

$$B(\text{nt/amp-m}) \quad \text{similar to} \quad E(\text{nt/C})$$

(Side note: if you know about antisymmetric $F^{\mu\nu}$ whose 6 non-zero elements are B and E, it sure is uncomfortable to have these fields have different units!)

The volt was defined as above, and we usually use E(volt/m) in common parlance. It was decided by someone that we should have a unit of magnetic flux (not field) called the Weber:

$$d\phi(\text{Weber}) = B(\text{nt/amp-m}) \cdot dA(\text{m}^2) \text{ so that} \quad 1 \text{ Weber} \equiv 1 \text{ nt-m/amp}$$

Then we can think of B(Weber/m²) in common speech.

The current SI unit system (see http://iserver.asa.edu/py/physicsweb/hisotry_of_units.htm) was adopted in a conference in 1971. At this time, the Tesla was introduced to give the B field its own private unit. Webers were not thrown out. Since Weber was already defined as above, we can regard the Tesla as a derived unit, or we could formally define it this way

$$B(\text{Tesla}) \quad 1 \text{ Tesla} \equiv 1 \text{ Weber/m}^2 = 1 \text{ nt/amp-m}$$

My Bleaney book is 1965 so does not mention Tesla, but Portis is 1978 and does. Jackson is 1962-7 and also does not mention Tesla. [by the way, Hz was adopted by SI in 1960. My 1956 and 1964 handbooks don't use it. I think common use of Hz phased in around 1966 in the US, was cycles per second.]

So, one more comment. At this point we have defined volts and amperes, so one more obvious unit remains:

$$R(\text{ohms}) = V(\text{volts})/I(\text{amps})$$

$$1 \text{ ohm} \equiv 1 \text{ volt/amp}$$

There are some other SI units that are a bit new to me:

$$\text{pascal} = 1 \text{ nt/m}^2 \text{ for pressure}$$

$$\text{siemens} = \text{amp/volt to replace mhos for conductance}$$

Magnetization M is interesting. Bleaney page 131 is talking magnetostatics and mentions the idea of a magnetic dipole moment $m = I(\text{current})S(\text{area of loop})$, similar to the electric dipole $p = qL$. If you talk about a volume density of moments, then you have $M = N m/V = > M(\text{current/length})$ like **Ampere/m**. Since M is related to H , we sometimes see H with units of current/length.

4. The cgs-esu-Gaussian system of units

In this system, the starting point is to set $k_1 = 1$ so that $F = qQ/r^2$ and this of course makes electrostatics very nice and simple. We regard this equation as defining the esu unit of charge also called a statcoulomb (Purcell calls this an esu and does not mention statcoulomb).

$$F(\text{dynes}) = q(\text{esu})Q(\text{esu})/r(\text{cm})^2$$

$$k_1(\text{dyne-cm}^2/\text{esu}^2) = 1 \quad \text{but no explicit dimensions!}$$

Since k_1 has no dimensions, we regard this as the formal definition of the esu in terms of base units:

$$1 \text{ dyne} = 1 \text{ esu}^2 / \text{cm}^2$$

Notice that $E = F/q$ is in dynes/esu or below it can be statvolts/cm.

Since we know that $k_1(\text{nt-m}^2/\text{C}^2) = 9 \times 10^9$ from the mksa system, we consider:

$$\begin{aligned} 9 \times 10^9 &= k_1(\text{nt-m}^2/\text{C}^2) = k_1(\text{dyne-cm}^2/\text{esu}^2) * \{\text{dyne/nt}\} * \{\text{cm/m}\}^2 * \{\text{C/esu}\}^2 \\ &= 1 * e^{-5} * e^{-4} * \{\text{C/esu}\}^2 \end{aligned}$$

$$\text{Thus:} \quad \text{Coulomb} = 3 \times 10^9 \text{ esu (statcoulomb)}$$

So the statcoulomb = esu is a much smaller thing than a Coulomb. That 1 meter sphere with an esu on it would only have a potential of about 3 volts, much more manageable!

The potential is going to be $dV = Edx = F/q dx$. The unit will be the **statvolt** and we have:

$$dV(\text{statvolts}) = F(\text{dynes})/q(\text{esu}) * dx(\text{cm})$$

$$1 \text{ statvolt} \equiv 1 \text{ dyne-cm/esu}$$

The conversion is therefore:

$$\begin{aligned} 1 \text{ statvolt} &= 1 \text{ dyne-cm/esu} = 1 \text{ nt-m/C} * \{\text{dyne/nt}\} \{\text{cm/m}\} \{\text{C/esu}\} \\ &= 1 \text{ volt} * e^{-5} * e^{-2} * 3e9 = 300 \text{ volts} \end{aligned}$$

Certainly we would say that a **statamp** = esu/sec, the unit of current. Then **statohm** = statvolt/statamp. The E field is measured in statvolts/cm.

Now moving to the magnetic equation, we have

$$dF/dx = k_2 * 2iI/D \quad k_2 = k_2(\text{dyne-sec}^2 / \text{esu}^2) = k_1/c^2 = 1/c^2 = 1/(9e20) \text{ sec}^2/\text{cm}^2$$

Remember that the force between currents was already pretty small even with amperes. Here with a current unit that is 3 billion times smaller, our force is really really tiny! Usually in equations one just shows the c factors as "c", and of course here it is in cgs units. Thus we have

$$dF(\text{dyne})/dx(\text{cm}) = (1/c(\text{cm/sec}))^2 * 2i(\text{esu/sec})I(\text{esu/sec})/D(\text{cm}) \quad // \text{ force between wires}$$

As noted in general above, in terms of units this equation also says $1 \text{ dyne} = 1 \text{ esu}^2 / \text{cm}^2$.

What about B and Biot-Savart? Recall that

$$dF = (1/\alpha) i d\mathbf{x} \times d\mathbf{B} \quad d\mathbf{B} = \alpha k_2 * (I d\mathbf{X} \times \mathbf{r})/r^3$$

Jackson in his Gaussian system (Purcell also) selects $\alpha = c$ so these become

$$dF = (1/c) i d\mathbf{x} \times d\mathbf{B} \quad d\mathbf{B} = (1/c) * (I d\mathbf{X} \times \mathbf{r})/r^3$$

This is nice in that we associate a $1/c$ factor with each current when we do the split here. The factor of c is taken to have its units with it in the Gaussian system. Thus we get

$$dF(\text{dynes}) = (1/c(\text{cm/sec})) i(\text{esu/sec}) d\mathbf{x}(\text{cm}) \times d\mathbf{B}(\text{dynes/esu})$$

The point here was to show that we have **B(dynes/esu)** in the Gauss system. In fact, this unit is called a "Gauss":

$$B(\text{gauss}) = B(\text{dynes/esu}) \quad 1 \text{ gauss} \equiv 1 \text{ dyne/esu}$$

Notice that E and B both have the same units in this system, dynes/esu, although they are usually called "gauss" for the B field. The term "oersted" is used for the H field, but it is the same unit as the gauss.

How are gauss and Tesla's related? I see no trivial way to compute this, just have to do brute force:

$$dF(\text{nt}) = i(\text{C/sec}) d\mathbf{x}(\text{m}) \times \mathbf{B}(\text{tesla})$$

$$dF(\text{dynes}) = (1/c(\text{cm/sec})) i(\text{esu/sec}) d\mathbf{x}(\text{cm}) \times \mathbf{B}(\text{gauss})$$

Divide these two equations to get:

$$\frac{dF(\text{nt})}{dF(\text{dynes})} = \frac{i(\text{C/sec})}{i(\text{esu/sec})} * \frac{d\mathbf{x}(\text{m})}{d\mathbf{x}(\text{cm})} * c(\text{cm/sec}) * \frac{B(\text{tesla})}{B(\text{gauss})} =$$

$$\{ \text{dyne/nt} \} = \{ \text{esu/C} \} * \{ \text{cm/m} \} * c(\text{cm/sec}) * \frac{B(\text{tesla})}{B(\text{gauss})}$$

$$1e-5 = 1/3e9 * 1e-2 * 3e10 * \frac{B(\text{tesla})}{B(\text{gauss})}$$

So

$$\frac{B(\text{tesla})}{B(\text{gauss})} = 10^{-4} \quad \Rightarrow \quad B(\text{gauss}) = 10,000 B(\text{Tesla}) \quad \Rightarrow \quad 1 \text{ Tesla} = 10,000 \text{ gauss}$$

We could also relate the E field units by comparing volts/meter to statvolts/cm. So

$$E(\text{volts/m}) = E(\text{statvolt/cm}) * \{ \text{statvolt/volt} \} * \{ \text{m/cm} \} = E(\text{statvolt/cm}) * 300 * 100 \\ = 30,000 E(\text{statvolt/cm})$$

$$\Rightarrow \quad 1 \text{ statvolt/cm} = 30,000 \text{ volts/m}$$

So in mksa the magnetic field unit is bigger, but the electric field unit is smaller.

What about magnetic flux. I think people used this

$$1 \text{ maxwell} = 1 \text{ gauss-cm}^2 \quad \text{compare to:} \quad 1 \text{ Weber} = \text{Tesla/m}^2$$

Divide and use $T = 10^4$ gauss to conclude that $1 \text{ Weber} = 10^8 \text{ maxwell}$.

There are really no other units that people use in this system. The inside back cover of Purcell gives a good summary of things.

5. The cgs-emu system of units

Now, on to the emu system. Sometimes units here are called "absolute" units, hence the terms like abvolt. We take as our starting point,

$$dF(\text{dyne})/dx(\text{cm}) = k_2 * 2i(\text{abC/sec})I(\text{abC/sec})/D(\text{cm}) \quad k_2(\text{dyne-sec}^2/\text{abC}^2) = 1 \text{ (dimensionless)}$$

$$dF(\text{nt})/dx(\text{m}) = k_2(\text{nt-sec}^2/\text{C}^2) * 2i(\text{C/sec})I(\text{C/sec})/D(\text{m}) \quad k_2(\text{nt-sec}^2/\text{C}^2) = 10^{-7} = \mu_0/4\pi$$

We write the second line only so we can do a division to find how abC are related to C:

$$10^5 = 10^7 * \{ \text{C/abC} \}^2 \quad \Rightarrow \quad \text{abcoulomb} = 10 \text{ coulomb}$$

This factor entirely arises from the nt/dyne shift being offset by the k_2 shift. Of course the abamp will be an abcoulomb/sec.

Now $k_1 = k_2 * c^2 = c^2 = (3 \times 10^{10} \text{ cm/sec})^2$. This we are going to have in the electrostatic side of things,

$$F(\text{dynes}) = c^2 (\text{cm}^2/\text{sec}^2) q(\text{abC})Q(\text{abC})/r(\text{cm})^2 \quad \text{and} \\ E(\text{dynes/abC}) = c^2 (\text{cm}^2/\text{sec}^2) Q(\text{abC})/r(\text{cm})^2$$

The potential will be $V = \int E dx$ so that

$$1 \text{ abvolt} = 1 \text{ dyne-cm/abC} \\ = 1 \text{ nt-m/C} * \{ \text{dyne/nt} \} * \{ \text{cm/m} \} * \{ \text{C/abC} \} = 1 \text{ volt} * 10^{-5} * 10^{-2} * 10^{-1} \\ = 10^{-8} \text{ volt}$$

Next we come to Biot-Savart and we have to face the α question: ($k_2=1$)

$$d\mathbf{F} = (1/\alpha) i d\mathbf{x} \times d\mathbf{B} \quad d\mathbf{B} = \alpha (I d\mathbf{X} \times \mathbf{r})/r^3 \quad // \text{ 1st is Biot-Savart} \quad (4)$$

If we set $\alpha = 1$ (as it is in mksa), then we could conclude that $B(\text{Tesla}) = 10^{-3} B(\text{emu})$. A factor of 10 comes from abcoulomb to coulomb, and a factor of 100 comes from 1/distance. However, I think no one uses an emu unit like this, I think they use the Gauss from the cgs-esu system instead! This could be fixed by making $\alpha = 1/10$. Then we would make no distinction between the two cgs systems for B. However, we would then require that $d\mathbf{F} = 10 i d\mathbf{x} \times d\mathbf{B}$. This has to be the case of $B = \text{Gauss}$ and $i(\text{abC/sec})$, no way around it! If B is the same in emu as esu, then so is B-flux, the maxwell.

Obviously if we have abamps and abvolts, we are going to have **abohms**.

6. Some History

The emu system is the counterpoint to the esu system because here we are going to take $k_2 = 1$ with no dimensions, whereas in esu we take $k_1 = 1$. What does this system use for α ? Was it the historical system used by Ampere et al? Do I know any sources that use this system? Geophysics may have historically used this system. The note below suggests that abvolts were used prior to 1881, so maybe the emu system with cgs was the standard for everything prior to that time!

My guess is that before 1881, everyone doing electric currents work was using these "ab" units and cgs base units, but was just calling them "electromagnetic units". Surely the "ab" for absolute is a retroactive addition from the modern day to distinguish current amps from historical amps which are today's abamps. The claim made below is that in 1881 abvolt was replaced with volt that was 10^8 times larger and more practical.

Why was the ampere defined as 1/10 of an abamp (with the result that the Coulomb became 1/10 of the emu unit (abcoulomb))? The reason just comes from $P = IV$ as noted earlier,

$$1 \text{ amp} = 1 \text{ joule/volt} = 10^7 \text{ erg} / (10^{-8} \text{ abvolt}) = (1/10) \text{ abamp}$$

So the potential unit (volt) determines the current unit (ampere) and thus the charge unit (coulomb). Also:

$$\text{abohm} = \text{abvolt/abamp} = 10^{-8} \text{ volt}/(10 \text{ amp}) = 10^{-7} \text{ ohm}$$

Some quotes of interest from the web:

" First to adopt a customary metric system was France in 1799. Progress towards its adoption elsewhere has been distinctly less rapid now that those opposed to it are no longer decapitated. Anyway, the CGS system enjoyed wide acceptance until 1881 when the then new electricity industry decided that thee.m.u. abvolt was too small for practical use and succeeded in introducing a value 10^8 times larger: our present value for the volt. The ohm and the amp were also re-scaled. "

" The First International Conference of Electricians (Paris, 1881) adopted the British Association definition of the ohm and added definitions for the volt, ampere, coulomb, and farad. Thus was born the "absolute practical system of electrical units":

- absolute, because the units were defined solely in terms of mechanical units (length, mass and time)
- practical, because the sizes were much more convenient than the cgs units.

The ampere was a derived unit, defined as the current produced in a conductor with a 1-ohm resistance when there was a potential difference of 1 volt between its ends"

" In 1948 it was decided that the centimetre, the gram and the erg (the CGS unit of energy) should be replaced by the larger metre, kilogram and joule. This was called the MKSA system and it restored coherence "

More on the order in which things were done (1780 - 1881)

- 1785 Coulomb's law for static charges
- 1800 Volta makes batteries, allowing people to have wires with currents in them for the first time
- 1820 Oersted sees current in wire deflect a compass needle
- 1820 Arago sees current in wire attract iron filings.
- 1820 Biot, Savart and Ampere get force on current from B and between two currents
- 1825 Ampere puts out a summarizing paper
- 1827 Ohm's law in Die Galvanische Kette, Mathematisch Bearbeitet paper

- 1874 BAAS adopts cgs
- 1881 conference also adopts practical units including amp, ohm, volt

During all this time, no one knew about electrons and protons. Charge Q was regarded as some kind of invisible fluid that came out of matter, maybe two fluids. The fluid could flow through a wire, and could be produced with a battery. The idea that the fluid was made of individual particles was unknown until 1897. The connection between light waves and this charge fluid through the Maxwell equations must have seemed very strange. That did not stop electric motors from being built in 1831.

Dalton summarized basic ideas about *atoms* around 1803, they were round solid balls with interesting properties. In 1897 Thomson showed that cathode rays were individual charged "corpuscles" much lighter than atoms, later called electrons, and they were assumed to be pieces of atoms. People made atomic models like the **Plum Pudding Model** with a mixture of + and - charges swimming around in the ball. Only in 1911 did **Rutherford** infer the presence of charged positive atomic nucleus. The fact that light was quantized in $h\nu$ was found separately from the photoelectric effect (**Einstein**), and this with the nucleus idea because the wave model for hydrogen of **Bohr** in 1913, and so started the probability idea and quantum mechanics. We now know that E&M is just quantum mechanics for a massless particle. Relativity (again Einstein) came along in 1905 and soon got mixed with QM.

RAINBOW

1

Rainbows

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Introduction

Most of us are familiar with the sight of a rainbow after a rainstorm. The sun is at our back, and we see an arc with violet innermost, blending into blue, green, yellow and orange, with a red outer band. This is the primary rainbow, and we see it as well in the spray of a waterfall. Sometimes after a rainshower we may be fortunate enough to see another rainbow, the secondary rainbow, higher in the sky than the primary rainbow. The sequence of colors in this secondary rainbow is reversed, with the red ring innermost, and the violet band outermost.

In this project you examine how rainbows are formed and see why the sequence of colors in the two bows is as described above. First we must study the path followed by light passing from one medium to another.

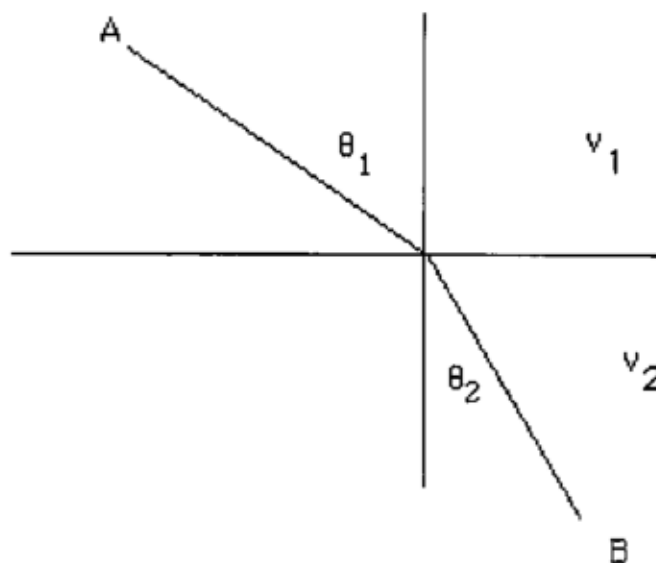


Figure 1

In traveling from A to B, the light is refracted, that is changes direction, at the interface between the two media. The angle of incidence, θ_1 , and the angle of refraction, θ_2 , satisfy *Snell's Law* (Willebrord Snell, 1621),

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2},$$

where v_1 is the velocity of light in the first medium, and v_2 is the velocity of light in the second. The ratio $\frac{v_{\text{water}}}{v_{\text{air}}}$ is 0.7508 for red light, and 0.7440 for violet light.

Part I. The primary rainbow

The theory of rainbows presented here was developed by Descartes. The primary rainbow is formed by rays of light which enter a drop of water (assumed to be spherical), are refracted at the boundary, reflect once internally off the back of the raindrop, and are again refracted as they leave the drop, headed toward the observer of the rainbow.

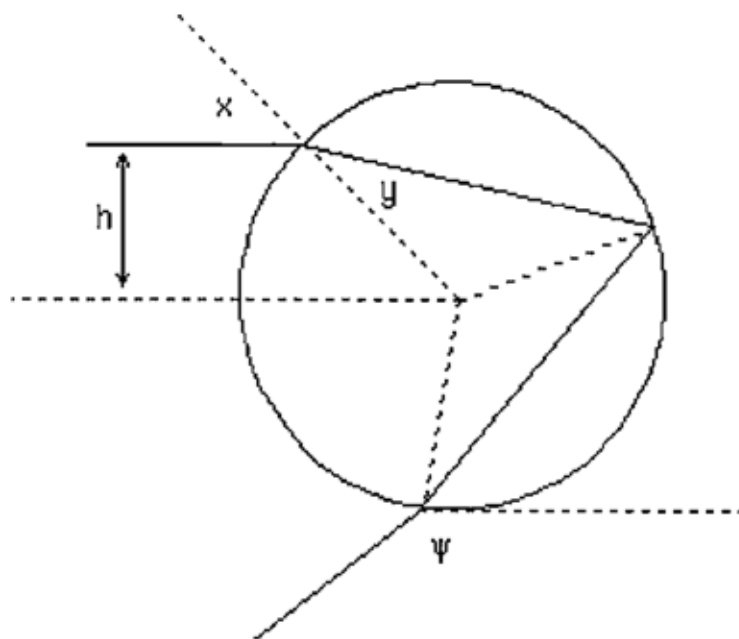


Figure 2

The distance h from the center line of the drop to the incoming ray is called the *impact parameter*, and the angle x is called the angle of incidence. The angle ψ is called the

scattering angle. If a ray is incident with impact parameter 0, it is reflected back on its original path, and the scattering angle is π . As the impact parameter increases, the scattering angle decreases, as seen in the above figure. As h increases, the scattering angle reaches a minimum of about 138° , and then begins increasing.

Descartes' reasoning was that light exiting the drop is concentrated near the scattering angle for which the scattering angle is least sensitive to changes in the impact parameter h . That is, it is concentrated near the scattering angle for which the rate of change of scattering angle with respect to h is as small as possible, namely where

$$\frac{d\psi}{dh} = 0.$$

Of course this happens where the scattering angle ψ assumes its minimum value, and Descartes calculated scattering angles for many values of the impact parameter h , thereby determining that the reflected light is concentrated near the scattering angle of 138° . The primary rainbow is formed by light from those drops for which the angle between the incident rays and the observer's line of sight is $180 - 138 = 42^\circ$.

Exercises

1. Fermat's Principle in optics is that light travels on the path which minimizes the time of travel (See Project 2.). Use Fermat's Principle to show that in traveling from A to B (see Figure 1), light follows the path for which Snell's Law

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

holds.

2. Show that the scattering angle ψ is given by the formula

$$\psi = \pi + 2x - 4y, \text{ or}$$

$$\psi = \pi + 2x - 4 \operatorname{Arcsin} \left[\frac{v_{\text{water}}}{v_{\text{air}}} \sin x \right]$$

This can be done by examining Figure 2 and either using plane geometry or by carefully calculating the ray's change of direction at each refraction or reflection.

3. Plot the scattering angle as a function of the angle of incidence, and calculate the minimum value of ψ for both red and violet light. Use this to explain why the rainbow

forms in a circular arc and why the violet arc is the innermost ring and the red arc is outermost.

Part II. The secondary rainbow

The secondary rainbow is formed by rays of light which undergo two internal reflections before being refracted towards the observer. This rainbow is fainter than the primary rainbow because at each reflection a portion of the light is transmitted into the air. The path of light undergoing two internal reflections is shown in Figure 3.

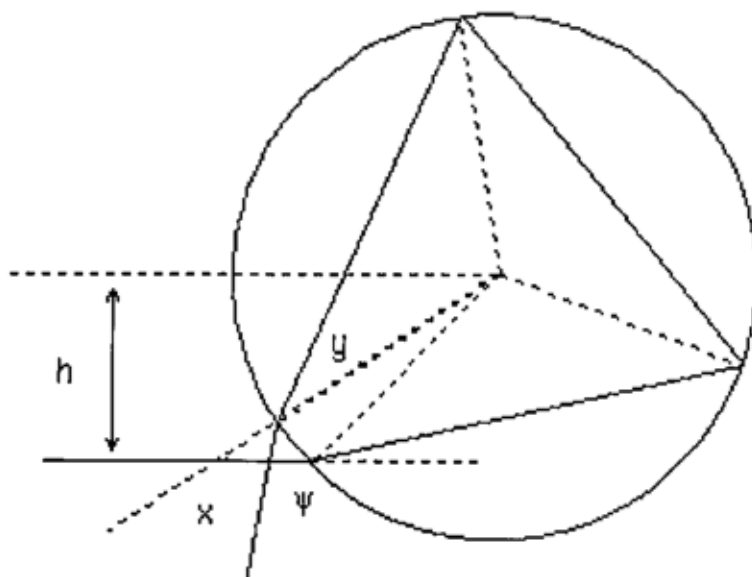


Figure 3

In this case, a ray entering a drop at impact parameter $h = 0$ is reflected twice, and leaves the drop in its original direction, with scattering angle $\psi = 0$. As h changes, the scattering angle increases to a maximum of about 130° , and then decreases. In this case, light is concentrated in the direction of this maximum scattering angle.

Exercises

4. Show that in the case of two internal reflections, the scattering angle ψ is given by the formula

$$\psi = 6y - 2x, \text{ or}$$

$$\psi = 6 \text{Arcsin} \left[\frac{v_{\text{water}}}{v_{\text{air}}} \sin x \right] - 2x$$

5. Plot ψ as a function of x , and determine the maximum value of ψ for both red and violet light. Explain why the secondary rainbow appears higher in the sky than the primary rainbow, and why the sequence of bands in the secondary rainbow is reversed, with the red band innermost and violet outermost.
6. Explain why the sky is particularly dark in the region between the primary and secondary rainbows. This dark area is known as *Alexander's Dark Band*.

References

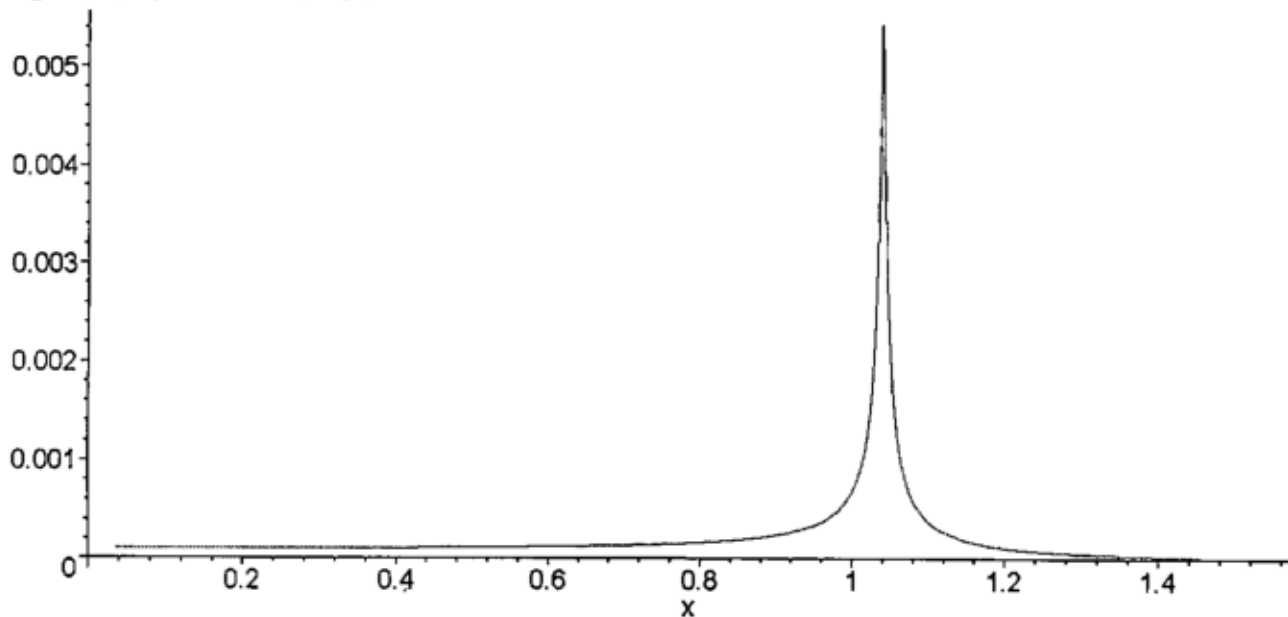
H. Moyses Nussenneig, *The Theory of the Rainbow*, Scientific American, April 1977, reprinted in Light from the Sky, W. H. Freeman and Company, San Francisco.

A. D. Andrew, G.L. Cain, S. Crum, T. D. Morley, *Calculus Projects Using Mathematica*, McGraw-Hill, 1996.

```

[ > restart;
[ > y := arcsin(n1*sin(x)/n2);
[ > p := Pi + 2*x - 4*y; // psi(x)
[ > p1 := diff(p,x); // psi'(x)
[ > p2 := diff(p1,x); // psi''(x)
[ > N1 :=sin(x)*cos(x)/(sin(p)); // first factor of dN solution
[ > N2 :=abs(-p1+signum(p1)*(p1^2 + 2*p2*f)^.5)/(p2); // second factor
[ > N :=N1*N2; // complete solution
[ > n1 := 1; // Now insert specific values for constants
[ > n2 :=1.33;
[ > f := 1e-4; // this is Rp/D, we want the eye far away, angularly speaking (and D >> R)
[ > plot(N,x=0..Pi/2);

```



```

> restart;
> y := arcsin(n1*sin(x)/n2);

$$y := \arcsin\left(\frac{n1 \sin(x)}{n2}\right)$$

> p := Pi + 2*x - 4*y; // psi(x)

$$p := \pi + 2x - 4 \arcsin\left(\frac{n1 \sin(x)}{n2}\right)$$

> p1 := diff(p,x); // psi'(x)

$$p1 := 2 - 4 \frac{n1 \cos(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}}$$

> p2 := diff(p1,x); // psi''(x)

$$p2 := 4 \frac{n1 \sin(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}} - 4 \frac{n1^3 \cos(x)^2 \sin(x)}{n2^3 \left(1 - \frac{n1^2 \sin(x)^2}{n2^2}\right)^{(3/2)}}$$

> N1 := sin(x)*cos(x)/(sin(p)); // first factor of dN solution

$$N1 := - \frac{\sin(x) \cos(x)}{\sin\left(2x - 4 \arcsin\left(\frac{n1 \sin(x)}{n2}\right)\right)}$$

> N2 := abs(-p1+signum(p1)*(p1^2 + 2*p2*f)^.5)/(p2); // second factor

$$N2 := \left| -2 + 4 \frac{n1 \cos(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}} + \text{signum}\left(2 - 4 \frac{n1 \cos(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}}\right) \left( \left(2 - 4 \frac{n1 \cos(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}}\right)^2 + 2 \left(4 \frac{n1 \sin(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}} - 4 \frac{n1^3 \cos(x)^2 \sin(x)}{n2^3 \left(1 - \frac{n1^2 \sin(x)^2}{n2^2}\right)^{(3/2)}}\right) f \right)^{.5} \right| \Bigg/ \left( 4 \frac{n1 \sin(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}} - 4 \frac{n1^3 \cos(x)^2 \sin(x)}{n2^3 \left(1 - \frac{n1^2 \sin(x)^2}{n2^2}\right)^{(3/2)}} \right)$$


```

```
> N := N1*N2; // complete solution
```

$$N := -\sin(x) \cos(x) \left| -2 + 4 \frac{n1 \cos(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}} + \operatorname{signum} \left(2 - 4 \frac{n1 \cos(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}} \right) \left(\left(2 - 4 \frac{n1 \cos(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}} \right)^2 + 2 \left(4 \frac{n1 \sin(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}} - 4 \frac{n1^3 \cos(x)^2 \sin(x)}{n2^3 \left(1 - \frac{n1^2 \sin(x)^2}{n2^2} \right)^{(3/2)}} \right) f \right)^5 \right| \left(\sin \left(2x - 4 \arcsin \left(\frac{n1 \sin(x)}{n2} \right) \right) \left(4 \frac{n1 \sin(x)}{n2 \sqrt{1 - \frac{n1^2 \sin(x)^2}{n2^2}}} - 4 \frac{n1^3 \cos(x)^2 \sin(x)}{n2^3 \left(1 - \frac{n1^2 \sin(x)^2}{n2^2} \right)^{(3/2)}} \right) \right)$$

```
> n1 := 1; // Now insert specific values for constants
```

```
n1 := 1
```

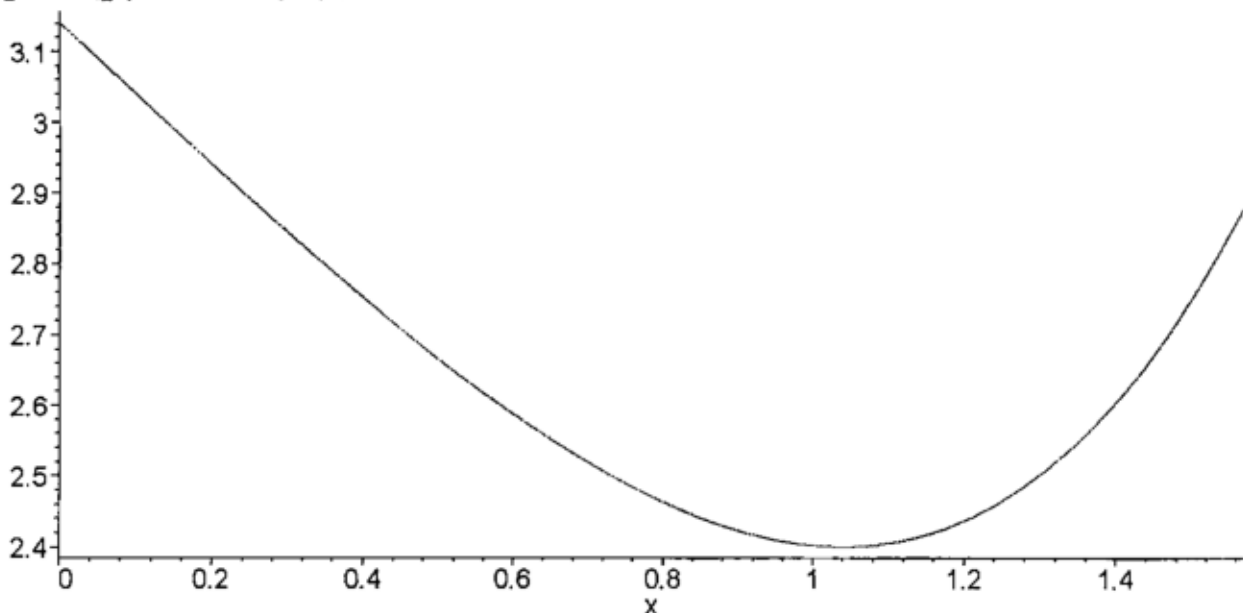
```
> n2 := 1.33;
```

```
n2 := 1.33
```

```
> f := 1e-4; // this is Rp/D, we want the eye far away, angularly speaking (and D >> R)
```

```
f := .0001
```

```
> plot(p, x=0..Pi/2);
```



Look at the pencil drawing. It shows a single ray coming in at impact parameter h (angle x) and emerging at an angle ψ relative to the initial direction. Now imagine an observing eye viewing the emerging ray at a great distance. Model the eye as a circular pupil that catches rays. The number of rays captured by the eye is proportional to $d\psi$ because the pupil has a finite size. Since we are so far away, the fact that the location on the sphere of the emerging ray varies slightly as ψ varies does not matter.

In the other direction, imagine that a ray comes in at some $d\phi$ azimuth (north pole is left end of sphere) below the plane of paper. Then just tilt the entire disk drawn in the figure by rotating it about the horizontal axis by angle ϕ . The angle between the emerging ray and the untilted emerging ray is $\delta\theta = \sin\psi \delta\phi$ as proven in our little appendix below. This is obvious when $\psi = \pi/2$ so $\sin\psi = 1$.

Imagine a square (edge d) into which is inscribed the pupil (diameter d) of a viewing eye (at distance D from the sphere). The square forces $\delta\theta = \delta\psi = d/D$. All rays within these small angles hit the square. Of these, the fraction $\pi/4$ hit the disk of the retina, but we don't bother with this factor in the following since we are not looking for an absolute answer, just the shape of the curve.

Now, the number of rays which pass through an area $R^2 d\Omega$ on the sphere ($x =$ polar angle, $\phi =$ azimuthal angle) is $\sigma(\text{rays/area}) * R^2 d\Omega * \cos(x)$, this last factor since $R^2 d\Omega$ is tilted relative to the incoming plane wave of rays. We have assumed a uniform incoming plane wave ray density σ ; these rays are coming from the left in the figure. We are also implicitly assuming polarization perpendicular to the plane of paper. The other polarization direction will have some extra angle complications and will probably have a weaker output, but the nature of the result won't be different. Thus we have

$$\delta N(x, \phi) = \sigma R^2 \cos x \sin x \delta x \delta \phi = \text{number of rays hitting our differential area.}$$

Then set $\delta x = \delta\psi/\psi'(x)$ and $\delta\phi = \delta\theta/\sin\psi$. Finally, use $\delta\theta = \delta\psi = d/D$ and the result is

$$\delta N(\text{number of rays reaching retina}) = \frac{\sigma R^2 \sin x \cos x d}{D \sin \psi} \frac{d}{D \psi'(x)} \quad (1)$$

where the second fraction is δx which we have kept separate for the moment. We have an equation for $\psi'(x)$ on our pencil sheet, and we know $\psi'(x) = 0$ at $\psi = 137.48^\circ$ or so. At this point, everything else in the equation is finite, so dN has an infinite peak here! This makes the point quite well that the critical angle completely dominates the reflected light, and this was Descartes idea. But we know the result cannot really be infinite. To correct the situation, let's replace $\delta x = \delta\psi/\psi'(x)$ with a more accurate expression. We can expand $\psi(x)$ around the point x and get this result, where now we keep an extra term:

$$\delta\psi = \delta x \{ \psi' + (\delta x/2) * \psi'' \}$$

Rewrite this as

$$(\psi''/2) (\delta x)^2 + (\psi') (\delta x) - \delta\psi = 0 \quad A (\delta x)^2 + B(\delta x) + C = 0$$

solve this for δx ,

$$\delta x = [- \psi' \pm \sqrt{(\psi')^2 + 2 \psi'' \delta\psi}] / \psi''$$

Which root do we want here? We want the root that causes cancellation between the two large terms because the result we are looking for is differentially small. We know that $\psi'' > 0$, so assume that $\delta\psi > 0$ so the inside of the square root is larger than $|\psi'|$. Then we want this solution:

$$\delta x = | -\psi' + \text{sign}(\psi') * \sqrt{(\psi')^2 + 2\psi''\delta\psi} | / \psi''$$

We have added $| |$ because we want $\delta x > 0$ since we are just doing a ray density scaling.

We set $\delta\psi = d/D$ inside and then our formula becomes

$$dN(\text{number of rays reaching retina}) = \frac{\sigma R^2 \sin x \cos x d}{D \sin \psi} \frac{|-\psi' + \text{sign}(\psi') * \sqrt{(\psi')^2 + 2\psi''(d/D)}|}{\psi''} \quad (2)$$

Right AT the critical point where $\psi' = 0$ we can write this as

$$dN(\text{number of rays reaching retina}) = \frac{\sigma R^2 \sin x \cos x d}{D \sin \psi} \frac{d}{D \sqrt{\psi'' d / 2D}} \quad (3)$$

Comparing with (1) above, we see that ψ' in the denominator has been replaced by $\sqrt{\psi'' d / 2D}$. This is small, but not 0, so the result for dN is large but finite.

So let's plot the above general result for dN using Maple! The result is very impressive and highly peaked as expected. We could compute the width of the peak, but the Maple plot shows it to be about .04 radians with our assumed $f = d/D = .0001$. This is about 2.4 degrees, very narrow, a thumb width which is 2.4 degrees as well for me (1" at 24"). This might agree with the observed width of a rainbow. Probably the color bands overlap a lot, etc.

Colors are there because the index of water varies with frequency. Since visible is below the electronics resonance area, index (and dielectric constant) increases with increasing frequency, so red light is bent least and has the lowest index, and will have a smaller ψ angle, so red appears at the bottom of the rainbow band (on the inside of the big circle). To see why a rainbow is a circle, hold the piece of paper with the figure on it and rotate it such that rays coming from behind you all parallel cause reflection into your eye. The paper must be attached as a paddle on a wheel whose axis goes through your eye.

In the single bounce situation which we have looked at, the output angle ψ ranges from 180 degrees down to 137 degrees and our drawing is typical. Suppose the ray does two integral bounces. I have shown a second bounce as a dotted line in the figure. You can see that this ray emerges on the top, not on the bottom of the figure. In this case, red light with less bend comes out at *larger* ψ because now ψ is measured around on the top side of the figure. So the second rainbow has the opposite coloring order. For n bounces, you have to study the geometry and see whether ray comes out on the top or the bottom. I have seen three rainbows in Torrey.

Appendix. What is the angle between the tilted emergent ray and the original untilted emergent ray? Go out to the eye a large distance D away. Make a little coordinate system centered on the emergence point, with x to the right and y down and z out of paper. The coordinates of the point at the eye that is hit by this ray are: $z = D \sin \psi \sin \phi$, $x = D \cos \psi$, $y = D \sin \psi$, or $r = a$. The coordinates of the point with no tilt are $z = 0$,

$x = D\cos\psi$, $y = D\sin\psi$ or $\mathbf{r} = b$. The corresponding rays are given by $\mathbf{r}_1 = \alpha\mathbf{a}$ and $\mathbf{r}_2 = \alpha\mathbf{b}$ where α is a scaling factor. The angle between these rays is given by $\cos\theta = \mathbf{r}_1\text{-hat} \cdot \mathbf{r}_2\text{-hat} = \mathbf{a}\text{-hat} \cdot \mathbf{b}\text{-hat}$. Now $|\mathbf{a}| = D\sqrt{1 + (\sin\psi d\phi)^2}$ while $|\mathbf{b}| = D$. And $\mathbf{a} \cdot \mathbf{b} = D$. Thus, $\cos\theta = 1/\sqrt{1 + (\sin\psi d\phi)^2}$. But $d\phi$ is small, so we can expand RHS to get $1 - 1/2((\sin\psi d\phi)^2)$. But we know that for small angle $d\theta$ we have $\cos d\theta = 1 - 1/2d\theta^2$. This we get our answer which is $d\theta = \sin\psi d\phi$.