

Most stuff is quite old, but a few things are recent.

Mechanics

- What makes tides
- Gut basics of Hamiltonian Mechanics
- Goldstein mechanics notes from Nov 1977
- Physics of Sprinting paper with Igor, March 1981
- Cars, torque, and brake horsepower
 - Acceleration Times based on car torque Oct 1985
 - Theory of Effective Horsepower Oct 1985
 - Computing Acceleration Times

Thermal Loss in Houses

- Thermal conduction Feb 1978
- Heat loss through windows, etc (refs)

Quantum Mechanics

- Messiah and the meaning of bras and kets April 1975
- Operators in Wavefunction Quantum Mechanics
- Clearing up Active vs Passive transformations
- Solution of 2D harmonic oscillator (Laguerre
- Harmonic oscillator in 3D
- Comments on Quantum Mechanics

Special Relativity

- Relativistic mechanics and massless particles
- Accelerating rockets
- E&M and special relativity
- Thomas rotation effect
- Thomas precession and the $\frac{1}{2}$ factor
- Foucault pendulum comment.
- Some other mechanics/magnetism problem.

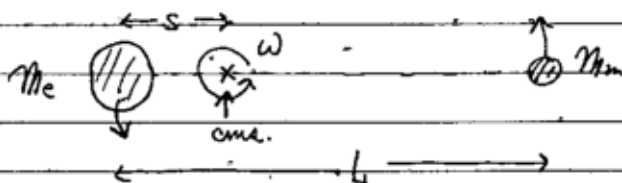
Particle Physics

- dates when certain things were found
- Goldberger-Watson footnote for page 347 equation 12b
- PCAC and GT (Partially Conserved Axial Current and Goldberger Trieman relation
- Relating A^μ and V^μ to the generator currents, more Goldberger Trieman
- Chiral comments
- An experiment to measure $g-2$.
- Standard Model summary page (from web)
- Chew 2008 paper Cosmological Hilbert Space

Mechanics

What makes tides

- ① Consider the earth (M_e) and moon (M_m) in orbit about each other.



It is clear that:

$$s = L \times \frac{M_m}{M_e + M_m} = \text{c.m.e. point}$$

- ② What is ω ? Do a force balance on the moon.

$$F_{\text{TOT}}^m = 0 = \frac{G M_e M_m}{L^2} - M_m \omega^2 (L - s)$$

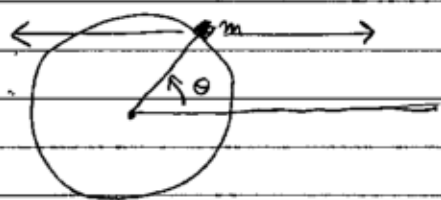
Easier to do balance on the earth:

$$F_{\text{TOT}}^e = 0 = \frac{G M_e M_m}{L^2} - M_e \omega^2 s$$

$$\omega^2 = \frac{G M_e}{L^2 (L - s)}$$

$$\Rightarrow \omega^2 = \frac{G M_m}{L^2 s}$$

- ③ Now consider a mass of water m on the earth at θ as s .



We want to know the net earth-horizontal force on this mass.

If we assume that $R_e, R_m \ll L$, then

The forces are as shown.

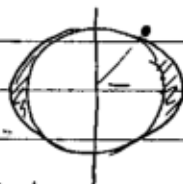
$$F_{\text{TOT}}(on\ m) = \frac{G M_m m}{L^2} - m \omega^2 \underbrace{[s - R_e \cos \theta]}_{\text{distance to c.m.e.}}$$

- ④ Using result of ② we get

$$F_{\text{TOT}}(on\ m) = \frac{G M_m m}{L^2} \cdot \frac{R_e \cos \theta}{s} = F(m\ on\ m) \times \frac{R_e \cos \theta}{s}$$

Once more: $F(m, m) = F(\text{moon on } m) \times \frac{R_e \cos \theta}{L}$

We can see that water on the back half of the earth [180° , $90^\circ < \theta < 270^\circ$] has $\cos \theta < 0$ and tidal force is to the left. Front half has $\cos \theta > 0$ so force to right. Hence:



⑤ For an infinitely massive earth, we can get the same result more quickly by:

a) balance on moon $\Rightarrow m_m \omega^2 L = G M_e m_m / L^2$

$$\Rightarrow \omega^2 = \frac{G M_e}{L^3}$$

b) then look at mass m + say

$$F = \frac{G m m_m}{L^2} + m \omega^2 R \cos \theta$$

$$= \frac{G m m_m}{L^2} + m \frac{G M_e}{L^3} R \cos \theta$$

$$= \frac{G m}{L^2} \left[m_m + M_e \frac{R \cos \theta}{L} \right] = \frac{G m m_m}{L^2} \left[1 + \frac{M_e R \cos \theta}{m_m L} \right]$$

c) This method gives the wrong answer. The fact that the cm is shifted away from earth center is crucial.

d) so, the correct heavy earth answer is:

$$F(m, m) = F(\text{moon on } m) \times \frac{R_e \cos \theta}{L} (M_e + m_m)$$

$$\boxed{\vec{F}_m \approx F(\text{moon on } m) \times \left[\frac{M_e + m_m}{m_m} \right] \times \left[\frac{R_e}{L} \right] \cdot \cos \theta}$$

(tides)

- ⑥ Now some numbers. Use 1 kg for water mass. Then look up:

$$M_e = 5.98 \times 10^{24} \text{ kg}$$

$$M_m = 7.35 \times 10^{22} \text{ kg}$$

$$L = 384,400 \text{ km} \\ = 3.84 \times 10^8 \text{ m}$$

$$G = 6.67 \times 10^{-11} \text{ mks.}$$

$$R_e = 6.38 \times 10^6 \text{ m}$$

$$R_m = 1.74 \times 10^6 \text{ m}$$

} see F1-50
of CRC, 53

- ⑦ To remember these, let's compute surface periods of satellites on earth/moon. Equation is:

$$\frac{GM_p \cdot m_{sat}}{R_p^2} = m_{sat} \left(\frac{2\pi}{T} \right)^2 R_p \Rightarrow T = 2\pi \sqrt{\frac{R_p^3}{GM_p}}$$

$$T_{earth} = 84.5 \text{ minutes}$$

$$T_{moon} = 108.6 \text{ minutes}$$

$$\frac{g_e}{g_m} = \frac{M_e}{M_m} \left(\frac{R_m}{R_e} \right)^2 = 6.05$$

- ⑧ Now, what is $(F_{\text{moon on m}})$ for $m = 1 \text{ kg}$?

$$F = \frac{GM_m m}{L^2} = \frac{GM_m \cdot 1}{R_m^2} \cdot \left(\frac{R_m}{L} \right)^2 = g_m \cdot \left(\frac{R_m}{L} \right)^2 = \frac{1}{6} \cdot (9.8) \cdot \left[\frac{1.74}{384} \right]^2$$

$$= 3 \times 10^{-5} \text{ nt} \quad (\text{compare to } 9.8 \text{ nt from the earth}).$$

- ⑨ Factor $\frac{M_e + M_m}{M_m} = 82.3$.

By the way, $s = L \cdot \frac{M_m}{M_e + M_m} = \frac{1}{82.3} L = 4.7 \times 10^6 \text{ km} = .74 R_e$!
ie, offset is large.

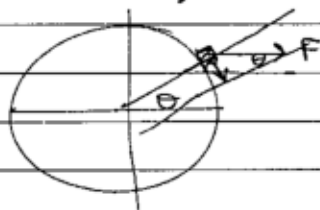
⑩ Finally, $\frac{R_e}{L} = \frac{6.38}{384} = \frac{1}{60.2}$ so.

$$F_{\text{tidal}} = 3 \times 10^{-5} \text{ nt} \times 82.3 \times \frac{1}{60.2} \times \cos \theta$$

$$\approx 4 \times 10^{-5} \text{ nt} \quad \text{maximum}$$

- ⑪ The tidal force is small, but any force will move water.

⑫ Actual horizontal force on m is

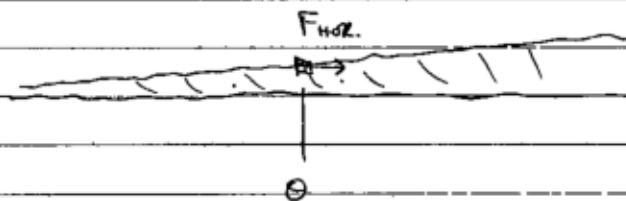


$$F_H = F \sin \theta, \text{ so}$$

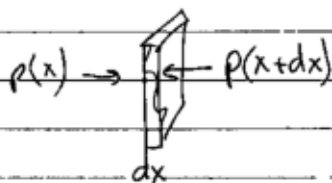
$$F_{HOR} = F(mom) \cdot \left(\frac{M_e + M_m}{M_m} \right) \cdot \frac{R_e}{2L} \cdot \sin(2\theta) = F_{HOR}$$

↑ force per unit mass

⑬ So look at piece of earth near θ



To get equilibrium on a mass of water, need pressure increase as you go to right.



$$F_{HOR} \cdot (g \cdot dA \cdot dx) = [p(x+dx) - p(x)] \cdot dA$$

$$\Rightarrow g F_H = \frac{dp}{dx} \quad p = \rho g h$$

$$\Rightarrow F_H = g \frac{dh}{dx}$$

$$x = -R_e \theta$$

$$dx = -R_e d\theta$$

Same fluid:

$$F_H = g \cdot \frac{dh(\theta)}{-R_e d\theta}$$

$$\Rightarrow dh = - \frac{R_e}{g} \cdot F_H(\theta) d\theta$$

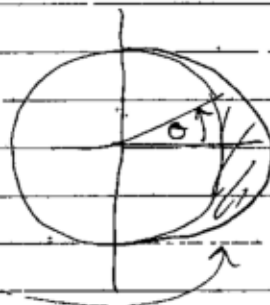
$$\Rightarrow h(\theta) = - \frac{R_e}{g} \cdot F(mom) \cdot \left(\frac{M_e + M_m}{M_m} \right) \cdot \frac{R_e}{2L} \int_{\pi/2}^{\theta} \sin(2\theta) d\theta$$

$$= - \frac{R_e}{g} \cdot F(mom) \cdot \left(\frac{M_e + M_m}{M_m} \right) \cdot \frac{R_e}{2L} \cdot \left[-\frac{1}{2} \cos \theta \right]$$

$$\Rightarrow h(\theta) = \text{const} \times \cos \theta$$



↑ tide is a cosine



(tides)

(14) My prediction for $h(\theta)$ due to tides from is:

$$h(\theta) = \frac{R_e}{g_e} \cdot F_{\text{moon}} \cdot \left(\frac{M_e + M_m}{M_m} \right) \cdot \frac{R_e}{2L} \cdot \frac{1}{2} \cos \theta.$$

$$\text{where } F_{\text{moon}} = \frac{GM_m}{L^2} = g_m \cdot \left(\frac{R_m}{L} \right)^2$$

$$\text{Thus, } h_{\text{max}} = \frac{1}{4} \cdot R_e^2 \cdot \frac{g_m}{g_e} \cdot \left(\frac{R_m}{L} \right)^2 \cdot \frac{1}{L} \cdot L \cdot f$$

$$= \frac{1}{4} \cdot (R_e/L)^2 \cdot (R_m/L)^2 \cdot (g_m/g_e) \cdot L \cdot f$$

$$= \frac{1}{4} \left(\frac{g_m}{g_e} \right) \cdot \left(\frac{R_e R_m}{L^2} \right)^2 \cdot L \cdot f = \frac{1}{4} \left(\frac{M_m}{M_e} \right) \cdot \left(\frac{R_e}{L} \right)^4 \cdot L \cdot f$$

$$= \frac{1}{4} \left(\frac{7.35}{598} \right) \cdot \left(\frac{6.38}{384} \right)^4 \cdot 3.84 \times 10^8 \text{ m} \cdot \left(\frac{M_e + M_m}{M_m} \right)$$

$$\approx \frac{1}{4} \left(\frac{M_m}{M_e} \right) \cdot \left(\frac{R_e}{L} \right)^4 \cdot L \cdot \frac{M_e}{M_m} \approx \frac{1}{4} \cdot \left(\frac{R_e}{L} \right)^4 \cdot L$$

$$\approx 7.3 \text{ meters}$$

$$\approx 24 \text{ feet.}$$

This sounds reasonable for the maximal tide

(15) Earth rotation changes g by $\omega_e^2 R_e = 4\pi^2 R_e / T^2 \approx .03 \text{ m/sec}^2$, so this should not affect h_{max} very much. (i.e., $\frac{1}{3}$ of 1%).

(16) Macropedia says greatest tide is in Fundy, ~ 50 feet. Note that there are other tides to think about like the sun.

Some gut basics of Hamiltonian mechanics:

- ① Given a Lagrangian $L(q, \dot{q}, t)$, one can render stationary, under variation of the path $\delta q_i(t)$, the action $S = \int L dt$. This yields the usual Euler-Lagrange equations which form the basis of Lagrangian mechanics:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}, \quad \forall i$$

- ② If one defines the canonical momentum as usual, one finds:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \frac{\partial L}{\partial q_i} = \dot{p}_i \quad / \text{Euler.}$$

If, eg., some q_j does not appear in L , then $\dot{p}_j = 0$ and $p_j = \text{constant of the motion}$. Then q_j is called a cyclic coordinate.

- ③ Now define a Hamiltonian: $H = \sum_i p_i \dot{q}_i - L$.

Notice that $\frac{\partial H}{\partial \dot{q}_j} = p_j - \frac{\partial L}{\partial \dot{q}_j} = p_j - p_j = 0$, so H does not depend on $\dot{q}_j$'s.

However, H is $H(q, p, t)$ because

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i \quad \text{and} \quad \frac{\partial H}{\partial p_i} = +\dot{q}_i$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad \text{and} \quad \frac{dH}{dt} = \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial t} = -\dot{p}_i \dot{q}_i + \dot{q}_i \dot{p}_i + \frac{\partial H}{\partial t}$$

So here are the main results of Hamiltonian formalism:

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \quad \frac{\partial H}{\partial p_i} = \dot{q}_i \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

- ④ If L has no explicit time dependence, then $H = \text{conserved!}$

Nov 1977

A Review of Classical Mechanics

- ① System is described by some generalized coordinates, q_i . Usually, a potential is a function of these, so $V(q_i)$. Then you have some generalized forces $Q_i \equiv -\partial V / \partial q_i$. This fact only true for conservative forces, otherwise $\nexists$ no potential.

- ② Newton's Law $\vec{F}_i = \dot{\vec{p}}_i$ can be recast in this way:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \left(\frac{\partial T}{\partial q_j} \right) = Q_j \equiv \sum_i \vec{F}_i \cdot \left(\frac{\partial \vec{r}_i}{\partial q_j} \right)$$

where here you have generalized force even without its being conservative. $T = K.E.$ In order to prove this, however, you must

- ③ ~~have holonomic constraints~~, i.e., ones of the form $f(\vec{r}_i, t) = 0$ which allows you to go from $3N$ variables $\vec{r}_i$ to $3N-k$ "generalized" coords q_i .

- ④ If forces are conservative, then if you define Lagrangian $L = T - V$ and if $V = V(q_i) \neq V(q_i, \dot{q}_i)$, then above result becomes

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0} \quad \text{"Lagrange equations of motion"}$$

Now $L = L(q_i, \dot{q}_i, t)$, all three types of variables.

- ⑤ The classical action is a definite time integral of Lagrangian

Thus:
$$I \equiv \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt = \text{"the action"}$$

⑤ Hamilton's Principle reads $\delta I = 0$ where you vary the path keeping endpoints fixed. Goldstein derives the Lagrange equations from this principle.

⑥ The "canonical momentum" conjugate to q_i is $p_i \equiv \left(\frac{\partial L}{\partial \dot{q}_i} \right)$.
One says that a coordinate q_i is cyclic if $\frac{dp_i}{dt} = 0$, i.e., if its conjugate momentum is a constant in time.

⑦ The Hamiltonian is defined as

$$H = \sum_i \dot{q}_i p_i - L \quad \text{usually} = T + V.$$

$$\text{or } \boxed{H(p_i, q_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t)}$$

Notice that $\frac{\partial H}{\partial \dot{q}_i} = p_i - p_i = 0$, so $H \neq H(\dot{q}_i, \dots)$

Note also:
$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = -\dot{p}_i$$

And
$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad \text{So} \quad \left[\begin{array}{l} \frac{\partial H}{\partial q_i} = -\dot{p}_i \\ \frac{\partial H}{\partial p_i} = +\dot{q}_i \\ \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \end{array} \right] \quad \text{"Hamilton's Equations"}$$

⑤ Consider $H(p_i, q_i, t)$. Maybe you can find $Q_i = Q_i(p_i, q_i, t)$
 $P_i = P_i(p_i, q_i, t)$

new coords + canonical momenta such that

The new Q_i are cyclic, thus simplifying things. If the new P_i are conjugate to the Q_i + satisfy a Hamilton set,

$$\frac{\partial K}{\partial Q_i} = -\dot{P}_i$$

$$\frac{\partial K}{\partial P_i} = +\dot{Q}_i$$

Then $p, q \rightarrow P, Q$ is called a canonical
~~point~~ transformation. $K = K(P_i, Q_i, t)$
 is new Hamiltonian.

In order for a C.T. to exist, there must exist a generating function like $F_i(q_i, Q_i, t)$, see page 240.

⑥ A point transformation is like $Q_i = Q_i(q_i, t)$ only. This is a special case of a canonical transformation.

⑩ If P, Q and p, q are related by a canonical transformation, then you can show that the "jacobian sum" is an invariant:

$$\sum_i \frac{\partial(q_i, p_i)}{\partial(u, v)} = \sum_i \frac{\partial(Q_i, P_i)}{\partial(u, v)}$$

where u, v denote a point on a two-dimensional surface in huge $2N$ dimensional phase space $dp_1 \dots dp_n dq_1 \dots dq_n$.

It follows that

$$J_1 = \iint_{\text{2-dim surface}} \sum_i [dq_i dp_i] = \iint_{\text{same}} \sum_i [dQ_i dP_i]$$

This is a
 "Poincaré
 invariant"

⑪ Notice that:

$$\sum_i \frac{\partial \mathcal{L}(q_i, p_i)}{\partial (u, v)} = \sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix} = \sum_i \left[\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \frac{\partial q_i}{\partial v} \right]$$

$$\equiv \{u, v\}_{q, p} = \text{Lagrange Bracket of } u \text{ and } v.$$

Note in particular that $\left. \begin{aligned} \{q_i, q_j\} &= \{p_i, p_j\} = 0 \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned} \right\} \text{ fundamental Lagrange brackets.}$

⑫ Turn upside down to get Poisson brackets:

$$[u, v]_{p, q} \equiv \sum_i \left[\left(\frac{\partial u}{\partial q_i} \right) \left(\frac{\partial v}{\partial p_i} \right) - \left(\frac{\partial u}{\partial p_i} \right) \left(\frac{\partial v}{\partial q_i} \right) \right]$$

Then you can show these results:

$$\left. \begin{aligned} [p_i, p_j] &= 0 \\ [q_i, q_j] &= 0 \\ [p_i, q_j] &= \delta_{ij} \end{aligned} \right\} \text{ "fundamental Poisson Brackets."}$$

⑬ Theorem: Poisson brackets are indep. of which (canonically related) coordinates you use. $[u, v]_{p, q} = [u, v]_{p, q}$.

⑭ Equations of motion:

$$[q_i, H] = \sum_j \left[\underbrace{\left(\frac{\partial q_i}{\partial q_j} \right)}_{\delta_{ij}} \left(\frac{\partial H}{\partial p_j} \right) - \underbrace{\left(\frac{\partial q_i}{\partial p_j} \right)}_0 \left(\frac{\partial H}{\partial q_j} \right) \right] = \frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$[p_i, H] = -\frac{\partial H}{\partial q_i} = -\dot{p}_i$$

In general,

$$\left[u, H \right] + \frac{\partial u}{\partial t} = \frac{du}{dt}$$

Physics of sprinting

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Sprinting is described by a simple physical model. The model is used to predict the differences between the recorded times for races on a straight track and on a curve. It is shown that the choice of the running lane makes a non-negligible difference.

I. INTRODUCTION

In the past few years there has been a growing interest in the application of physical models to problems of animal locomotion in general^{1,2} and to human athletic performance in particular.³⁻⁵

A problem of special interest is to devise a model that represents with reasonable accuracy the propulsive and resistive forces acting on a human runner.

One common model assumes that the sum total of resistive forces F_r acting on a runner can be represented by

$$F_r = -\sigma Mv, \quad (1)$$

where M is the mass of the runner, v is the speed, and σ is a parameter of the model, presumed constant or nearly constant for a given runner.⁶ Although such a form is suggested by analogy with viscous forces, it is far from clear that it can adequately represent human runners where the forces are rapidly varying and, in general, complex. Nonetheless such a form was used by Keller³ to model competitive running and draw certain conclusions regarding optimal running strategy and maximal future performances. One significant conclusion reached by Keller is that a well-conditioned athlete is able to sustain a maximum and nearly constant muscular effort for races over distances of 290 m or less. We shall define all such races as sprints.

A basic question is: What are typical values of the parameter σ ? It is to be expected that σ will depend on the runner, his speed, the type race, and possibly on external conditions such as the track surface, altitude, etc. Keller³ uses a value of $\sigma = 0.44 \text{ sec}^{-1}$ while Whitt and Wilson⁵ quote similar values. The latter, however, were measured at walking speeds. To our knowledge there exists no measurement of σ at sprinting speeds.

We propose that for a given sprinter a measurement of σ and the propulsive force parameter f (defined below) can be performed in principle by measuring that sprinter's running times for two straight line races of different distances. Our model will then allow us to use the values of these parameters to predict that runner's performance for a sprint run on the curve.

II. MODEL

We assume as does Keller³ that a sprinter of mass M is subject to two horizontal forces. One is the resistive force expressed by Eq. (1) and the other is a constant propulsive force F_p which we write as

$$F_p = fM, \quad (2)$$

where f is the force per unit mass of the athlete.⁶ The equation of motion for the sprinter is then

In second part of 200m race, use this to get t assuming $v_0 = v_t(1-\epsilon)$.
Result: $t_{2nd100} = \frac{x}{v_t} + \frac{\epsilon}{\sigma}$

$$\frac{d^2x(t)}{dt^2} = \frac{dv(t)}{dt} = f - \sigma v(t), \quad (3)$$

where $x(t)$ is the distance measured from the start of the race and $v(t)$ is the sprinter's instantaneous velocity. The solution to this equation subject to the initial conditions $x(0) = 0$, $v(0) = v_0$ is

$$v(t) = (f/\sigma)(1 - e^{-\sigma t}) + v_0 e^{-\sigma t}, \quad (4)$$

$$x(t) = (f/\sigma)t + (f/\sigma^2 - v_0/\sigma)(e^{-\sigma t} - 1). \quad (5)$$

These well-known solutions describe the motion of a particle sedimenting under the action of a constant force of "gravity" Mf and a viscous force $-\sigma Mv$; the particle (sprinter) approaches the terminal velocity $v_t = f/\sigma$ asymptotically.

Let us suppose that the times t_1 and t_2 for an athlete's performances over two different straight-line sprinting distances x_1 and x_2 are known. Then Eq. (5) can be used (with $v_0 = 0$) to solve for f and σ (see Appendix).

We give in Table I the best running times over two different distances for four world-class sprinters from the 1967-69 period.^{7,8} (We note that two of them are still holders or coholders of three separate world records.) The computed values of σ and f for each runner are shown in columns 6 and 7, respectively.

The last entry in Table I is Tommie Smith who has the distinction of having a recorded time for the straight-track 100-m race and also for the rarely run straight-track 220-yard race for which he holds the world's record. In addition he is the holder of the world's record for the 200 m on the curve which he established in Mexico City in 1968.⁷

The computed values of the parameters f and σ differ widely from runner to runner. We have no way of knowing how reliable the data are but if we assume that the computed values of f and σ for Tommie Smith are accurate, then we can extend our theory to predict his performance in a 200-m race on the curve. The closeness of our prediction to the actual recorded time will be a test of the validity of the basic assumptions of the theory embodied in Eqs. (1) and (2).

III. RUNNING THE CURVE

In a typical 200-m race the starts are staggered so that each runner runs the first 100 m of the race on the curve. However, the radius of curvature of the curved portion of each lane is different. The lanes are 1.22 m wide so that the inner radius of the n th lane is

$$R(n) = 100/\pi + (n-1)(1.22) \text{ m}.$$

Let the runner be running with speed v on a curve of

- ① get f and σ from (13) and (14) and get R from $R = \frac{100}{\pi} + (n-1) \times 1.22$
 ② use $v_t = f/\sigma$ and $\epsilon = \frac{1}{2}(fR^{-1}\sigma^{-2})^2$
 ③ then (12) tells you x_{TOTAL} on 200m race, 1st part on curve.

Table I. Computed values of the parameters f and σ for various runners (*indicates a current world record holder). The times are the runners best times over the given distances.^{7,8}

Runner	x_1	$t_1(\text{sec})$	x_2	$t_2(\text{sec})$	$\sigma(\text{sec}^{-1})$	$f(\text{N/kg})$	$v_t(\text{m/sec})$	$ \delta\sigma/\sigma $	$ \delta f/f $	$ \delta v_t/v_t $
John Carlos	60 yards	6.0	100 yards	9.0	0.667	8.13	12.19	13%	10%	3%
Bill Gaines	60 yards	5.9	100 yards	9.3	1.250	13.45	10.76	25%	23%	3%
Jim Hines*	100 yards	9.2	100 m	9.9	0.581	7.10	12.22	65%	51%	14%
Tommie Smith*	100 m	10.1	220 yards (straight)	19.51	1.252	13.46	10.75	19%	18%	1%

constant radius R and subject to the forces (1) and (2). Now, however, the propulsive force must supply a component that gives rise to the centripetal acceleration of magnitude v^2/R . The athlete's equation of motion then becomes

$$dv/dt + \sigma v = (f^2 - v^4/R^2)^{1/2}. \quad (6)$$

Equation (6) is to be solved with given parameters f , σ , and R subject to the initial conditions $v(0) = 0$, $x(0) = 0$.

An approximate analytical solution can be obtained by noticing that for typical situations the second term under the square root is small, $v^4/R^2 \ll f^2$, and by expanding the square root we obtain

$$dv/dt + \sigma v \cong f - (1/2)v^4/fR^2. \quad (7)$$

As it stands, Eq. (7) is still not subject to a simple analytical solution. We can, however, solve it by treating the term in v^4 as a small perturbation (see Appendix). The solution so obtained subject to the initial conditions $v_c(0) = 0$, $x_c(0) = 0$ gives the speed $v_c(t)$ and the distance travelled along the curve $x_c(t)$ as:

$$v_c(t) = (f/\sigma)(1 - Z^{-1}) - 1/2(f/\sigma)(fR^{-1}\sigma^{-2})^2 F_1(Z), \quad (8)$$

$$x_c(t) = (f/\sigma^2)(\ln Z - 1 + Z^{-1}) - (1/2)(f/\sigma^2)(fR^{-1}\sigma^{-2})^2 F_2(Z), \quad (9)$$

where $Z = e^{\sigma t}$ and

$$F_1(Z) = 1 + (10/3 - 4\ln Z)Z^{-1} - 6Z^{-2} + 2Z^{-3} - 1/3 Z^{-4},$$

$$F_2(Z) = -37/12 + \ln Z + (2/3 + 4\ln Z)Z^{-1} + 3Z^{-2} - 2/3 Z^{-3} + 1/12 Z^{-4}.$$

The accuracy of our perturbation procedure depends on the smallness of the dimensionless parameter $\epsilon = (fR^{-1}\sigma^{-2})^2/2$ which for the case of interest to us has a value of 0.03139. For running times on the order of 10 sec, terms in Z^{-1} and higher powers of Z^{-1} are negligible and can be dropped from the expressions for F_1 and F_2 .

With these approximations we can write expressions for $x_c(t)$ and $v_c(\infty)$, the latter being the terminal speed of the runner on the curve. We note that with our typical value of σ the sprinter attains 98% of his terminal speed within approximately 3 sec:

$$v_c(\infty) \cong (1 - \epsilon)v_t. \quad (10)$$

$$x_c(t) \cong v_t(1 - \epsilon)t - (v_t/\sigma)(1 - 37/12 \epsilon). \quad (11)$$

Equations (10) and (11) are useful in that they allow us to bring out the effect of curvature of the track represented by the parameter ϵ . Physically, 2ϵ is the square of the ratio of the centripetal acceleration v^2/R calculated at terminal speed to the propulsive force per unit mass f . With our pa-

rameters the difference between v_t and $v_c(\infty)$ is of the order of 0.3 m sec⁻¹. Thus the runner accelerates slightly upon entering the straightaway.¹⁰

Equations (10) and (11) allow us to compute the total time t_T for the race in a closed (approximate) form

$$t_T = t_{100} + [100/v_t + \epsilon/\sigma] \leftarrow \text{time on 2nd 100} \\ = 2(100)/v_t + 1/\sigma + \epsilon[(100/v_t) - 13/(12\sigma)] \\ + \epsilon^2[(100/v_t) - 25/(12\sigma)] + O(\epsilon^3). \quad (12)$$

Equation (12) then can be used to predict the running time, for example, for Tommie Smith running a 200-m race on the curve. We use the parameters of Table I and assign him to lane 3, the lane in which he ran for his world record. Then $R = 34.27$ m, $\epsilon = 0.0314$, $v_t = 10.75$. We obtain

$$t_{100} = 10.35 \text{ sec}, \\ v_{100} = 10.41 \text{ m/sec}, \\ \text{time to run the last 100 m} = 9.33 \text{ sec}, \\ \text{total predicted time for the race} = 19.68 \text{ sec}.$$

In the Mexico City Olympics on 16 October 1968 in the men's 200-m final, Tommie Smith running in lane 3 set the still-current world record of 19.83 sec.

In order to test the accuracy of our calculation we numerically integrated the exact Eq. (6) and computed the time necessary to cover the first 100 m of the course on the curve. The comparison of this calculation with the results of our approximate analytical solution is shown in Table II. The two procedures give results that differ at most by 0.3%. This surprisingly close agreement lends support to the validity of the approximations made in the calculations. The close agreement between our "predicted" and the measured value of Tommie Smith's performance in the 200-m race on the curve comes as somewhat of a surprise considering the meagerness of the data and the crudeness of the model.

Finally, we apply our model to the following problem. Since the value of the radius of curvature R enters Eqs. (8)-(12) we would expect that athletes running in the outside lane (lane 8) would enjoy an advantage over those in the inside lane.

In fact, it is well known that sprinters dislike running in

Table II. Comparison of numerical solutions with approximate analytical solutions. Times are calculated for the first 100 m of the race on the curve; $f = 13.46$ N/kg, $\sigma = 1.252$ sec⁻¹.

Lane	$R(\text{m})$	$t_{100}(\text{sec})$ numerical solution	$t_{100}(\text{sec})$ approximate solution	difference (%)
1	31.83	10.361	10.389	0.27%
3	34.27	10.327	10.348	0.20%
8	40.37	10.266	10.277	0.11%
straight track	∞	10.100	10.100	

→ Solve for t : $t = \left(\frac{x_c}{v_t} + \frac{1}{\sigma}\right) + \epsilon \left[\frac{x_c}{v_t} - \frac{25}{12\sigma}\right] + \epsilon^2 \left[\frac{x_c}{v_t} - \frac{25}{12\sigma}\right] + O(\epsilon^3)$

the inside lane complaining that the turn is "too tight." Our calculation bears this out. We find that, with the parameters we used for our test case, the total recorded time for a runner in lane 1 would have been 19.72 sec whereas for lane 8 it would have been 19.60 sec. At finishing speeds of about 11 m/sec that difference in time translates into a distance difference of over a meter. Since many sprint races are lost or won by mere centimeters the above effect is quite significant.

Curiously enough sprinters do not consider lane 8 to be the most advantageous one. Because of the staggered start runners in lane 8 run for half the race ahead of the other competitors. This, they claim, puts them at the psychological disadvantage of not being able to see what their competition is doing.

IV. ERRORS

We wish to make a few brief comments about the "experimental" errors. Since neither the records books nor the sport press include error bars with their reported results we will limit ourselves to a few crude estimates. We assume that most of the error is the result of inaccurate time measurements. Many of the times given in Table I were probably the averages of several hand-recorded times. Other times might have been recorded electronically and then rounded off to the nearest tenth of a second.

For the purposes of this estimate, we assume that the accuracy in the recorded times is ± 0.05 sec. We notice that the resulting variation in the calculated values of f and σ are quite large. This is due in a large part to the fact that the calculation involves a denominator which is a difference between two numbers numerically close to one another. One consequence of this effect is that data for races over similar distances such as 50 and 60 yards or 100 yards and 100 m are practically useless.

However, as is apparent from Eq. (12), the total time t_T depends primarily on v_i , so to some degree the errors in f and σ tend to cancel one another. The uncertainties included in Table I were calculated according to the assumption $\Delta t_1 = -\Delta t_2 = \pm 0.05$ sec. Assuming this error in the measurement of time the resulting error in the predicted t_T for T. Smith is

$$\Delta t_T = \pm 0.054 \text{ sec.}$$

A less conservative error estimate of $\Delta t_1 = -\Delta t_2 = \pm 0.11$ sec brings the predicted and observed t_T within the range of "experimental" error.

V. CONCLUSIONS

In summary we have found that the forces acting on a sprinter at typical sprinting speeds can be reasonably well modeled by Eq. (3). The use of that equation over two different distances can determine the sprinter's parameters f and σ . The values of the parameters that we calculate for a few sample cases are in some disagreement with those commonly found in the literature. Our model, extended to include centripetal effects, then allows us to predict an athlete's time for a race run on the curve. This in turn can be used to compute the difference in the times for two comparable athletes running in different lanes. This difference is solely due to the different radii of curvature of

their respective lanes. We find its magnitude to be non-negligible.

Clearly our results suggest a line of further investigation for those interested in the biomechanics of sports. To the extent that a sprinting human can be characterized by parameters f and σ it might be of interest to measure these parameters under carefully controlled conditions and correlate them with the physical characteristics of the sprinters, such as height, weight, type of build, reaction time, strength of leg muscle, etc. It is conceivable that these parameters f and σ might also depend on external conditions such as track surface, altitude, and other factors.

Finally, we wish to comment that it has been our experience that there is a fairly large audience among today's college students interested in learning about the biomechanics of sports from the point of view of fairly sophisticated physics. Such a trend is certainly indicated by the amount of research done in many countries in the area of sports medicine and technique.

APPENDIX

In order to determine the parameters f and σ for a given sprinter, let us assume that the sprinter has recorded times t_1 and t_2 over two straightaway distances of lengths x_1 and x_2 . Using Eq. (5) twice with $v_0 = 0$ and neglecting the exponential term we solve for f and σ to obtain

$$\sigma = (1/t_1)[(x_2/x_1) - 1] / [(x_2/x_1) - (t_2/t_1)]. \quad (13)$$

$$f = \sigma^2 x_2 / (\sigma t_2 - 1). \quad (14)$$

In solving Eq. (7) we proceed as follows. First, we solve it with the v^4 term set equal to zero. This solution which we call w is the same as Eq. (4) with $v_0 = 0$,

$$w(t) = (f/\sigma)(1 - e^{-\sigma t}).$$

Next, we write the solution to Eq. (7) as $v = w + u$ where we treat u as a small perturbation. u then satisfies

$$du/dt + \sigma u = -1/2 v^4 / (fR^2). \quad (15)$$

Replacing v in Eq. (15) by its approximation w , we integrate twice to obtain the solutions (8) and (9).

For our test case (Tommie Smith) the parameters f and σ are directly determined from Eqs. (13) and (14) using the data given in Table I.

Equations (8) and (9) can be further simplified if we notice that for our test case $\sigma \approx 1.25 \text{ sec}^{-1}$ and for running times of the order of 10 sec, $Z^{-1} \sim 10^{-6}$ so that F_1 and F_2 can be approximated by

$$F_1 \approx 1, F_2 \approx t\sigma - 37/12.$$

This gives Eqs. (10)–(12). These equations then predict the time to complete the curve portion of the race t_{100} , the speed at the end of 100 m, and the total time.

For comparison we find that, using the same values of f and σ , a 200-m race run entirely on the straightaway would have taken 19.40 sec as compared with our lane 3 calculated time 19.68 sec. Even for a runner in lane 8 the straightaway time is faster by 0.20 sec.

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³J. B. Keller, *Phys. Today* **26**(9), 42 (1973).

⁴H. Lin, *Am. J. Phys.* **46**(1), (1978).

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⁶We use mks units throughout so that $M = \text{kg}$, $x = \text{m}$, $v = \text{m/sec}$, F

$= \text{N}$, $f = \text{N/kg}$, and $\sigma = \text{sec}^{-1}$.

⁷N. McWhirter, *Guinness Book of World Records* (Sterling, New York, 1979).

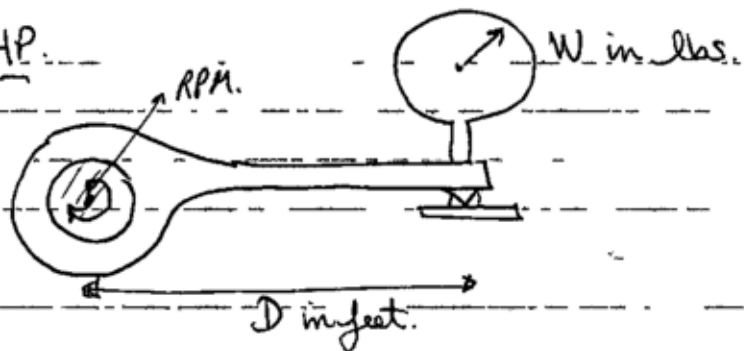
⁸*Sports Illustrated* (New York, NY) and *Track and Field News* (Los Altos, CA) miscellaneous issues between 1967 and 1969.

⁹Personal observation (IA).

¹⁰We wish to thank an anonymous referee for pointing this out to us.

Cars and Torque and BHP.

Measuring device:



This type of device was actually used to measure engine horsepower at one time. You open the throttle all the way, then tighten the friction brake to get your desired RPM. Then you read your force scale in pounds. A typical set of numbers might be $D=4$ feet and $W=30$ lbs. This means the torque is 120 lb-feet. ft/lbs. That is, the test fixture is applying that much torque on the flywheel to keep no angular acceleration, so flywheel of course is doing opposite and equal torque. To get the brake horsepower, you multiply by angular velocity of course. This is 2π f. Since $1 \text{ HP} = 33,000$ pound-feet of work in 1 minute, you get little equation:

$$\text{BHP} = \frac{(2\pi D) \times (W) \times f}{33,000} = \frac{DWf}{5252} = \left(\frac{2 \cdot f}{5250} \right) = \text{BHP}$$

The formula is freshman physics: power is distance times force divided by time, with units converted to horsepower. The force distance factor is the torque which is invariant, whether measured at flywheel or at scale. Think of the distance that a point on the flywheel goes, working against the frictional force.

Normally torque decreases a little as RPM increase. If torque were constant, the BHP would simply increase with RPM linearly. The point of max BHP occurs at max RPM you can use engine at; max torque is at slower RPM.

Acceleration of your vehicle is of course directly affected by torque. I.e., $F = ma$ and torque $= I \alpha$. Car acceleration a is linear in torque, inverse in weight.

Example: Saab turbo does 160 HP at 5500 RPM. But it does 188 ft-lbs at 3000 RPM. This torque gives BHP $= 188 \times 3000 / 5250 = 107$ HP at 3000 RPM.

1. Car engine is rated at horsepower as function of RPM, and also torque. Torque is the more constant of the two versus RPM so lets use "average torque" delivered by the engine as you shift through the gears.

2. Units. Lets use good old American car units. So:

V	ft/sec	M	slugs
N	lb-feet	t	seconds
E	ft-lbs	P	ft-lb/sec

3. Now, lets say you want to accelerate a car from 0 to some velocity V. In doing so, you keep your engine operating at average torque of N and average RPM of RPM. (ie, this is related to omega). Assume that energy output of engine per second is:

$$\text{Power} = \omega \times \tau \quad (\text{recall } v \times F \text{ for linear motion})$$

The energy you put into your vehicle, assuming no frictional losses, is this times time. Set that equal to your kinetic energy at speed V and you can solve for t:

$$\frac{1}{2} m v^2 = \omega \tau t \Rightarrow t = \frac{\frac{1}{2} m (\text{slugs}) \cdot [v (\text{ft/sec})]^2}{2\pi f (\text{sec}^{-1}) \tau (\text{ft-lb})}$$

Now convert this to more common car units:

$$t = \frac{\frac{1}{2} \left[\frac{W(\text{lb})}{32} \right] \cdot [v (\text{mph}) \times 1.47]^2}{2\pi \left[\frac{f (\text{min}^{-1})}{60} \right] \cdot \tau (\text{ft-lb})} = \underbrace{\left\{ \frac{60}{64} \cdot \frac{(1.47)^2}{2\pi} \right\}}_{.32} \cdot \frac{W(\text{lb}) \cdot [v (\text{mph})]^2}{\text{RPM} \cdot \tau (\text{ft-lb})}$$

So here is our result for acceleration with no friction due to air resistance or transmission train losses (engine losses are taken into account since we use actual measured engine torque; shifting losses also not accounted for):

$$t = \frac{(.32) W \cdot v^2}{\text{RPM} \cdot \tau}$$

Example: you are sitting in a 1986 RX, so total weight is 2800 lbs. Engine is 138 ft-lbs at 3500 RPM. Assume this product can be maintained. Then we could compute a theoretical 0-60 time: the result is 6.7 seconds, the quote is 8.0 seconds, so you add about 20% to your computed time.

Horsepower. The BHP is correct at its rated RPM, but the effective HP over driving speeds may differ. If you have a torque at an RPM, here is your HP :

$$P(\text{ft-lb/sec}) = 2\pi f(\text{sec}^{-1}) \cdot \tau(\text{lb-ft})$$

$$P(\text{BHP}) \times 550 = 2\pi \cdot \left[\frac{f(\text{min}^{-1})}{60} \right] \cdot \tau$$

$$P(\text{BHP}) = \left\{ \frac{2\pi}{60 \cdot 550} \right\} \cdot (\text{RPM} \cdot \tau) \quad \frac{2\pi}{33000} = \frac{1}{5252}$$

$$P(\text{BHP}) = \frac{\text{RPM} \cdot \tau}{5252}$$

Example: if you run your RX enginer at 138 ft-lbs and 3500 RPM, you are doing 92 horsepower. Rating is 146 HP at 6500 RPM. (its a rotary engine). If you could get 120 ft-lbs at 4500 RPM, that is 103 HP. You really need to know the torque as function of RPM. For the RX, you know that at 6500 you have toruq of $146 \cdot 5252 / 6500 = 118$ ft-lbs. Thus, fairly constant, characteristic of a rotary engine.

Example: consider the prelude enginer:

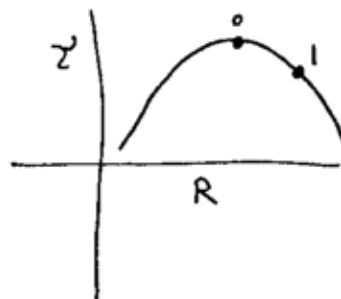
4000 RPM	107 ft-lbs	implies BHP = 81.5 hp
5500 RPM	100 hp	implies torque = 95 lb-feet

Again, fiarly constant torque. For any vehicle you are getting two torque,RPM points by reading the two specs.

A theory of "effective horsepower".

1. Assume that a car's torque curve is a parabola. You can determine this curve because the torque spec gives you the peak of the curve, and the horsepower spec gives you another point on the curve. Once you have the curve, you can compute the average horsepower over some range of RPM. Having computed this figure, you can compare different vehicles.

$$\tau = \tau_0 - (\tau_0 - \tau_1) \cdot \left[\frac{R - R_0}{R_1 - R_0} \right]^2$$



$$\text{HP} = \tau \cdot R \cdot \frac{1}{5252}$$

$$\langle P \rangle = \frac{\int_{R_a}^{R_b} (\tau(R) \cdot R) dR}{R_b - R_a} = \frac{1}{5252(R_b - R_a)} \left[\int_{R_a}^{R_b} [\tau_0 - K(R - R_0)^2] R dR \right]$$

$$K = \frac{\tau_0 - \tau_1}{(R_1 - R_0)^2}$$

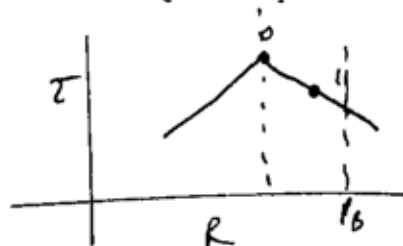
$$\text{int} = \tau_0 \cdot \frac{1}{2} (R_b^2 - R_a^2) - K \int_{R_a}^{R_b} [R^3 - 2R_0 R^2 + R_0^2 R] dR$$

$$\frac{1}{4} (R_b^4 - R_a^4) - 2R_0 \cdot \frac{1}{3} (R_b^3 - R_a^3) + R_0^2 \cdot \frac{1}{2} (R_b^2 - R_a^2)$$

$$\langle \text{HP} \rangle = \frac{1}{5252} \cdot \frac{1}{(R_b - R_a)} \left[\frac{1}{2} \tau_0 (R_b^2 - R_a^2) - \frac{(\tau_0 - \tau_1)}{(R_1 - R_0)^2} \left\{ \frac{1}{4} (R_b^4 - R_a^4) - \frac{2}{3} R_0 (R_b^3 - R_a^3) + \frac{1}{2} R_0^2 (R_b^2 - R_a^2) \right\} \right]$$

At this point I need a computer. You would enter the various numbers and get an answer.

2. Lets try a simpler model. A Bartlett curve:



$$\tau = \tau_0 - (\tau_0 - \tau_1) \left[\frac{R - R_0}{R_1 - R_0} \right]$$

$$\langle \text{HP} \rangle = \frac{1}{5252} \int_{R_0}^{R_b} [\tau_0 - K(R - R_0)] R dR$$

$$= \frac{1}{5252} \int_{R_0}^{R_b} [(\tau_0 + K R_0) - K R] R dR$$

$$\langle \text{HP} \rangle = (\tau_0 + K R_0) \frac{1}{2} (R_b^2 - R_0^2) - K \frac{1}{3} (R_b^3 - R_0^3) \frac{(\tau_0 - \tau_1)}{(R_1 - R_0)}$$

Again I need a computer. So:

3. Lets go even simpler. Lets assume that average horsepower is average RPM times average torque and use the linear model again:

$$\langle \tau \rangle = \frac{1}{\Delta R} \int_{R_0}^{R_B} dR = \frac{1}{\Delta R} \int_{R_0}^{R_B} [\tau_0 - k(R - R_0)] dR$$

$$= \frac{(\tau_0 + kR_0)(R_B - R_0) - k \frac{1}{2} (R_B^2 - R_0^2)}{(R_B - R_0)}$$

$$= (\tau_0 + kR_0) - \frac{1}{2} k (R_B + R_0)$$

$$\langle \tau \rangle = \tau_0 + k [R_0 - \bar{R}] = \tau_0 - k [\bar{R} - R_0]$$

$$\langle R \rangle = \frac{1}{\Delta R} \int_{R_0}^{R_B} R dR = \frac{\frac{1}{2} (R_B^2 - R_0^2)}{R_B - R_0} = \langle R \rangle$$

So $\langle HP \rangle = \frac{1}{5252} \langle \tau \rangle \langle R \rangle$ where $\langle \tau \rangle = \tau_0 - \left| \frac{\Delta \tau}{\Delta R} \right| [\bar{R} - R_0]$

$$\bar{R} = \frac{1}{2} [R_0 + R_1]$$

$$\Delta \tau = \tau_0 - \tau_1 \quad \bar{R} = \langle R \rangle = \frac{1}{2} [R_0 + R_B]$$

$$\Delta R = R_B - R_0$$

$$R_B = 5500$$

Ex: Turbo: $R_0 = 3000$ $\tau_0 = 188$ $\Delta \tau = 35$ $\bar{R} = \frac{1}{2} [3000 + 5500] = 4250$
 $R_1 = 5500$ $\tau_1 = 153$ $\Delta R = 2500$

$$\langle \tau \rangle = 188 - \frac{35}{2500} [4250 - 3000] = 170.5 \quad \langle HP \rangle = 138.0$$

BMW: $R_0 = 3250$ $\tau_0 = 170$ $\Delta \tau = 20$ $\bar{R} = \frac{1}{2} [3250 + 5500]$
 $R_1 = 4250$ $\tau_0 = 150$ $\Delta R = 1000$ $= 4375$

$$\langle \tau \rangle = 170 - \frac{20}{1000} [4375 - 3250] = 147.5 \quad \langle HP \rangle = 122.8$$

How to compute acceleration times

Assume:

- ① car can deliver f % of peak HP on average thru shifts. (efficiency)
- ② weight is curb weight + 175 lb driver.
- ③ frictional losses is included in f of ① above.

Then:

$$\frac{1}{2} m v^2 = P \cdot t \cdot f \Rightarrow t = \frac{\frac{1}{2} m v^2}{f \cdot P}$$

Again: $t(\text{sec}) = \frac{\frac{1}{2} m(\text{slugs}) \cdot [V(\text{ft/sec})]^2}{f \cdot P[\text{ft-lb/sec}]}$

Now $P[\text{ft-lb/sec}] = P[\text{HP}] \cdot 33000 \text{ lb-ft/min}$

$$P[\text{ft-lb/sec}] = P[\text{ft-lb/min}] \cdot \frac{1}{60} = P[\text{HP}] \cdot \frac{33000}{60} = 550 P$$

So $P[\text{ft-lb/sec}] = 550 \cdot P[\text{HP}]$

So, $t = \frac{\frac{1}{2} \cdot \left[\frac{M(\text{lbs})}{32} \right] \cdot [1.47(V_{\text{mph}})]^2}{f \cdot 550 \cdot P[\text{HP}]}$

$$t = \frac{M(\text{lbs}) \times V(\text{mph})^2}{f \cdot P(\text{HP}) \cdot 16290}$$

Or: (ft) =

3.96	for	Jeep 0-50	(act = 7.1)	$f = 56\%$	(1.5 $\Rightarrow$ 53%)
4.68	for	Celica 0-60	(act = 8.21)	$f = 57\%$	
5.45	for	Prelude 0-60	(act = 10.1)	$f = 54\%$	
5.71	for	Ford 0-60	(act = 10.9)	$f = 52\%$	
3.91	for	318 0-50	(act = 7.9)	$f = 47\%$	
4.24	for	RX 0-60	(act = 8.0)	$f = 53\%$	

$$t = \frac{M(\text{lbs}) \cdot V(\text{mph})^2 \cdot \eta(\text{eff})}{P(\text{BHP}) \cdot 16290} \quad \eta \approx 53\%$$

CRX predict: $t(0-60) = \frac{(1995) \times (60)^2}{.53 \times 91 \times 16290} = \frac{9.14 \text{ seconds}}{6.35 \text{ 0-50}}$

Corrugation of Roads

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Abstract

We present a one dimensional model for the development of corrugations in roads subjected to compressive forces from a flux of cars. The cars are modeled as damped harmonic oscillators translating with constant horizontal velocity across the surface, and the road surface is subject to diffusive relaxation. We derive dimensionless coupled equations of motion for the positions of the cars and the road surface $H(x, t)$, which contain two phenomenological variables: an effective diffusion constant $\Delta(H)$ that characterizes the relaxation of the road surface, and a function $a(H)$ that characterizes the plasticity or erodibility of the road bed. Linear stability analysis shows that corrugations grow if the speed of the cars exceeds a critical value, which decreases if the flux of cars is increased. Modifying the model to enforce the simple fact that the normal force exerted by the road can never be negative seems to lead to restabilized, quasi-steady road shapes, in which the corrugation amplitude and phase velocity remain fixed.

1 Introduction

It is commonly observed that under the influence of a flow of traffic, dirt roads develop regular corrugations of longitudinal "pitch," or wavelength, between 0.5 and 1 m, and amplitude up to 50 mm [1]. At first glance such an instability of the road surface might seem counterintuitive, as one might guess that a flow of traffic would exert downward forces on the surface which tend to smooth and compact the road bed, thereby suppressing pattern development rather than promoting it. Yet irregular, rough roads do not in fact heal themselves, but instead become progressively rougher and more corrugated under a flow of cars. Similar phenomena also occur, though sometimes on different length and time scales, on paved roads, on railroad rails [2], and

on the rollers used to calendar paper [3]. The purpose of this paper is to investigate this phenomenon using a simple, tractable physical model.

An early attempt at explaining road corrugation is due to Relton [4], who proposed that the underlying instability mechanism is a "relaxation oscillation," caused essentially by stick-slip dynamics. According to this view, a moving wheel pushes grains ahead of it. The grains pile up in front of the wheel and form a heap. When the heap grows large enough, the wheel sticks momentarily, and then slips, running over the heap and leaving it behind as a ridge. For a given uniform speed, this stick-slip process will be fairly periodic, and so will generate equidistant ridges.

A different picture was provided by Mather [2], who argued that the origin of the road instability is the bouncing motion of the wheel, caused by random irregularities on the ground. When the bouncing occurs, the car is projected upward along a certain angle and is airborne for a brief time. When it then strikes the ground, the car creates a crater and the motion then repeats itself. According to this picture, it is not the piling up of grains ahead of the vehicle that is responsible for the instability, but the impact stress of the vehicle on the ground. The wavelength of the resulting corrugations will be determined by the competition between the typical distance the car flies over the ground and the size of the crater generated by the impact stress, which should in turn depend on the hardness of the ground and the relaxation time of the ejected grains. Mather's picture is similar to other surface instabilities involving granular materials, in particular the ripple patterns in wind blown sand [5, 6, 7, 8], where ejected grains are carried away by the wind and land in a place far from the ejection point. In such a nonlocal model, what sets the wavelength of the ripple is the ratio of the flux of the grains to the appropriately scaled saltation length, i.e., the distance that an ejected grain is carried by the wind.

The model we present in this paper builds on Mather's picture, but we ignore any nonlocal transport of grains along the road, and we assume that the cars, and more specifically their wheels, generally do not lose contact with the road. Instead, we model the cars simply as masses attached to damped springs. We assume that the downward contact forces exerted on the road by the wheels causes a permanent downward deflection of the road surface, by either erosion or plastic deformation. In either case we model the effect of the contact force on the road as a simple proportionality between the deformation at any point on the road and the force exerted on that point, with a phenomenological "softness" parameter as the proportionality

$$25 \text{ m/sec} \\ \frac{1}{50} \text{ sec} \Rightarrow \frac{1}{2} \text{ m}$$

constant. We also include a diffusive relaxation that tends to smooth the road surface, which may come about, for instance, as a result of rain. We find that the diffusive relaxation is a stabilizing effect, as one would expect, but that its presence or absence generally has no qualitative effect on the corrugation phenomena. What is important, however, is the phenomenon of hardening. We expect that the softness parameter, and also the diffusion coefficient if it is present, will decrease as the road compacts, so that the road becomes less susceptible to the passage of more cars once the first several have compacted it and perhaps produced corrugations. This turns out to be a stabilizing effect, narrowing the parameter range in which corrugations occur.

Since we are modeling the cars, which in reality are complicated mechanical systems, by simple damped harmonic oscillators, the question quickly arises as to what the appropriate natural frequency might be. Given the observed pitch of road corrugations of 0.5 to 1 m, and the presumed speeds in the range of, say, 10 to 25 m/s of the cars that produce them, we expect that the important oscillation modes must have natural frequencies of about 0.02 to 0.1 s. This is one or two orders of magnitude shorter than the frequency of the car body bouncing on its suspension, so that is unlikely to be the relevant oscillation; indeed drivers normally adjust to the roughness of the road they are on and slow down to avoid such bouncing. This suggests that the relevant mode may be that of the wheel attached to the suspension. Alternatively, the important oscillation may be an elastic deformation of the wheel itself [9]. This could account for an observed difference in the pitch of corrugations produced by hard and soft tires [1].

$$\frac{1}{50} \text{ sec} - \frac{1}{10} \text{ sec}$$

2 Model

Imagine a flux of cars traveling with some average horizontal velocity v_x along a road surface whose height above some arbitrary zero level is given as a function of time t and position x along the road by $H(x, t)$; see Fig. 1. The cars are supported on springs, such that the natural angular frequency of vertical oscillation of the cars is ω_0 . Further, we assume that the springs are damped with damping constant b . Let $Z(x, t)$ be the height of a moving car, relative to a zero level chosen such that $H(x, t) - Z(x, t)$ is the amount by which the springs are compressed. In a frame of reference moving horizontally with the car, the vertical component of the car's equation of motion comes

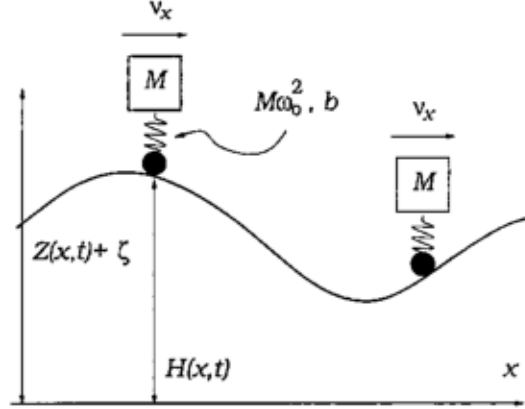


Figure 1: A schematic picture of the corrugation of roads. The road surface is subjected to a flux of cars with horizontal speed v_x , and the cars are modeled as damped harmonic oscillators with mass M , spring constant $M\omega_0^2$, and damping coefficient b . The heights of the cars and the surface are $Z(x, t) + \zeta$ and $H(x, t)$ respectively, where ζ is the equilibrium length of the spring.

simply from Newton's second law,

$$M \frac{d^2}{dt^2} Z(x, t) + b \frac{d}{dt} [Z(x, t) - H(x, t)] = M\omega_0^2 [H(x, t) - Z(x, t)]. \quad (1)$$

The time derivatives here are total derivatives; that is, the relevant value of x is time-dependent, since this equation follows the vertical motion of a single car. We may convert this into an equation for the height of the car at a fixed x by replacing the total time derivative d/dt with $\partial/\partial t + v_x(\partial/\partial x)$. Thus in a reference frame that is fixed to the road bed, the vertical equation of motion is

$$M \left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} \right)^2 Z + b \left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} \right) (Z - H) + M\omega_0^2 (Z - H) = 0. \quad (2)$$

Note that we have neglected any possible variation in the horizontal velocity v_x of the cars. ✓

We must now write an evolution equation for the height $H(x, t)$ of the road surface. We assume first that the road surface sinks at a rate which is proportional to the downward force on the road. This may be due to compaction of the road bed under the surface or to ejection of loose grains at the surface as a car passes; in either case we expect that the proportionality constant between the downward force on the road and its rate of sinking will decrease as the road sinks. That is, the road should harden as more and more cars pass over it. In addition, we assume that there is a "diffusive" relaxation process which tends to even out any roughness in the road surface. This can result from the action of wind or rain, and may also contain a contribution from the passage of the cars, as the loosely connected grains at the surface are fluidized by the flow of cars. Again, we expect the effective diffusion coefficient to decrease as cars pass and the road hardens. With these assumptions, the equation of motion for the road height reads

$$\frac{\partial H}{\partial t} = D(H) \frac{\partial^2 H}{\partial x^2} - a(H) [Mg + M\omega_0^2(H - Z)], \quad (3)$$

where Mg is the weight of a car. We are neglecting the geometrical distinction between $\partial H/\partial t$ and the rate of motion of the road surface normal to itself, and also the fact that the compressive force is not really vertical when H is not constant. These are satisfactory approximations provided the vertical scale of the surface corrugations is much smaller than the horizontal scale. However, these effects should be included in any model of road corrugation which is nonlinear in the amplitude of the corrugation.

The proportionality factor $a(H)$ above represents the softness of the road, that is, its responsiveness to compressing forces. It should include the flux of cars as a multiplicative factor, as the rate of road sinking should also depend on the density of the traffic it sustains. The compression term should be replaced by zero whenever the quantity in brackets is negative, since that represents the situation in which the cars are airborne, so that the compressive force on the road would really be zero rather than negative. Requiring that the road harden as it compacts means that a (and probably also D) should decrease as H decreases, so we want $a(H)$ to be positive and an increasing function of H . A physically acceptable form of $a(H)$ is shown in Fig. 2.

Before proceeding, we first nondimensionalize the equations of motion for Z and H . We choose the time scale to be $1/\omega_0$, the horizontal length scale to be v_x/ω_0 , and the vertical length scale to be g/ω_0^2 . The equation of motion

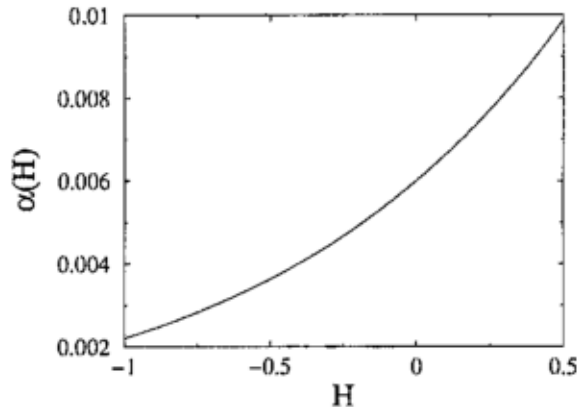


Figure 2: A physically plausible form for the softness or compactivity function $a(H)$. We require a to be an increasing function of H which either approaches zero for $H \rightarrow -\infty$ or vanishes for H below some fixed, finite limit. ✓

for the cars then becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 Z + 2\Gamma \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) (Z - H) + Z - H = 0, \quad (4)$$

and the equation for the road surface is

$$\frac{\partial H}{\partial t} = \Delta(H) \frac{\partial^2 H}{\partial x^2} - \alpha(H)(1 + H - Z), \quad (5)$$

where the new dimensionless parameters are given by

$$\Gamma = b/2M\omega_0, \quad \Delta(H) = (\omega_0/v_x^2) D(H), \\ \alpha(H) = M\omega_0 a(H). \quad (6)$$

The parameter Γ is a property of the cars alone; the cars' springs are underdamped for $\Gamma < 1$. Since the instability is controlled by the interplay of essentially two mechanisms, ejection of grains from the surface or compaction of the road by the flux of cars and subsequent relaxation of the road surface due to diffusion, Γ and Δ will control the dynamics of the surface. As we will see, the hardening of the road also plays an important role in the development of the instability.

3 Linear Stability Analysis

We may solve the system of partial differential equations (4) and (5) numerically, and we will discuss this below, but useful information regarding the behavior of the system may also be extracted from an approximate linear stability analysis. The first step in the analysis is to recognize that as time increases, the spatially averaged values of $Z(x, t)$ and $H(x, t)$ will tend to decrease, reflecting the gradual settling of the road bed. We denote this spatially averaged, i.e. spatially independent, quiescent solution by $H_0(t)$ and $Z_0(t)$. Substituting this solution into the equations of motion, we find that it satisfies

$$\ddot{Z}_0 + 2\Gamma(\dot{Z}_0 - \dot{H}_0) + Z_0 - H_0 = 0. \quad (7)$$

and

$$\dot{H}_0 = -\alpha(H_0)(1 + H_0 - Z_0), \quad (8)$$

The full solution may now be written as

$$H(x, t) = H_0(t) + h(x, t), \quad Z(x, t) = Z_0(t) + z(x, t). \quad (9)$$

in which the functions $h(x, t)$ and $z(x, t)$ will carry information about pattern development. Substituting these forms of $H(x, t)$ and $Z(x, t)$ into (4) and (5), using (7) and (8) for the time evolution of Z_0 and H_0 , and expanding to first order in h and z , we obtain the following linearized equations of motion:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 z + 2\Gamma\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)(z - h) + z - h = 0, \quad (10)$$

$$\frac{\partial h}{\partial t} = \Delta \frac{\partial^2 h}{\partial x^2} - \alpha(h - z) - \beta h, \quad (11)$$

where α and Δ are evaluated at $H = H_0$, and β is given by

$$\beta = (1 + H_0 - Z_0) \alpha'(H_0) = -\frac{d \ln \alpha(H_0)}{dt}. \quad (12)$$

The last equality here follows from (8). Note that β should be positive, since we expect that α will increase with H and decrease with time.

If α and Δ are nontrivial functions of H , then it is not simple to solve the linearized equations (10) and (11), because the coefficients in the latter equation are functions of time. However, we may perform an *approximate*

OK.

stability analysis by regarding α , β , and Δ as constants, at least for short time intervals, and calculating the linear growth rates that obtain when those parameters have their current values. Thus we will determine *time dependent* growth rates and phase velocities, provided we can ascertain the time dependent forms of α and Δ .

We proceed by assuming both h and z to be proportional to $\exp(ikx + \sigma t)$, where the linear growth rate σ is in general complex. As usual, $\text{Re}[\sigma]$ is the exponential growth or decay rate of the amplitude of a perturbation with wave number k , and $\text{Im}[\sigma]$ is related to the phase velocity c of that mode through $c = -\text{Im}[\sigma]/k$. This substitution yields

$$[(\sigma + ik)^2 + 2\Gamma(\sigma + ik) + 1]z = [1 + 2\Gamma(\sigma + ik)]h \quad (13)$$

and

$$\sigma h = -\Delta k^2 h - \alpha(h - z) - \beta h. \quad (14)$$

Eliminating h between these equations gives the stability relation

$$[(\sigma + ik)^2 + 2\Gamma(\sigma + ik) + 1](\beta + \sigma + \Delta k^2) + \alpha(\sigma + ik)^2 = 0, \quad (15)$$

from which we can find the growth rates and phase velocities of spatially sinusoidal perturbations in terms of their wave numbers.

It is possible to determine the stability boundary for the model, at least parametrically, from (15). That is, we could determine the locus in parameter space on which the real part of the linear growth rate σ , as a function of k , has a global maximum at height $\text{Re}[\sigma] = 0$. However, it is much more instructive to simplify the already approximate problem further by taking α and Δ to be small, as we expect to be the case once the road has had a chance to harden sufficiently. To set the stage for this calculation, imagine setting $\alpha = 0$ in (15). The cubic equation for σ would then factor. Two of the solutions would always have negative real parts, since $\sigma + ik$ for these two solutions would be a root of a quadratic with positive coefficients; these represent decaying perturbation modes. The third would be $\sigma = -\beta - \Delta k^2$, so this mode too would always be stable unless β is also small, on the same order as α or smaller. Note that small α does *not* necessarily imply that β must be small, because β is the *logarithmic* derivative of α .

Since we are interested in finding parameter ranges for which corrugations grow, we now focus on the case where α , β , and Δ are all small. As we just saw, only one of the three solutions for σ can possibly be positive; to leading

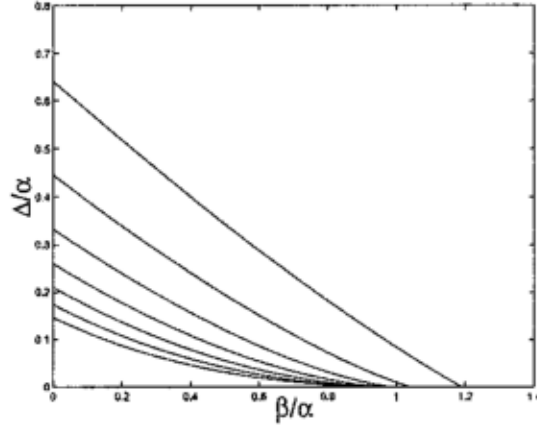


Figure 3: The approximate stability diagram in the (β/α) - (Δ/α) plane, obtained from (16), for values of the damping parameter Γ , ranging from 0.3 (top) to 0.9 (bottom) in steps of 0.1. All stability boundaries For $\Gamma \geq 0.5$ pass through the point (0,1). The flat road surface is stable above and to the right of the boundary curve for the appropriate Γ value.

order in the small parameters it is given by

$$\sigma = -\beta - \Delta k^2 + \alpha k^2 \frac{(1 - k^2) - 2i\Gamma k}{(1 - k^2)^2 + 4\Gamma^2 k^2}. \quad (16)$$

We may locate an approximate stability boundary by seeking combinations of parameters for which the real part of this σ has a maximum, as a function of wave number k , at $Re[\sigma] = 0$. Thus we solve the equations $Re[\sigma] = 0$ and $d(Re[\sigma])/dk = 0$ to obtain the critical values of β/α and Δ/α parametrically in terms of k . The results are shown for several values of the damping parameter Γ in Fig. 3. As is clear from (16), both β and Δ represent stabilizing effects, so for a given Γ , the flat road surface is stable above and to the right of the stability boundary curve for that Γ . The stability boundaries move down and left as Γ is increased, so damping of the car's springs also tends to stabilize the flat road surface. One can show explicitly that the stability boundary reaches $\Delta = 0$ at $\beta/\alpha = [4\Gamma(1 + \Gamma)]^{-1}$, and it reaches $\beta = 0$ at $\Delta/\alpha = [4\Gamma(1 - \Gamma)]^{-1}$ for $\Gamma < 1/2$ or $\Delta/\alpha = 1$ for $\Gamma > 1/2$. Moreover, for small Γ - that is, when the oscillations of the car springs are very slightly damped - the stability boundary is given approximately by the

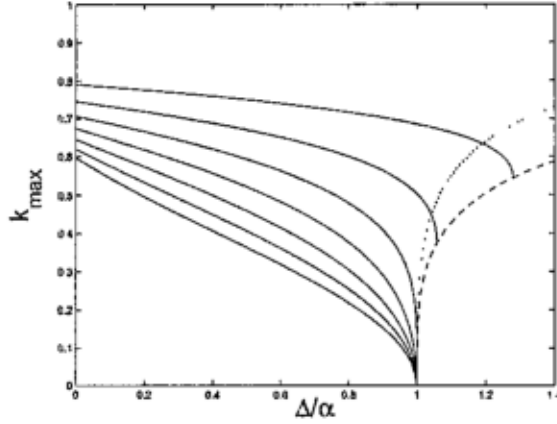


Figure 4: The wave number of the perturbation whose linear growth rate is maximal, plotted as a function of Δ/α for various values of the damping parameter Γ ranging from 0.3 (top) to 0.9 (bottom) in steps of 0.1. The dashed curve marks where the local maximum in $\text{Re}[\sigma(k)]$ merges with a local minimum and vanishes. Between it and the dotted curve, $\text{Re}[\sigma]$ has a local maximum at the indicated k , but the *global* maximum is at $k = 0$.

line $\Delta + \beta = \alpha/4\Gamma$.

To find the wave numbers of the most important perturbations, we first note from (16) that the parameter β only enters the expression for the growth rate additively. As a result, the wave number k_{max} at which the real part of σ has its maximum is independent of β ; it is given explicitly by

$$\frac{\Delta}{\alpha} = \frac{(1 - k_{max}^2)^2 - 4\Gamma^2 k_{max}^4}{[(1 - k_{max}^2)^2 + 4\Gamma^2 k_{max}^2]^2}. \quad (17)$$

Fig. 4 shows k_{max} as a function of Δ/α for several values of the damping constant Γ . (Below and to the right of the dotted curve, the maximum growth rate is negative for any positive value of β , so the flat road surface can only be unstable when Δ/α is to the left of the dotted curve.) We see that the wave number of the most rapidly growing mode decreases with increasing damping Γ ; it also decreases with increasing Δ/α , although the dependence on Δ/α is weak when Γ is small. In fact, for small Γ we always have $k_{max} = 1 - \Gamma + O(\Gamma^2)$. In no case do we find k_{max} to be above 1. Recall that k is the wave number of the perturbation in units of ω_0/v_x . Thus the

fact that k_{max} is always less than 1 means that the wavelength of the most rapidly growing perturbation is always greater than $(2\pi/\omega_0) v_x$, the distance a car travels in one period of its natural oscillation.

From the imaginary part of σ we obtain the drift velocity c of a perturbation of wave number k (in units of v_x),

$$c = -Im[\sigma]/k = \frac{2\alpha\Gamma k^2}{(1 - k^2)^2 + 4\Gamma^2 k^2}. \quad (18)$$

Note the factor α : since α is small, the drift velocity is small compared to v_x , the speed of the cars. Also, since α contains a factor of the number of cars passing per unit time, the drift is actually a fixed distance per passing car, rather than per unit time. The drift velocity is positive, so that corrugations are pushed in the direction that traffic is moving as they develop. It reaches its maximum value of $\alpha/2\Gamma$ at $k = 1$.

4 Numerical Results

Since the linear stability analysis of the preceding section is only approximate, we have found it useful to check its results by carrying out numerical simulations of the full model given by Eqns. (4) and (5). We choose a wave number k , choose a sinusoidal form for the initial $H(x)$ and $Z(x)$ with small amplitudes, and then integrate the equations on an interval of length $2\pi/k$ with periodic boundary conditions. We then monitor the amplitudes and phase velocities of H and Z as time passes. The logarithmic derivative of the amplitudes with respect to time gives the instantaneous exponential growth rate $\sigma(t)$.

For the non-constant α cases we use the simple ansatz

$$\alpha(H) = \alpha_0 \exp(\epsilon H), \quad \epsilon > 0, \quad (19)$$

which has the qualitative form shown in Fig. 2 and is mathematically tractable. The parameter ϵ controls the rate at which the soil hardens. We may solve for the time evolution of the quiescent road level H_0 provided α is small, for then Z_0 varies on a much shorter time scale than H_0 , and so Z_0 should always remain near H_0 . Then (8) reduces to

$$\dot{H}_0 = -\alpha(H_0) = -\alpha_0 \exp(\epsilon H_0). \quad (20)$$

This can be integrated immediately to yield $\exp(-\epsilon H_0) = \epsilon \alpha_0(t + t_0)$, where t_0 is a constant of integration. From this we get

$$\alpha = \frac{1}{\epsilon(t + t_0)}, \quad (21)$$

and from the definition (12) of β we then get

$$\beta = \frac{1}{t + t_0} = \epsilon \alpha, \quad (22)$$

so that the combination β/α that appears in the approximate linear stability analysis is just the constant ϵ .

Figs. 5a and 5b show typical numerical and theoretical determinations of the growth rate $Re[\sigma(t)]$ and phase velocity $c(t) = -Im[\sigma(t)]/k$ for α given by (19) above and Δ constant, for various values of the hardening parameter $\epsilon = \beta/\alpha$. As expected, for a given value of k the agreement is quite strong; this continues to hold over a range of α_0 and ϵ .

It is clear from the the linear stability analysis that if the diffusion parameter Δ remains finite as α decreases to zero, or more generally if α/Δ goes to zero as the road compacts, then we will eventually reach a situation in which diffusion dominates the dynamics. From that point on, any corrugation in the road will decay. On the other hand, we certainly expect that as the road compacts and its surface hardens, the hardening should inhibit the lateral transfer of material which we have modeled as diffusion, as well as further compaction of the road. We see, then, that in order to get nontrivial corrugations on the road surface, we must have Δ decrease at least as rapidly as α as the road compacts.

To investigate the possibility of generating corrugation patterns, we examine the simplest nontrivial case, namely where Δ and α have the same H -dependence, $\Delta(H) = \Delta_0 \exp(\epsilon H)$ and $\alpha(H) = \alpha_0 \exp(\epsilon H)$. As above, this ansatz leads to β/α being constant – specifically, equal to ϵ – and all three parameters α , β , and Δ decreasing as t^{-1} at long times t . From (16) we then see that the linear growth rate σ is also proportional to t^{-1} for large t . Figures 6a and 6b show this slow decrease of the real and imaginary parts of $\sigma(t)$ for a few single mode solutions, arrived at by numerical solution and stability analysis.

Since σ is the logarithmic time derivative of the amplitude of a perturbation, we see that the amplitude itself grows or decays *algebraically* in the

long-time limit:

$$\frac{d \ln A}{dt} = \sigma = \frac{\sigma_0}{t + t_0} \quad \rightarrow \quad A \propto (t + t_0)^{\sigma_0}. \quad (23)$$

Thus the growth rate of the corrugation amplitude becomes small at long times in this model, but the corrugation does not reach a true steady state.

It is clearly of interest to determine whether restabilized steady states exist. However, such states would depend on a host of nonlinear effects which are omitted in the model embodied in Eqns. (4) and (5). While we do not believe our model as presented is capable of predicting steady states, we have observed a remarkable feature in some of our numerical calculations. We have investigated cases in which α and Δ are constants. These cases neglect the hardening effect which, as we saw above, tends to stabilize the flat road surface; the parameter β vanishes. The linear stability analysis is then exact, and we expect to find purely exponential growth or decay of h and z . Our numerical calculations show that modes that are stable according to the linear stability analysis do indeed decay in amplitude. Modes which are linearly unstable grow until their amplitudes are so large that we have $1 + H(x, t) - Z(x, t) < 0$ for part of the cycle. When this happens, (5) suggests – unphysically – that the compression of the road is negative. What is happening here is that the cars are losing contact with the road. To avoid having the road surface spontaneously rise when an airborne car passes over it, we set $\alpha = 0$ whenever we are in this situation.

Such a detachment in fact does occur in real situations, and has been termed “bouncing” [9]. In a laboratory experiment involving two rotating disks that are in contact with each other under static compression, the two disks lose contact and bounce against each other when the amplitude of the corrugation along the perimeter of the disks exceeds about 1/3 the static compression. In the case of a car moving on a corrugated surface, such bouncing will kick the car into the air, but the car will quickly land in a different place, a mechanism suggested by Mather [2]. Such a bouncing motion should involve a local flux of the cars and a saltation function that relates the landing position to the start-off position, as in the case of wind-blown sand [5, 6, 7]. We note that such a nonlocal behavior would be difficult to account for in the model completely, but our mechanism of setting α locally to zero captures some of its flavor. Our expectations are that the modes that grow large enough to be subject to the local removal of the force term in (5) will obviously have their growth rates reduced from those predicted

by the linear stability analysis in the absence of such a force removal, and that this reduction in growth rate may be sufficient to generate steady states with finite amplitudes. To test this intuition, we numerically solve the system using single cycles of sinusoidal modes k , in periodic systems whose lengths are a single wavelength. The initial disturbances are chosen with amplitudes large enough to cause $1 + H(x, t) - Z(x, t)$ to be negative on occasion.

Provided a given mode is unstable according to the linear stability analysis, our numerical analysis indicates that it evolves toward a quasi-steady state in which the amplitude and phase velocity c of the road corrugations remain fixed, even while the average height of the road bed continues to sink. Fig. 7 shows the steady state amplitudes of h as a function of k for Γ , Δ , and α as given in the figure caption. For fixed k , the steady state amplitudes increase with increasing α . Thus we see for this choice of parameters that softer roads (i.e., larger α) support larger disturbances in the steady state than do harder ones. This result is intuitively satisfying, as we expect the ease of deformation of the material to correlate with the amplitude of the disturbance it supports. The phase velocities of the quasi-steady states exhibit particularly interesting behavior. Fig. 8a shows the observed quasi-steady state phase velocities, plotted as symbols, and an "envelope" in the $k - c$ plane. If there were no step function in the term in α , the phase velocities would lie in the region bounded by the envelope. The top curve would be the velocity curve for the largest α used (0.0055) and the bottom curve would be the velocity curve for the smallest α used (0.0040). The step function in the compactivity has forced the phase velocities to "collapse" so that they seem to lie along or near a single curve in the $k - c$ plane. Examination of the quasi-steady state velocities on a finer scale, as shown in Fig. 8b, indicates a persistence of the kind of variation of phase velocity with softness we have typically seen (that is, the softer the road, or equivalently the larger α is, the greater the phase velocity, all other things being equal), though this variation exists on a much finer scale in the steady state case than in the case in which the step function is not invoked.

Even though we have identified interesting steady state behavior in the solution of the differential equations in this limiting case of α being constant, we must reiterate that setting α to a constant is certainly unphysical, and that many nonlinear effects have been omitted from our model equations. Moreover, our model does not account carefully for what happens when contact with the road and cars is lost; it merely turns off the coupling between cars and road in the equation of motion for $H(x, t)$, and furthermore as-

sumes the evolution of $Z(x, t)$ remains well described by Eq. (4). Thus we cannot expect that our nonlinear calculations will speak to the behavior of real roads. Nonetheless, we find it remarkable that simply setting $\alpha = 0$ when $1 + H(x, t) - Z(x, t)$ is negative seems to contain the growth of the unstable modes.

5 Conclusions and Prospects

In this work, we have explored the origin of the corrugation instability of dirt roads subjected to a constant flux of cars. We have presented a simple phenomenological model for the evolution of the road surface and carried out a linear stability analysis to uncover the gross features of the instability. From our approximate stability analysis and its numerical verification, we find that the dynamical processes of diffusion and compression in a road bed are sufficient to generate instabilities that can give rise to pattern formation. Small-amplitude corrugations on the road surface grow when the diffusion parameter Δ/α and the hardening parameter β/α lie below the stability boundary (for the appropriate damping coefficient Γ for the cars) shown in Figure 3. From the definitions (6) of α and Δ in terms of the original, dimensional softness and diffusion coefficients $a(H)$ and $D(H)$, we see that the dimensionless combination Δ/α is inversely proportional to the square of v_x , the horizontal speed of the cars across the road. Thus we find that there is a critical car speed above which the flat road surface is unstable. Also, since $a(H)$ is proportional to the flux of cars, there is a critical flux for any given speed, above which the flat road is unstable. Both diffusive relaxation and hardening of the surface are stabilizing effects. The wave number of the most rapidly growing mode is given implicitly in dimensionless terms by (17), which is plotted in Fig. 4.

Our model is clearly schematic; a quantitative theory of road corrugations would need to incorporate a number of effects we left out of our calculations. We have not attempted to account, for instance, for the geometric distinction between $\partial h/\partial t$ and the normal velocity of the road surface as it compacts, nor for the fact that the force exerted by a moving car on a corrugated road surface is not purely vertical. For these reasons alone we have not carried out any nonlinear analysis on our model – the model is explicitly not valid beyond linear order in the amplitude of the incipient corrugation. We have also ignored all details of the physical processes by which the road compacts,

so it is not clear how well our phenomenological picture of compaction being proportional to applied vertical force or impulse can describe the dynamics of a real road. Our model for the cars is the simplest possible, neglecting the multiplicity of oscillation modes of real cars and the Hertzian nature of the contact force between the tires and the road. In particular, the fact that real vehicles have two or more pair of tires separated by fixed distances may introduce a new, relevant length scale into the problem. Finally, a quantitative theory would have to incorporate distributions of vehicle sizes, weights, oscillation frequencies, speeds, and the like.

Despite the fact that most nonlinear effects are omitted from our model, we find particularly interesting the numerical result that simply setting the contact force to zero when the equations would make it negative can lead to an apparently steady state. At least this results in a drastic slowing of the dynamics once a certain road configuration has been established. Continuing work will focus on determining the selection mechanism of such "frozen" states, and their response to local perturbations.

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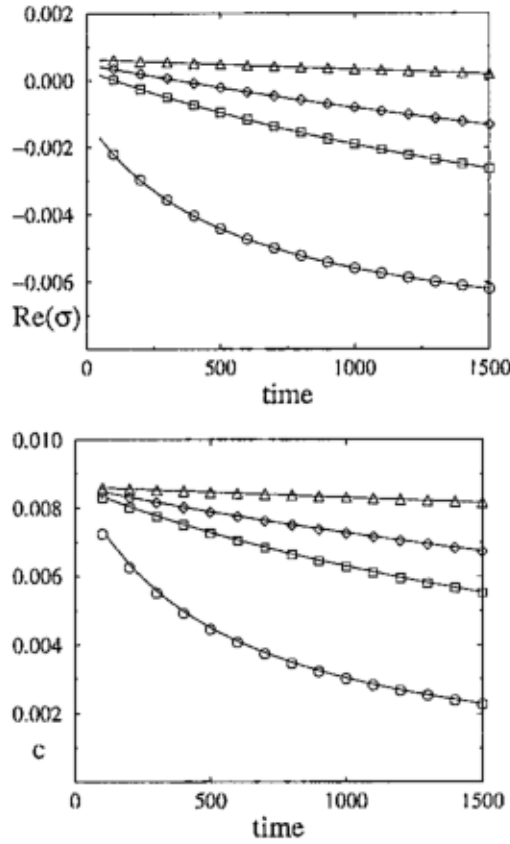


Figure 5: (a) Linear growth rate $Re[\sigma]$ and (b) phase velocity c vs. time for single mode solutions in the case $\alpha = \alpha_0 \exp(\epsilon H)$ with $k = 0.90$, $\alpha_0 = 0.004$, $\Gamma = 0.1$, $\Delta = 0.01$, $\epsilon = 0.5$ circle, 0.1 square, 0.05 diamond, 0.01 triangle. Continuous lines are from numerical solution of the full equations, symbols from the approximate linear stability analysis.

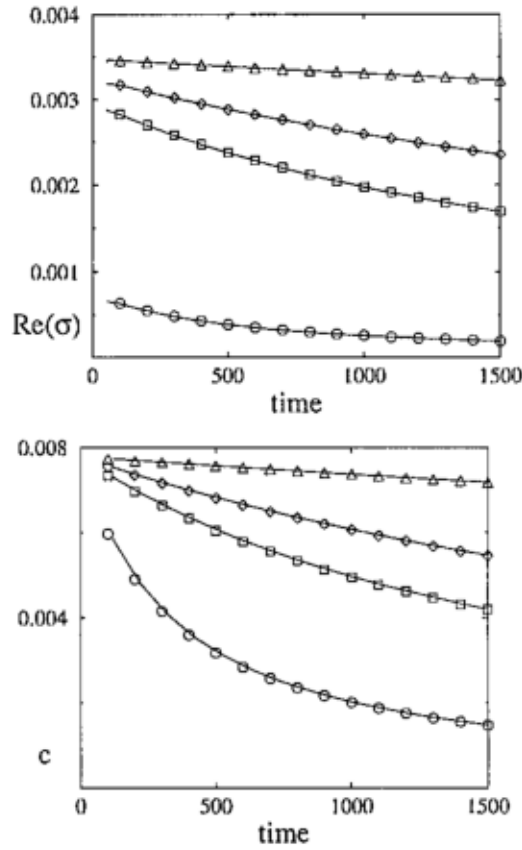


Figure 6: (a) Linear growth rate $Re[\sigma]$ and (b) phase velocity c vs. time for single mode solutions with $\alpha = \alpha_0 \exp(\epsilon H)$ and $\Delta = \Delta_0 \exp(\epsilon H)$. Parameter values are $k = 0.85$, $\alpha_0 = 0.006$, $\Delta_0 = 0.01$, $\Gamma = 0.1$, $\epsilon = 0.5$ circle, 0.1 square, 0.05 diamond, 0.01 triangle. Continuous lines are from numerical solution of the full equations, symbols from the approximate linear stability analysis.

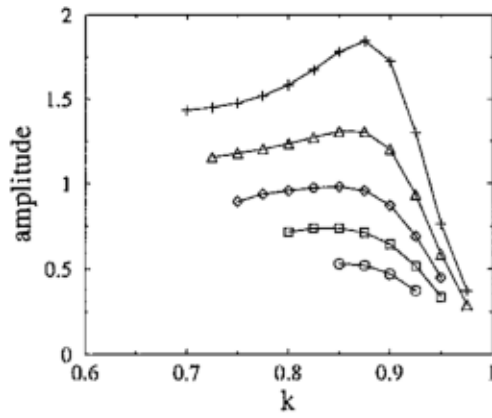


Figure 7: Quasi-steady state amplitude vs. k in the case of constant α , but setting $\alpha = 0$ when $1 + H - Z < 0$. Parameter values are $\Gamma = 0.10$, $\Delta = 0.01$, $\alpha = 0.0040$ circle, 0.0045 square, 0.0050 diamond, 0.0055 triangle, 0.0060 plus.

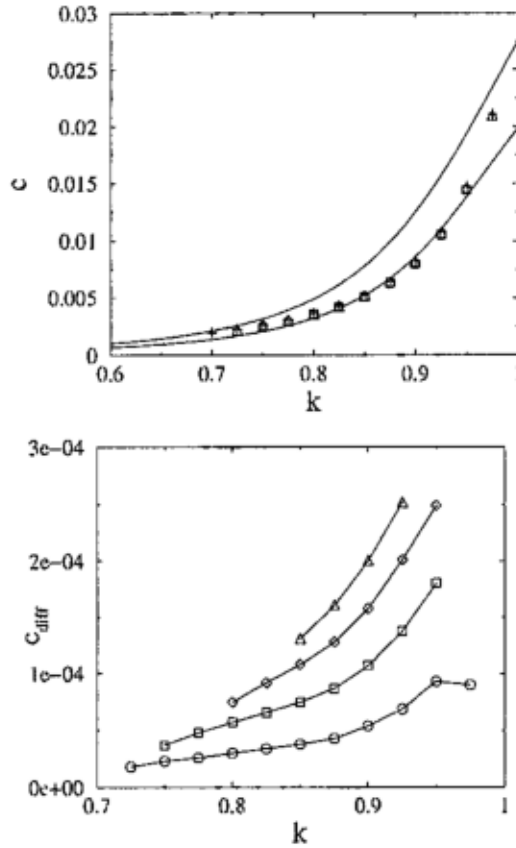


Figure 8: (a) Quasi-steady state phase velocity (symbols) c vs. k in the cases described in Fig. 7. The upper and lower solid curves mark where $c(k)$ would lie if the force were not removed when $1 + H - Z < 0$, for $\alpha = 0.0055$ and 0.0040 respectively. (b) The finer structure in the data presented in (a) is visible here. We plot $c_{diff} = c(\alpha = 0.0060) - c(\alpha)$ vs. k . Parameters are $\alpha = 0.0055$ circle, 0.0050 square, 0.0045 diamond, 0.0040 triangle.

Thermo

① Ref says:

$$Q_z = \text{heat flux} = \frac{dQ}{dA dt} = \frac{\text{heat}}{\text{area-sec}} = -k \frac{\partial T}{\partial z}$$

But he gives no data.

coefficient of thermal conductivity.

Implication is:

$$\frac{d^2 E}{dA dt} = -k \frac{\Delta T}{\Delta z}$$

$$\text{Power loss/area} = k (T_2 - T_1) / (\text{thickness})$$

② For radiation loss we have:

$$\text{Power loss/area} = \alpha \sigma (T_2^4 - T_1^4)$$

$$\text{where } \sigma = 5.7 \times 10^{-8} \frac{\text{watts}}{\text{m}^2 \cdot \text{deg}^4}$$

Kelvin!

 α = absorptivity

= .04 polished metal

= .95 rubber

③ What do we know about k ? Depends on T if you look 0-1000K, but rel. constant at "room temp"

$$k = 4 \frac{\text{watts}}{\text{cm-deg}} \quad \text{/copper}$$

$$= 10^{-2} \frac{\text{watts}}{\text{cm-deg}} \quad \text{/glass} \quad \times \left[\frac{100 \text{ cm}}{\text{m}} \right] \approx 1 \text{ W/m/deg}$$

$$= 2.6 \times 10^{-4} \frac{\text{watt}}{\text{cm} \cdot ^\circ \text{K}} \quad \text{/air at } 300^\circ \text{K.}$$

$$= 1.5 \times 10^{-3} \text{ w/cm} \cdot ^\circ \text{K} \quad \text{/oil}$$

Table of Room Temp k' - (all in $W/cm \cdot ^\circ K$):

brick (dry)	4×10^{-3}	@ $0^\circ C$	/source: AIP handbook
concrete	8×10^{-3}	@ $0^\circ C$	
granite	22×10^{-3}	@ $0^\circ C$	
ice	22×10^{-3}	@ $0^\circ C$	
✓ glass	10×10^{-3}	@ $0^\circ C$	
snow	1.6×10^{-3}	@ $0^\circ C$	
wood	2×10^{-3}	@ $0^\circ C$	
✓ air	0.0001 2.6×10^{-3}	@ $0^\circ C$	
oil	1.5×10^{-3}		
water	6×10^{-3}		
copper	4000×10^{-3}		

Conclusions: select in $mW/cm \cdot ^\circ K$:

copper	4000
granite, ice	22
concrete, glass	10
water	6
wood, oil, snow	2
brick	0.5
air	0.3

Calculation:

① A $1\text{ m} \times 1\text{ m}$ pane of glass, thickness 0.5 cm . Temp is ~~38~~ difference is 70°F inside, 32°F outside, so $\Delta^\circ\text{F} = 38^\circ\text{F} \times \left[\frac{5}{9}\right] = 21^\circ\text{C}$

a) what is total conduction loss?

b) what is total radiation loss?

$$(a) \quad \dot{Q} \text{ in watts} = K \frac{\Delta T}{\Delta z} = 10 \times 10^{-3} \left(\frac{\text{watt}}{\text{cm} \cdot ^\circ\text{K}} \right) \cdot \frac{21^\circ\text{K}}{0.5\text{cm}} \times 1\text{m}^2$$

$$= \frac{10^2 \times 21}{.5} \frac{\text{watt} \cdot ^\circ\text{K} \cdot \text{m}^2}{\text{cm} \cdot ^\circ\text{K} \cdot \text{cm}} \left[\frac{100\text{ cm}}{1\text{ m}} \right]^2 = \frac{10^2 \times 21}{.5}$$

$$= 42 \times 10^2 \text{ watts} = 4,200 \text{ watts!} \quad \checkmark$$

(b) assume obvious then:

$$\dot{Q} \text{ in watts} = 6 \times 10^{-8} \left(\frac{\text{watts}}{\text{m}^2 \cdot ^\circ\text{K}^4} \right) \cdot (1.0) \times \left(\frac{[299]^4 - [273]^4}{1.9 \times 10^9} \right) \times 1\text{m}^2$$

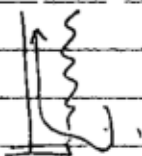
$$= 114 \text{ watts}$$

Assumes "outdoors" is 0°C , "indoor" is object.

② Loss thru brick? K is smaller by $\times 10$, wall thicker by $\times 40$, so loss down by $\times 400$

1m^2 of brick wall passes 11 watts

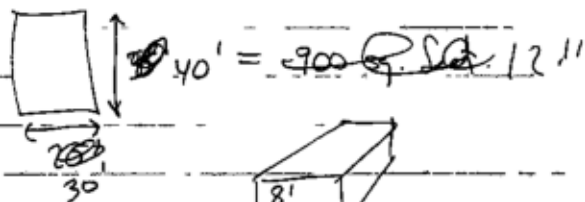
Remarks: air however is good insulator. If curtain can put a $2'' = 5\text{ cm}$ air shield, then window loss is much less. But hot air will tend to connect over window if curtain has floor gap?



A 5 cm air shield would ideally reduce window loss by $\frac{\times 10}{\times 40} \times \frac{\times 10}{\times 10} = \text{down to } \frac{42}{10} \text{ watts}$

③ So, estimate total house heat loss rate:

walls + ceiling + ~~floor~~:



$$\begin{aligned} \text{wall area} &= 2 \times 8 \times 30 + 2 \times 8 \times 40 + 30 \times 40 = \\ &= 480 + 640 + 1200 = 2320 \text{ sq feet.} \\ &= 219 \text{ m}^2 \end{aligned}$$

So loss is $219 \text{ m}^2 \times \frac{114 \text{ watt}}{\text{m}^2} = 25,000 \text{ watts.}$

Windows:

$2 \text{ m}^2 = 8,000 \text{ watts.}$

So:	walls + ceiling	25,000 watts
	windows	8,000 watts

40,000 watts when 70°F in 30°F out.

$S = 1.35 \text{ kW/m}^2$
above atmos.

1 BTU = 1055 ~~cal~~ joules

So, need about $38 \text{ BTU/sec} = 136,000 \text{ BTU/hr.}$

my estimate of needed furnace output to keep house at 70°F when it is 32°F outside.

Electric cost: 40,000 watts = 40 kW

This means about 80¢/hr based on 2¢/kWh. But I know my furnace is about 20¢/hr when running. Seems about right.

④ Solar insulation effect: at most, 1000 W/m^2 comes in, direct sunlight. But loss rate is 4200 W/m^2 by conduction. Thermopane of course is another story.

⑤ Door opening effect. Say each door opening - catastrophic - fills kitchen with cold air, by convection. Total loss is

$$\Delta T (^{\circ}\text{K}) \rho \left(\frac{\text{J}}{\text{mole} \cdot \text{K}} \right) \times V \times \rho_{\text{air}} \left(\frac{\text{g}}{\text{cm}^3} \right) \times \left[\frac{\text{Joule}}{\text{cal}} \right] \times \left[\frac{\text{mole}}{28 \text{ gm}} \right]$$

$$(20^{\circ}\text{K}) \left(\frac{1 \text{ cal}}{\text{mole} \cdot \text{K}} \right) (64 \times 10^3 \text{ cm}^3) \left(\frac{1.3 \times 10^3 \text{ gm}}{10^6 \text{ cm}^3} \right) \left(\frac{1000 \text{ J}}{252 \text{ cal}} \right) \left(\frac{1 \text{ mole}}{28 \text{ gm}} \right)$$

$$= 20 \times 7 \times 64 \times 1.3 \times 10^3 \times \frac{4}{28} = 1.9 \times 10^6 \text{ joules} = \frac{1}{2} \text{ kWh}$$

$$1 \text{ kWh} = 1000 \times 3600 = 3.6 \times 10^6 \text{ joules}$$

⑥ Conclusion: To lose all air in kitchen costs $\approx 1 \text{ kWh}$.

Going loss is 40 kWh/hr, so door opening not too significant.

⑦ Note: what costs \$ is not heating the air of your house. This cost is low + one time. What costs is the heat flow rate out!

● Review 11.26.83 in 457 LR.

Glass seems $\frac{1}{8}$ " thick = .32" call it $\frac{1}{3}$ cm.

Rule for glass $\frac{1}{3}$ cm thick is this:

$$\frac{dQ}{dA dt} = \frac{dP \text{ (watts)}}{dA \text{ (cm}^2\text{)}} = \left\{ \frac{10^{-7} \text{ watts}}{\text{cm-deg}} \right\} \frac{\Delta T \text{ (deg)}}{(\frac{1}{3} \text{ cm})} \times \left(\frac{100 \text{ cm}}{\text{m}} \right)^2$$

$$= 300 \cdot \Delta T \text{ watts/m}^2.$$

Let's take $\Delta T = 65^\circ\text{F} - 32^\circ\text{F} = 33^\circ\text{F} = 18^\circ\text{K}$.

$$\begin{array}{l} 68^\circ\text{F inside} \\ 32^\circ\text{F outside} \end{array} \quad ? \text{ Change } \Delta T_c = \frac{5}{9} \Delta T_F = 300 \frac{5}{9} \Delta T_F = 167 \Delta T_F$$

So far: $dP/dA = 167 \Delta T_F \text{ watts/m}^2$.

$$\frac{1000 \text{ watts}}{1000 \text{ joules/sec.}} = \frac{3600 \text{ sec.}}{\text{hr}} \cdot \frac{\text{BTU}}{1055 \text{ joules}} = 3412 \text{ BTU/hr.}$$

$$\text{Front top windows} = 1 + 1\frac{1}{2} + 1\frac{1}{2} \approx 3.5 \text{ m}^2. \quad \text{so } 3.4 \text{ wt}$$

Use $65^\circ - 35^\circ = 30^\circ\text{F} = \Delta T_F$, so

$$\Delta \dot{Q} = 167 \times 30 \times 3.4 \text{ BTU/hr} = 17,000 \text{ BTU/hr. leakage!}$$

$$15 \text{ min/hr} \approx 25,000 \text{ BTU/hr}$$

697 015 bu

TJ 163.5 .B84 O23 1978

Pergamon Press, 1978. 8 x 200

An excellent book, lots of math + detail, written by a prof.
Lots of references, etc.

Conductivity p37

$$Q = \frac{kA \Delta T}{L} = \frac{\Delta T}{R} \quad R = \frac{L}{kA} = \left(\frac{K^\circ}{W} \right)$$

$k = \text{Thermal cond. (watts/meter/}^{\circ}\text{K)}$

$$R' = \frac{L}{K}$$

* "Glazing" 6mm thick

$$k = 0.76 \text{ W/m/K}^\circ \checkmark$$

120 m² in typ ha.

loss? * $R' = 0.49 \text{ m}^2 \text{ K/W}$. 50% of house loss (single pane)

$$U' = 1/R' \quad U = 2.00 \text{ W/m}^2/\text{K} \quad U = \left(\frac{kA}{L} \right)$$

$$Q = \frac{kA\Delta T}{L} = \frac{A \cdot \Delta T}{R'} \leftarrow m^2-K/W.$$

$$R' = \frac{L}{k} = \frac{6 \text{ mm} \times \text{m} \times \text{°K}}{.76 \text{ W}} \left[\frac{1 \text{ m}}{1000 \text{ mm}} \right] = \frac{6}{760} \text{ m}^2 \text{K/W} \approx .01 \text{ m}^2 \text{K/W}.$$

$$\dot{Q} = A \cdot \Delta T \cdot u' = 120 \text{ m}^2 \times 14^\circ\text{K} \times 2.00 \frac{\text{W}}{\text{m}^2 \cdot ^\circ\text{K}} = 3360 \text{ watts} \checkmark$$

Note: K is material only, does not acnt in air layers!

double-glazing: 19mm is optimal air gap. You add $\frac{1}{h} = 0.8 \text{ m}^2 \text{ K/W}$ to get $R_{\text{TOT}} = 0.8 + 0.49 = 1.3$. It, you roughly, triple the resistance!!!

curtains: 10 mm thick pleated curtain (1 cm) has $R = .23 \text{ m}^2\text{K/W}$.
a 25 cm = 10" air gap is $0.5 \text{ m}^2 \text{K/W} \Rightarrow 1.22$ total, about
same as double glaze!

$$\times R' = \frac{L}{\kappa} = \frac{6 \times 10^{-3} \text{ m}}{.76 \text{ W m}^{-1} \text{ K}^{-1}} = 7.8 \times 10^{-3} \text{ m}^2 \text{ K/W, not .49! (But see } \rightarrow)$$



Model for a "wall"

$$\dot{Q} = U A \Delta T \quad \text{where} \quad U = \frac{q}{A \Delta T}$$

$$U = \frac{1}{h_1} + \frac{\delta}{K_s} + \frac{1}{h_2}$$

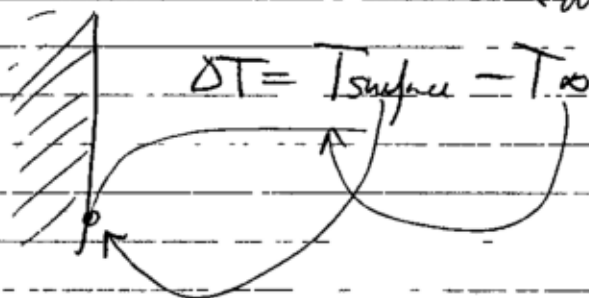
K_s = thermal conduct of material (W/m/K)

δ = thickness (m)

$$R_{tot} = R_1 + R_2 + R_3$$

h_1, h_2 = partial convective heat xfer coeffs.

Convection aside: $\dot{Q} = \bar{h}_c A \Delta T$
 $\bar{h}_c$ = average coeff.



$$+3) \begin{array}{r} 1.00 \\ 31 \\ \hline 96 \end{array}$$

Then roughly $\bar{h}_c \approx 5-25 \text{ W/m}^2/\text{K}$ for free air convection

$$C = \bar{h}_c A$$

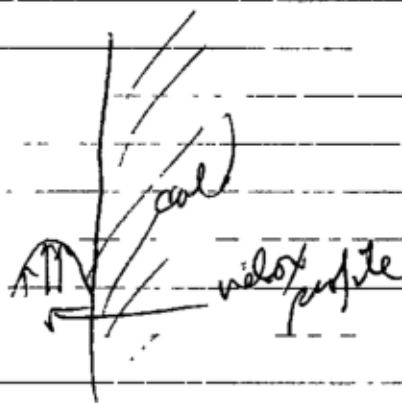
$$R = 1/C$$

laminar flow minimizes heat loss, no eddies.

$$\frac{K_s}{\delta} = \frac{0.76 \text{ W/m/K}}{6 \times 10^{-3} \text{ m}} \approx 130 \text{ W/m}^2/\text{K}$$

$$\frac{\delta}{K_s} \approx .007 \approx .01 \text{ m}^2 \text{ K/W} \quad (\text{resistance of glass})$$

$$\frac{1}{h_1} \approx \frac{1}{5} \approx .20 \text{ m}^2 \text{ K/W}, \quad 20\times \text{ more than the glass!}$$



Some summary equations:

$$\bar{h}_L = 1.37 \left(\frac{\Delta T}{L} \right)^{0.25}$$

↑ laminar flow.

L = length of plate in meters.
(height of wall)

example: $\Delta T = 10^\circ\text{C}$, $L = 1\text{m}$ so $\bar{h}_L \approx 3 \Rightarrow \bar{h}_L^{-1} = .33$.

One layer ΔT , the smaller is $\bar{h}_L^{-1}$ (res).

turbulence sets in at $(\Delta T)L^3 \approx 6$, however.

$$\bar{h}_L = 1.75 (\Delta T)^{0.33}$$

↑ turbulent flow

example: $\Delta T = 10^\circ\text{C}$, $\bar{h}_L \approx 3.5$ so $\bar{h}_L^{-1} \approx .29$ (less res!).

airspace

(no convection if $(\Delta T)\delta^3 < 10^{-5}$)

ex: $\delta^3 < 10^{-6}$ $\delta < 10^{-2} = 1\text{cm}$. (then use regular air conduction)

if $\Delta T\delta^3 > 10^{-5}$, then

laminar: $\bar{h}_r = .51 \left(\frac{\Delta T}{\delta} \right)^{.25} \left(\frac{L}{\delta} \right)^{-.11}$

turbulence: $\bar{h}_r = .96 (\Delta T)^{.33} \left(\frac{L}{\delta} \right)^{-.11}$

transitions at

$$\Delta T\delta^3 \approx 2 \times 10^{-3}$$

example: $\Delta T = 10^\circ\text{C}$ $\delta = 25\text{cm}$ $L \approx 1\text{m}$

$$\bar{h}_r (\text{lam}) = .51 (40)^{.25} (4)^{-.11} \approx .51 (3) (4)^{.11}$$

$$\bar{h}_L^{-1} \approx .60$$

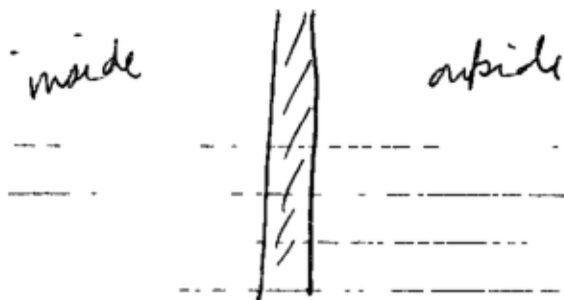
$$\frac{10}{1.1} = 9$$

example: turb $\bar{h}_L \approx 1 \times (10)^{.33} \approx 2.5$ $\bar{h}_L^{-1} \approx .40$

So, generally speaking you get $\bar{h}^{-1} = .3$ to $.6$ range.

[So most of window insulation is in that air layer!] ←

Summary:



~~7660~~
76

$$R = .3 \quad R = .007 \quad R = 0 \text{ (windy, cold)}$$

↑ natural convection air layer, one side only.

so $R \approx .3 \text{ m}^2 \text{ K/W}$ ✓

(earlier quote was $R = .49 \text{ m}^2 \text{ K/W}$) ✓

Layer width: $\Delta T \delta^3 \approx 10^{-5}$ so $\Delta T = 10^\circ \text{C} \Rightarrow \delta^3 = 10^{-6}$
 $\delta = 10^{-2} = 1 \text{ cm}$

So the convection layer is about 1 cm.

Air barrier 25 cm? $R = \frac{L}{k} \approx \frac{\frac{1}{4} \text{ m}}{0.025 \text{ W/m/K}}$
 $\approx \frac{40}{4} \approx 10 \text{ m}^2 \text{ K/W}$ $\uparrow$ k_{air}

This calc.

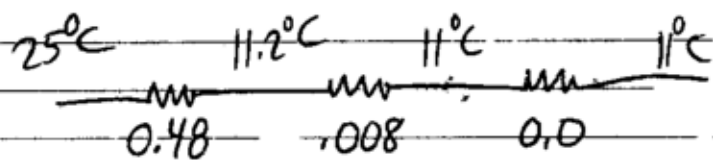
but .5 claimed in curtains ...

fails! You cannot stop convection.



body layer

His window analysis



↑
glass

$$R_{\text{gas}} = .49 *$$

I started into this subject by looking at the Utah handbook and also at Dave Andrews computer program. The first thing you learn is the definition of "R" :

$$\frac{\text{BTU}}{\text{HR}} \rightarrow \dot{Q} = \frac{A \times \Delta T}{R} \quad R \left(\frac{\text{ft}^2 \cdot ^\circ\text{F}}{\text{BTU/HR}} \right)$$

Insulating materials are rated by their R value. For example, a window is "approximately R=1" while 3.5 inches of batting is R = 11. You quickly learn that books and technical materials are in SI units, so you have another similar equation,

$$\text{watts} \rightarrow \dot{Q} = \frac{A \times \Delta T}{R} \quad R \left(\frac{\text{m}^2 \cdot ^\circ\text{K}}{\text{Watt}} \right)$$

The basic conversion factors are as follows:

$$\begin{aligned} 1 \text{ watt} &= 3.4 \text{ BTU/hr} \\ 1 \text{ m}^2 &= 10.76 \text{ ft}^2 \\ 1^\circ\text{K} &= \frac{5}{9} ^\circ\text{F} \end{aligned} \quad R \left(\frac{\text{ft}^2 \cdot ^\circ\text{F}}{\text{BTU/hr}} \right) = 5.27 R \left(\frac{\text{m}^2 \cdot \text{K}}{\text{W}} \right)$$

$$R \left(\frac{\text{ft}^2 \cdot ^\circ\text{F}}{\text{BTU/hr}} \right) = \frac{A(\text{ft}^2) \times \Delta T(^{\circ}\text{F})}{\dot{Q}(\text{BTU/hr})} \times \left[\frac{A(\text{m}^2)}{A(\text{ft}^2)} \times 10.76 \right] \times \left[\frac{\Delta T(^{\circ}\text{K})}{\Delta T(^{\circ}\text{F})} \times \frac{5}{9} \right] \times \left[\frac{\dot{Q}(\text{Watt})}{\dot{Q}(\text{BTU/hr})} \times 3.4 \right]$$

In the case that the thermal resistance is caused simply by the conduction through a non-convecting substance, you can relate R to the thermal conductivity k as follows:

$$\dot{Q} = k \left(\frac{\text{W}}{\text{m} \cdot \text{K}} \right) \times A \times \frac{\Delta T}{\Delta z}$$

$$k = \frac{\Delta z}{R} \quad R = \frac{\Delta z}{k}$$

For example, consider the 3.5" batting as if it were just air. Then you can compute as follows:

$$R = \frac{3.5''}{.26 \times 10^{-1} \frac{\text{Watt}}{\text{m} \cdot \text{K}}} \times \left[\frac{2.54 \times 10^{-2} \text{ m}}{''} \right] = 3.4 \frac{\text{m}^2 \cdot \text{K}}{\text{W}} = 18.0 \frac{\text{ft}^2 \cdot ^\circ\text{F}}{\text{BTU/hr}}$$

We compute 18, but the actual value is 11. Presumably because the batting contains solid material which conducts much better than air, air is humid, etc etc. Ball park OK.

Now, if you compute the conduction resistance through a 0.25" thick window, you get a very low R value:

$$R = \frac{0.25 \text{ in}}{1 \frac{\text{watt}}{\text{m-K}}} \times \left[\frac{25 \text{ W/m}^2 \cdot \text{m}}{\text{in}} \right] = .0064 \frac{\text{m}^2 \text{K}}{\text{W}} = .033 \frac{\text{ft}^2 \cdot ^\circ \text{F}}{\text{Btu/h}} \approx \frac{1}{30} \text{ unit}$$

If this were the actual resistance of a window, you would be in real trouble. Luckily the main resistance comes from the convection layers on each side of the glass, in particular on the more controlled inside layer. (Handbook says R about 1 for window...)

Convection Layers .

According to the quoted equations of O'Callahans book, here are the equations for the effective R value (in meters, K,watt units) of a vertical convection layer:

$$R = .7 \left(\frac{L}{\Delta T} \right)^{\frac{1}{4}} \text{ laminae}$$

$$\text{transition at } (\Delta T) L^3 = 6$$

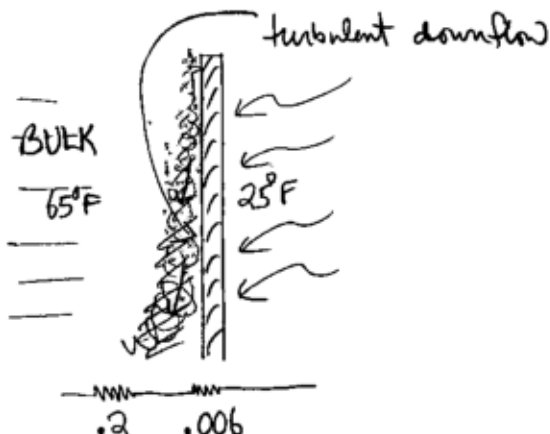
$$R = .6 \left(\frac{L}{\Delta T} \right)^{\frac{1}{3}} \text{ turbulent}$$

The first thing you notice is that the resistance decreases as the temperature difference increases! (Totally unlike conduction where R = indep of temperature roughly). Luckily the power is a weak one. For 65F inside and 25F outside, you have $\Delta T = 40^\circ \text{F} = 25^\circ \text{C}$. Also windows are about 1m high (this is L). Thus, folow is turbulent which means slightly lower R. For 25C get $\Delta T^{1/3} = 2.9$, for 10C get 2.0 so call it 2.5, no 3.0. Thus, you get $R = .6/2.5 = .2$ SI units for a single layer. In British units multiply by 5.27 to get $R = 1$, as claimed! So, with a 40F degree difference, you get $R=1$ for a single window. You should regard this as resistance between isothermal inside window surface and the bulk air temp of the room.

single window $\Delta T = 40^\circ \text{F}$ (25°F outside)

$$R \approx .2 \text{ SI}$$

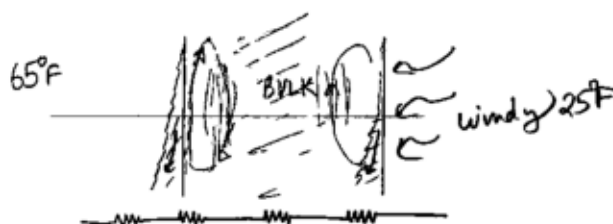
$$\approx 1.0 \text{ American Units.}$$



The convection layer is perhaps only 1 cm thick. We have assumed natural convection. If something acts to make this not so, eg, a heat vent blows directly on a glass window, you have forced convection and resistance decreases, perhaps a lot! One assumes that there is a windy exterior so you cannot assume a layer on the outside of the window. Nevertheless lets compute it:

Convection layer on each side of window. If total temp diff is 40F, then only 20F drop in each convection layer, so $dT = 12C$ for each layer. This gives $R = .26$ rather than .20 for each layer, so total is $R = .52$ and $R = 2.5$ in British units. You have more than doubled R by having a layer on each side. Unfortunately you cannot really control the external layer due to wind.

Storm window. (wide) Lets assume that storm window is thick enough that you have "bulk" in between so you can treat as series stuff:



Now you have not one but three surface convection layers and a bulk still air layer as well. If the central area is only a few inches, you may not have any "still air" in there but rather a single cell as shown in red for convection. In this case you cannot really include a central bulk air insulator. So lets assume that is the case. Then roughly you get 8C across each layer which means $R = .3$ for each layer so $R=0.9$ for total system which means $R = 5$ roughly for the combined storm window. If there are large leaks where air can flow from the room into the air space, then you are really back to the single window again, and the second layer has no effect, ie, both sides are at the same temperature. Thus I think it is important to seal storm layers.

Curtains: really the same as the storm window analysis except you have to make sure no gaps at top and bottom and air cannot flow through them! Otherwise much less effect. (We have not yet done radiative effects).

Air Gap windows: lets assume gap is so small that convection cannot occur inside the window. Assume a 0.25" gap of air. Air insulates about 40 times better than glass (see K values), so we would get $R = 1.3$ from the gap. (British). Add to this $R = 1.3$ for roomside convection layer so that $R = 2.6$ British for such a window, as compared to 1.0 for single window. I am not sure what commercial double glass air gaps are. If it is only 1/8th inch gap, then you get $R = 2.0$ for double glass. This agrees with Utah book and Andrews program, more or less.

Heat Loss thru front wall glass at 457 Club. Assume 9 panes at 1 meter each so area = 9 m^2 . Assume 40F difference so $R=1$ British. This means $R = .2$ SI. Temp diff is 25C. Then $Q \text{ dot} = 9 \times 25 / .2 = 1125 \text{ watts} = 3800 \text{ BTU/hour}$. This is 92,000 BTU/day or about .9 therm per day which means 27 therms/month = \$13.50/month.

Heat Loss through ceiling. Assume 1000 square feet of ceiling for upstairs. Assume $R=11$ mainly from batting. Then $Q \text{ dot} = 1000 \times 40 / 11 = 3600 \text{ BTU/hour}$. This is another .86 therms/day. *Surface units etc, add 1*
1.80

Heat Loss thru walls. I looked through back room speaker. Walls are brick + 1.5" batting crushed in there + sheetrock so $R = .4 + 4.0 + .3 = 4.7$ say. Assume 1500 ft of wall, so $Q \text{ dot} = 1500 \times 40 / 4.7 = 12,700 \text{ BTU/hour}$ out the walls. (I assumed windows = wall). This is 3.0 therms/day. *Inside convection layer add another 1.3 say?*
2.4

Heat Loss thru floor: call it same as ceiling because batting is the same. So about 1.0 therm/day through upstairs floor.

Summary: counting only conductive heat loss, we have this summary: (upstairs only)
25F outside
65F inside

ceiling loss	1.0 therm/day
floor loss	1.0 therm/day
wall loss	3.0 therm/day (windows all curtained)
front glass	1.0 therm/day
kitchen glass	0.6 therm/day

Kitchen windows: big ones are 50" x 60" each = 21 square feet = 2 meters each. Other is 60' x 34' = 14 feet = 1.3 m^2 so total is about 5.3 meters, so about 0.6 therms leaks out there if not covered. How good is the covering???

Ball Park? Gas use is about 10 therms/day very worst case, both furnaces, so maybe 7 therms/day upstairs, and this agrees nicely with above sum.

Conclusions: a square meter of window assuming $R=1$ lets 3 therms/month leak out which is \$1.50 month, assuming 25°F outside, ie, a cold month. If you cover that window perfectly, you might reduce the loss by factor of 5. A crude cover like loose curtains probably buys you a factor of 2.

What improvements can be made?

1. By shutting off that back room, probably saving 10-15% of that \$100.00 month upstairs bill because that much less ceiling walls etc to heat. Save \$15.00/month.

2. By keeping all curtains closed over all ²²~~35~~ meters of glass in house, save perhaps effective ~~$R=1$ or 15000 BTU/hour or 9 therms/day or 27 therms/month~~
Check this: backbed has $1.3 \times 2 = 2.6$ meters. Bed has 1.7 meters. Kitchen has 5.3 meters. Front is 9 meters. Slide door is 3.3 meters. Total upstairs is 22 meters. With $R=1$ loss through total upstairs glass would be $2\frac{1}{4}$ therms/day. With curtains raising R to 2, say, we save 1.1 therms/day or 33 therms/month or \$16.00/month.

Radiative Loss through Window: I think ^{seems wrong!! Jim thinks northsky is 37°F , southsky is 60°F .} I read somewhere that the night sky has effective radiation temperature of about -45°C , even though this is not the external air temp. Since 0°C is 273 K , we can compute what 1 meter of glass does in this regard:

$$\dot{Q} \approx 5.7 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \cdot \text{deg}} \times \left[\frac{(273+18)^4}{7.2 \times 10^9} - \frac{(273-45)^4}{2.7 \times 10^9} \right] \approx 250 \text{ watts.}$$

This is about double the conduction loss, by the way, all the more reason to keep curtains closed at night. During the day things are different, especially for southern windows.

Infiltration Losses. Assume a leak that is very small, perhaps only $1\text{ ft}^3/\text{minute}$ or $60\text{ ft}^3/\text{hour}$. Assume 40°F temp difference. Each cubic foot of air requires .02 BTU to raise 1°F (Utah book page 25). Thus $60\text{ ft}^3/\text{hour}$ 40 degrees means about 50 BTU/hr or 1200 BTU/day or .01 therm/day or $1/2^{\circ}\text{C/day}$ not very significant. (15°C/month). Of course 100 such small leaks could add up to \$15.00/month. Based on this, I see no reason to go crazy plugging little leaks everywhere.

Fireplace as energy leak:

- ① damper Assume leakage area when closed is edge of page $\rightarrow$
 $12" \times \frac{1}{4}" \approx 3 \text{ in}^2$. Typical num-windy airflow?

$$\frac{P_i}{P_e} \rightarrow \begin{matrix} \uparrow 25^\circ\text{F} \\ \downarrow 65^\circ\text{F} \end{matrix}$$

$$p = nkT$$

$\uparrow$
 $1.38 \times 10^{-16} \text{ erg/deg.}$

$$\Delta p = nk \cdot \Delta T$$

Who knows! Assume flow rate about $v = 10 \text{ cm/sec.} \approx 4 \text{ in/sec.}$

Then infiltration volume $\approx 12 \text{ in}^3/\text{sec} \approx 720 \text{ in}^3/\text{min} \approx \frac{1}{2} \text{ ft}^3/\text{min}$.
 This converts to $\frac{1}{100}$ therm/day, can neglect.

- ② open damper: Now $a \approx$ this page $\approx 100 \text{ in}^2$, so maybe 30X larger loss. Probably worse since big hole has low R. Even at 30X, we raise to $\frac{1}{3}$ therm/day. This is a lot compared to say ceiling loss.

I have been thinking about glass fireplace doors as a "fix" for damper leakage. Lets try to make quantitative cost analysis of doing this. We know that our cold months therms use is about 7 therms/day. Roughly a therm costs \$20/month. In summer, you use about 1 therm/month for hot water and gas dryer use, that is why summer bill is about \$20/month. In winter this rises to \$140/month in the worst few months. Lets think only about these months.

The entire house leaks 7 therms/day on one of these months. If I were to leave the damper open all the time, and if downdraft is about 10 cm/sec (move your finger to see that this is reasonable) or 4 inch/sec, and if damper is 100 square inches area, then you would lose about 1/3 therm/day in this way. That is fully one third the complete loss through the ceiling, or floor. That represents an added \$7/month on winter bill. Even in this case, you would have to go a lot of years to pay back a \$1000 glass screen.

With the damper closed, yes there is leakage. I have eyeballed it now. It is roughly the area shown on right edge of this page, its in two pieces along one edge of the damper. I call it $1/4"$ by $12"$ = 3 square inches. This is 100 times smaller than when damper is open! Thus, loss is maybe 7¢ / month. Truly this is something we need not worry about. Yes, there will be a slight draft. Candle shows how draft is much more when damper is open.

First, the big picture. Background gas use for hot water and dryer seems to be about 1 therm/day. This is 100,000 BTU/day. This gas use costs about \$20 per month. Nearly all of this is hot water use, mainly for showers. Dryer is 25000 BTU gas on, so one hour/week at 15000 BTU means 15,000 BTU/week or 1/50 therm day, completely negligible. Hot water heater holds 40 gallons and recovers at 33.6 gallons/hour for 100F and fires at 40000 BTU, so you might say it takes 50000 BTU to heat up 40 gallons. I would guess than my showers almost use the whole tank, so maybe 1 shower = 0.4 therm. The rest is dishes and cleaning and other hot water use, laundry, dishwasher, etc. All in all, I probably use about 1 therm/month hot water + gas drier and this gives a bill of \$20/month or \$240/year.

Now, total gas bill is about \$700/year. You may conclude that the heating bill is therefor about \$460/year. Roughly the heating season is Nov thru April or 6 months. Average monthly heating bill is therefore \$77.

Consider that \$460/year heating bill. We think we know how that heat "leaks" out of the house. House is a box. Top = 15%, floor = 15%, walls = 35% including draped windows except LR and K, 15% thru front LR glass, and 10% thru kitchen glass.

Crude estimate: if you were to replace entire LR glass, you would raise R from 1 to 2, and heat loss would be cut from 15% to 7%, saving about \$35/year. However, cut this in half because I have plastic on 2/3 of this glass and curtain use, so maybe save about \$15/year. Probably each window is \$200, or total wall would be \$2000 including sliding glass door. Pay back time is then about 133 years! Room would be a little warmer, but I just can't see the economy of doing this! Much of the cost is of course labor to install those new windows. Its like factory air versus installed air in a car. It might be worth doing in a new house, but not as a retrofit. If you want it warmer in there, just up the heat.

My top three plastic sheets probably save 5% of total heating bill or \$23/year, so it does pay for itself.

I think this 100 year payback time is typical for a window analysis. Goodman said \$200 for a big window installed, and I think you only save \$5/year at most, so call it 40 year payback, not worth it.

The fireplace doesnt even register! With damper closed, I figured 7% loss in worst winter month. Definitely glass \$1000 deal there not worth it from that point of view. While fire is actually burning, there can be a difference, but not really much.

So OK here is the conclusion. You have inherited an energy non-efficient house. Walls are not very good, nor is ceiling, nor are windows. The cost of really upgrading the whole thing is probably on the order of \$5000-\$10000. You might save \$230 per year, for pay back of 20-40 years. I would rather some other owner worried about this and not me. I will just pass it on. Probably cheapest thing you could do is double attic insulation. This would cut maybe 7% or \$35 orr annual bill and would cost maybe \$300, so 7 year payback. Even that is marginal. Moreover, heating costs are now dropping.

plastic installed < '87

HOME ENERGY CALCULATOR

David Swaim, Atlanta

You (and your computer) could become pretty popular when word gets out that you can analyze the benefits of home improvements on fuel bills. This program is in Microsoft (Apple, PET, OSI, etc.) and Atari BASIC.

Lately there has been a great deal of interest in saving energy in the home. Nobody needs to be reminded that fuel costs are rising. We all want to reduce our energy bills. The way to do this is simple: reduce household energy consumption. There are a number of ways this can be done.

The cheapest way is to change habits. An example would be setting the thermostat back to a lower temperature and wearing heavier clothes. If you're not too keen on that, the next alternative is to improve the ability of the house to protect you from the elements. Insulation could be added to the walls, floors, attic, and heat ducts. Weatherstripping could be applied to windows and doors. Storm windows and doors could be added.

Improvements such as these reduce the amount of heat that the house will lose to the outside. But which of the above items would save us the most money? Which one would cost the least to implement? Or, better yet, which will give the greatest savings for the least amount of cost? It's this last question we really want to answer.

The best measure of the cost effectiveness of an energy saving improvement is the payback period. That is simply the amount of time (in years) it takes for the savings in energy costs to add up to the total cost of installing the improvement. Obviously, the item with the shortest payback period is the best candidate for implementation. To determine the payback period, we must know two things: how much it will cost to make the improvement, and how much it will save us on utility bills for a year (a heating season). Obtaining the improvement cost requires consulting a contractor or, if we plan to do it ourselves, a building supply store.

Predicting Effectiveness

Finding out how much the improvement will

save us in heating costs over a season is not quite as easy to determine. One way would be to keep records of our heating bills for one season, make the improvement, and then keep records of our heating bills for the next heating season. There are two drawbacks to this method.

First, the severity of the weather will vary from one year to the next. If the first year is severe and the second is mild, our heating bills would be less even if we made no improvements. This problem can be corrected by adjusting the heating costs using weather data for the two years.

The second and biggest drawback to this method is that you can't find out if an improvement is cost effective until after you have installed it. If it turns out not to be cost effective, it is too late to decide not to implement it!

What we need is a way of predicting savings. If we know the weather and the heat loss characteristics of the house, we can estimate the heating cost. By calculating the heating costs based on heat loss characteristics of the house both before and after the improvements, we can obtain the estimated savings due to the improvements. This is what the program here does.

To gather the data needed by the program, you will need to make some measurements and observe insulation levels in your house. The first thing the program calculates is the heat loss of the house. Heat loss of a house depends on three things: the thermal resistance, known as the R-value, of the structure; the total area of the structure exposed to the elements; and the temperature difference between the inside and outside of the house. So we simply need the area, R-value, and the difference in temperature.

The only problem is that different parts of the house have different R-values. Windows will have a lower R-value than walls, for example. In general, you can divide the external area of the house into five categories: windows, doors, walls, ceiling, and floor. The program requests information on each of these five categories in turn.

For windows it requests height, width, number of windows (it calculates total window

area from these items), and type of frame and number of layers of glass. The number of types and/or sizes of windows is requested first. Most houses will have several sizes of windows, and there may be storm windows on some and not on others. The program allows for up to ten different types and/or sizes of windows. If you need more, change the dimension of S in statement 180.

Only one size and type of door is allowed. If you have sliding glass doors, you should consider them another type of window. You need to get the height, width, and number of doors. Remember: these are exterior doors only.

Information needed for the walls consists of type of construction and R-value of the insulation in the wall. If you enter a negative number for the R-value of the wall insulation, the program will give you a list of typical R-values for wall insulation. To get the area of the wall, the program asks for the ceiling height, total perimeter of the house, and the number of stories in the house. The program will calculate the gross wall area from this data and subtract the total window and door area to obtain the proper wall area.

One Hand Calculation

The only time you have to calculate area yourself is for ceiling and floor. For the ceiling, you will be asked for the number of inches of insulation in the attic and the type of insulating material. For the floor, the type of foundation is requested.

In addition to the heat losses mentioned so far, there are two others. The first of these is infiltration of outside air through cracks in windows and doors. The program asks if the windows and doors are weather-stripped. It uses this information and the total length of the cracks around windows and doors to calculate infiltration. The other heat loss is in the heat ducts from the furnace to the heat registers. The program asks if your heat ducts are insulated and where they are located. This concludes the input needed for calculating the total heat loss of the house. At this point the heat losses are displayed, and you are asked if you wish to make improvements to the house.

If the answer is "Y", you will be asked if you wish to improve each item. You can make improvements to one item or to any number of items. As you probably noticed, the first question you are asked is what the outside design temperature is. The outside design temperature for my area (Atlanta, Georgia) is 23 degrees. The outside design temperatures for other areas are tabulated in Table 1. For a more complete list, consult one of the references listed at the end of this article.

Actually, you do not need to put any specific temperature in here as long as it is less than 75 degrees, the inside design temperature used by

the program. The program will still give you valid results for savings and payback. However, using the correct outside design temperature gives you the advantage of seeing what the furnace size would be for your house with and without the improvements. In fact, heating engineers use the same basic method as this program does to size furnaces for houses.

When the program finishes calculating the heat loss of the house after improvements, it is ready to do the cost analysis. First you are asked for the type of heating fuel you use: electricity,

Table 1: Winter Design Temperatures

CITY	TEMPERATURE
MONTGOMERY AL	26
JUNEAU AK	-4
PHOENIX AZ	34
LITTLE ROCK AR	23
SACRAMENTO CA	32
DENVER CO	3
HARTFORD CONN	5
DOVER DEL	15
TALLAHASSEE FL	-29
ATLANTA GA	23
HONOLULU HI	62
BOISE ID	10
SPRINGFIELD IL	4
INDIANAPOLIS IN	4
DES MOINES IA	-3
TOPEKA KS	6
LEXINGTON KY	10
BATON ROUGE LA	30
AUGUSTA ME	-3
BALTIMORE MD	20
BOSTON MA	10
LANSING MI	-6
ST. PAUL MN	-10
JACKSON MS	24
JEFFERSON CITY MO	6
HELENA MT	-13
LINCOLN NE	0
CARSON CITY NV	7
CONCORD NH	-7
TRENTON NJ	16
SANTA FE NM	11
ALBANY NY	5
RALEIGH NC	20
BISMARCK ND	-19
COLUMBUS OH	7
OKLAHOMA CITY OK	15
SALEM OR	25
HARRISBURG PA	13
PROVIDENCE RI	10
COLUMBIA SC	23
PIERRE SD	-9
NASHVILLE TN	16
AUSTIN TX	29
SALT LAKE CITY UT	9
BURLINGTON VT	-7
RICHMOND VA	18
OLYMPIA WA	25
CHARLESTON WV	14
MADISON WS	-5
CHEYENNE WY	-2

fuel oil, or natural gas. Next you must input the cost per fuel unit of the heating fuel.

Note that this unit cost is in dollars, so if natural gas in your area is 35 cents per therm, you should input .35 dollars per therm.

Using this data and the heating degree days, the program calculates the total energy needed to heat the house for the entire heating season. The degree days and name of the city are on line 7010. You should change this line to reflect your own location. Some sample degree days for different cities are listed in Table 2, and a more complete

list can be found in any of the references. The last thing you must input is the total cost of the improvements you made. From this data the program calculates the payback period in years.

I got pretty popular in my neighborhood when word got out that my home computer could calculate how cost effective it would be to add insulation. I have also learned a great deal about my own home from running this program. Much of what I concluded was what I expected, but some conclusions surprised me. The program can definitely help home owners in assessing home energy improvements; it can also enable a home owner to spot dishonest "energy-saving" schemes pretty quickly.

Table 2: Yearly Heating Degree Days

CITY	DEGREE DAYS
MONTGOMERY AL	2291
JUNEAU AK	9075
PHOENIX AZ	1765
LITTLE ROCK AR	3219
SACRAMENTO CA	2419
DENVER CO	5524
HARTFORD CONN	6235
WILMINGTON DEL	4930
TALLAHASSEE FL	1485
ATLANTA GA	2961
HONOLULU HI	0
BOISE ID	5809
SPRINGFIELD IL	5429
INDIANAPOLIS IN	5699
DES MOINES IA	6588
TOPEKA KS	5182
LEXINGTON KY	4683
BATON ROUGE LA	1560
PORTLAND ME	7511
BALTIMORE MD	4111
BOSTON MA	5634
LANSING MI	6909
MINNEAPOLIS MN	8382
JACKSON MS	2239
ST. LOUIS MO	4484
HELENA MT	8129
LINCOLN NE	5864
RENO NV	6332
CONCORD NH	7383
TRENTON NJ	4980
ALBUQUERQUE NM	4348
ALBANY NY	6201
RALEIGH NC	3393
BISMARCK ND	8851
COLUMBUS OH	5211
OKLAHOMA CITY OK	3725
SALEM OR	4754
HARRISBURG PA	5251
PROVIDENCE RI	5954
COLUMBIA SC	2484
RAPID CITY SD	7345
NASHVILLE TN	3578
AUSTIN TX	1711
SALT LAKE CITY UT	6052
BURLINGTON VT	8269
RICHMOND VA	3865
OLYMPIA WA	5236
CHARLESTON WV	4476
MADISON WI	7863
CHEYENNE WY	7381

References

1. *ASHRAE Handbook 1981 Fundamentals*. Atlanta, Georgia: American Society of Heating, Refrigerating and Air-conditioning Engineers, Incorporated, 1981.
2. *Other Homes and Garbage*, Jim Leckie, Gil Masters, Harry Whitehouse, and Lilly Young. San Francisco, California: Sierra Club Books, 1975.
3. *Refrigeration and Air-Conditioning*, Air-Conditioning and Refrigeration Institute. Englewood Cliffs, New Jersey: Prentice-Hall, 1979.

Program 1: Microsoft BASIC

```

100 PRINT {CLEAR}{02 DOWN} HOME EN
    ERGY PROGRAM
110 PRINT:PRINT
120 PRINT BY DAVID SWAIM
130 PRINT P. O. BOX 720126
140 PRINT ATLANTA, GEORGIA 303
    58
150 GOSUB 8000
160 REM COPYRIGHT 1981 DAVID C. SWA
    IM II
170 REM
180 DIM A(6),Q(6),R(6),RW(4,3),D(4)
    ,IW(2,3),S(10)
190 DIM RF(3),TC(3),N$(5),IC(5),DM(
    2,3,3),IN(2)
200 REM WINDOW R VALUES
210 DATA 1.01,2.22,1.815,3.155
220 DATA .909,1.667,1.437,2.137
230 DATA .909,2,1.724,2.564
240 REM DOOR R VALUES
250 DATA .41,.75,.95,1.1
260 REM FLOOR R VALUES AND TEMP COR
    R
270 DATA 3.2,0,3.2,30,1.23,0
280 REM CEILING INSULATION R PER IN
    CH
290 DATA 3.5,3,2.5,4.5,5.5
300 N$(1)="WINDOWS":N$(2)="DOORS":N
    $(3)="WALLS"
310 N$(4)="CEILING":N$(5)="FLOOR"
320 REM DUCT MULTIPLIERS
330 DATA .2,.15,.1,.15,.1,.05,.1,.0
    5,.05

```



```

340 DATA .2,.15,.1,.1,.1,.05,.05,.0
5,.05
350 REM AIR CHANGES PER FOOT OF CRA
CK
360 DATA 39,74,52,24,32,33
370 REM READ WINDOW R VALUES
380 FOR F=1 TO 3
390 FOR G=1 TO 4
400 READ RW(G,F)
410 NEXT G,F
420 REM READ DOOR R VALUES
430 FOR I=1 TO 4:READ D(I):NEXT I
440 REM READ FLOOR R VAL AND TEMP C
ORR
450 FOR I=1 TO 3:READ RF(I),TC(I):N
EXT I
460 REM READ INSULATION R PER INCH
470 FOR I=1 TO 5:READ IC(I):NEXT I
480 REM READ DUCT MULTIPLIERS
490 FOR KD=1 TO 2
500 FOR K=1 TO 3
510 FOR J=1 TO 3
520 READ DM(KD,J,K)
530 NEXT J,K,KD
540 REM READ AIR CHANGES FOR INFILT
RATION
550 FOR I=1 TO 2
560 FOR J=1 TO 3
570 READ IW(I,J)
580 NEXT J,I
590 REM INSIDE DESIGN TEMPERATURE
600 IT=75:PK=1
605 GETA$:IFA$=""THEN605
610 PRINT"{CLEAR}{DOWN}WINTER OUTSI
DE DESIGN TEMPERATURE";
620 INPUT OT
630 DT=IT-OT
640 GOSUB 1000:REM WINDOWS
650 GOSUB 2000:REM DOORS
660 GOSUB 3000:REM WALLS
670 GOSUB 4000:REM CEILING
680 GOSUB 5000:REM FLOOR
690 GOSUB 5200:REM DUCTS
700 GOSUB 6000:REM REPORT RESULTS
710 Q1=TQ/DT
720 PRINT"{DOWN}DO YOU WISH TO MAKE
IMPROVEMENTS?"
730 GET A$:IF A$="" THEN 730
740 PK=2:IF A$="N" THEN 999
750 INPUT"{CLEAR}{02 DOWN}DO YOU WI
SH TO IMPROVE WINDOWS";A$
760 IF LEFT$(A$,1)="Y" THEN GOSUB 1
000
770 INPUT"{CLEAR}{02 DOWN}DO YOU WI
SH TO IMPROVE DOORS";A$
780 IF LEFT$(A$,1)="Y" THEN GOSUB 2
000
790 INPUT"{CLEAR}{02 DOWN}DO YOU WI
SH TO IMPROVE WALLS";A$
800 IF LEFT$(A$,1)="Y" THEN GOSUB 3
000
810 INPUT"{CLEAR}{02 DOWN}DO YOU WI
SH TO IMPROVE CEILING";A$
820 IF LEFT$(A$,1)="Y" THEN GOSUB 4000

```

```

830 INPUT"{CLEAR}{02 DOWN}DO YOU WI
SH TO IMPROVE FLOOR";A$
840 IF LEFT$(A$,1)="Y" THEN GOSUB 5
000
850 INPUT"{CLEAR}{02 DOWN}DO YOU WI
SH TO IMPROVE DUCTS";A$
860 IF LEFT$(A$,1)="Y" THEN GOSUB 5
200
870 GOSUB 6000:REM REPORT RESULTS
880 Q2=TQ/DT
890 PRINT:PRINT"HIT RETURN TO GET S
AVINGS"
900 GET A$:IF A$="" THEN 900
910 GOSUB 7000:REM CALCULATE A YEAR
OF SAVINGS
999 END
1000 REM WINDOW SUBROUTINE
1010 I=1:IF PK>1 THEN 1040
1020 PRINT"{CLEAR}{DOWN}HOW MANY DIF
FERENT TYPES OF WINDOWS";
1030 INPUT NX
1040 IX=1:CW=0:A(I)=0:Q(I)=0
1050 PRINT"{DOWN} ARE WINDOWS WEATHE
RSTRIPPED";
1060 INPUT WWS$
1070 IF LEFT$(WWS$,1)="Y" THEN IX=2
1080 FOR J=1 TO NX
1090 PRINT"SIZE";J:IF PK>1 THEN 1160
1100 PRINT"NUMBER OF WINDOWS";
1110 INPUT NW
1120 PRINT"SIZE OF WINDOWS (H,W) FT"
;
1130 INPUT H,W
1140 S(J)=H*W*NW
1150 CW=CW+(H+W)*NW
1160 A(I)=A(I)+S(J)
1170 PRINT"TYPE OF WINDOWS"
1180 PRINT" 1. SINGLE GLASS"
1190 PRINT" 2. SINGLE + STORM"
1200 PRINT" 3. DOUBLE PANE"
1210 PRINT" 4. TRIPLE (DOUBLE + ST
ORM)"
1220 INPUT G
1230 PRINT"TYPE OF WINDOW FRAME"
1240 PRINT" 1. WOOD"
1250 PRINT" 2. METAL OR JALOUSE"
1260 PRINT" 3. FIXED"
1270 INPUT F
1280 RM=RW(G,F)
1290 Q(I)=Q(I)+S(J)*DT/RM
1300 R(I)=RM
1310 PRINT"{CLEAR}{DOWN}";
1320 NEXT J
1330 IN(I)=0.018*DT*IW(IX,F)*CW
1340 RETURN
2000 REM DOORS SUBROUTINE
2010 I=2:IF PK>1 THEN 2080
2020 PRINT"{CLEAR}{DOWN}NUMBER OF DO
ORS";
2030 INPUT N
2040 PRINT"SIZE OF DOORS (H,W) FT";
2050 INPUT H,W
2060 A(I)=H*W*N
2070 CD=(H+W)*N

```

```

2080 PRINT{DOWN}TYPE OF DOORS"
2090 PRINT 1. WOOD"
2100 PRINT 2. WOOD + STORM"
2110 PRINT 3. METAL URETHANE CORE
2120 PRINT 4. METAL POLYSTYRENE C
ORE"
2130 INPUT T
2140 R(I)=D(T)
2150 Q(I)=A(I)*DT/R(I)
2160 DW=138
2170 PRINT{DOWN}ARE DOORS WEATHERST
RIPPED";
2180 INPUT DW$
2190 IF LEFT$(DW$,1)="Y" THEN DW=69
2200 IN(I)=0.018*DT*DW*CD
2210 RETURN
3000 REM WALLS SUBROUTINE
3010 I=3:HO=.17:HI=.68
3020 PRINT{CLEAR}{DOWN}TYPE OF WALL
CONSTRUCTION"
3030 PRINT{DOWN} 1. BRICK VENEER"
3040 PRINT " 2. STONE"
3050 PRINT " 3. WOOD SHINGLES"
3060 PRINT " 4. STUCCO"
3070 PRINT " 5. MASONRY BLOCK"
3080 PRINT " 6. LOG"
3090 PRINT " 7. OTHER:"
3100 PRINT " ENTER CALCULATED R
VALUE DIRECTLY"
3110 PRINT " WHEN ASKED FOR INS
ULATION R VALUE"
3120 INPUT TY
3130 ON TY GOTO 3140,3150,3160,3170,
3180,3190,3200
3140 RM=.2*3.5:GOTO 3210:REM BRICK
3150 RM=.08*5: GOTO 3210:REM STONE
3160 RM=.87: GOTO 3210:REM WOOD
3170 RM=.2*2: GOTO 3210:REM STUCCO
3180 RM=2: GOTO 3210:REM MASONR
Y
3190 RM=1.25*8:GOTO 3210:REM LOG
3200 RM=0:REM OTHER
3210 PRINT FOR LIST OF R VALUES F
OR INSULATION"
3220 PRINT ENTER -1 FOR INSULATIO
N R VALUE"
3230 PRINT"INSULATION R VALUE";
3240 INPUT RI
3250 IF RI<0 THEN GOSUB 3500:GOTO 32
30
3260 R(I)=HO+RM+RI+HI:IF PK>1 THEN 3
340
3270 PRINT"HOW MANY STORIES IN HOUSE
";
3280 INPUT NT
3290 PRINT"WHAT IS THE CEILING HEIGH
T (FT)";
3300 INPUT CH
3310 PRINT"WHAT IS TOTAL PERIMETER (
FT)";
3320 INPUT P
3330 A(I)=NT*CH*P-A(1)-A(2)
3340 Q(I)=A(I)*DT/R(I)

```

```

3350 RETURN
3500 REM LIST OF INSULATION R VALUES
3510 PRINT{CLEAR}{DOWN}LIST OF INSU
LATION R VALUES, WALLS"
3520 PRINT{DOWN} NO INSULATI
ON (AIR) = .94"
3530 PRINT BATT INSULATION IN WA
LL = 11"
3540 PRINT HALF INCH ASPHALT BOA
RD = 2.4
3550 PRINT 1/2 IN GYPSUM OR PLAST
ER = 1.39
3560 PRINT 1/4 IN WOOD FIBER BOA
RD = 1.12
3570 PRINT FIR OR PINE SHEATHI
NG = 1.92
3580 PRINT 3/4 IN PLYWOOD PANE
LS = 1.88
3590 PRINT 1/2 IN PLYWO
OD = 1.57
3600 PRINT:PRINT
3610 RETURN
4000 REM CEILING ROUTINE
4010 I=4
4020 HI=.61:HO=.61:IF PK>1 THEN 4060
4030 PRINT{CLEAR}{DOWN}WHAT IS TOTA
L CEILING AREA"
4040 PRINT"OF THE HOUSE";
4050 INPUT A(I)
4060 PRINT"HOW MANY INCHES OF INSULA
TION IN CEILING";
4070 INPUT CI
4080 PRINT"TYPE OF INSULATING MATERI
AL"
4090 PRINT{DOWN} 1. FIBERGLASS"
4100 PRINT " 2. MINERAL WOOL"
4110 PRINT " 3. VERMICULITE OR PERL
ITE"
4120 PRINT " 4. CELLULOSE FIBER"
4130 PRINT " 5. U-F FOAM{DOWN}"
4140 INPUT T
4150 RM=CI*IC(T)
4160 R(I)=HO+RM+HI
4170 Q(I)=A(I)*DT/R(I)
4180 RETURN
5000 REM FLOOR ROUTINE
5010 I=5:IF PK>1 THEN 5040
5020 PRINT{CLEAR}{DOWN}WHAT IS TOTA
L FLOOR AREA";
5030 INPUT A(I)
5040 PRINT"HOW MANY INS OF INSULATIO
N IN FLOOR";
5050 INPUT FI:IF PK>1 THEN 5110
5060 PRINT"TYPE OF FOUNDATION"
5070 PRINT 1. OPEN CRAWLSPACE"
5080 PRINT 2. ENCLOSED CRAWLSPACE
OR BASEMENT"
5090 PRINT 3. CONCRETE SLAB"
5100 INPUT TF
5110 R(I)=HO+FI*3.1+RF(TF)+HI
5120 Q(I)=A(I)*(DT-TC(TF))/R(I)
5130 RETURN
5200 REM DUCTS
5210 DI=.1

```

```

5220 IF TF=3 THEN KD=3:RETURN
5230 PRINT"{DOWN}IS YOUR DUCTWORK IN
      SULATED";
5240 INPUT D$:IF PK>1 THEN 5310
5250 PRINT"{DOWN}LOCATION OF HEAT DU
      CTS:"
5260 PRINT"      1. ATTIC OR CRAWLSPAC
      E"
5270 PRINT"      2. UNCONDITIONED BASE
      MENT"
5280 PRINT"      3. IN SLAB FLOOR"
5290 PRINT"      4. INSIDE CONDITIONED
      SPACE"
5300 INPUT KD
5310 RETURN
6000 REM WRITE A REPORT
6010 PRINT"{CLEAR}","HEAT LOSS EVALU
      ATION"
6020 PRINT:PRINT:TQ=0
6030 PRINT" ITEM"," AREA"," R-VALUE"
      ,"HEAT LOSS"
6040 PRINT"," SQ.FT.",," BTU/HR":PRI
      NT
6050 FOR I=1 TO 5
6060 A(I)=INT(A(I)*100+.5)/100
6070 R(I)=INT(R(I)*100+.5)/100
6080 Q(I)=INT(Q(I)+.5)
6090 PRINT N$(I),A(I),R(I),Q(I)
6100 TA=TA+A(I):TQ=TQ+Q(I)
6110 NEXT I
6120 REM PRINT INFILTRATION LOSS
6130 PRINT"INFILTRATION",,INT((IN(1)
      +IN(2))/2+.5)
6140 TQ=TQ+(IN(1)+IN(2))/2
6150 REM CALCULATE DUCT LOSS
6160 X=TQ/(A(5)*CH*NT):J=3:K=3
6170 IF X<45 THEN K=2
6180 IF X<35 THEN K=1
6190 DI=.15+.05*(3-K)
6200 IF LEFT$(D$,1)="N" AND KD<2 THE
      N 6240
6205 IF KD>2 THEN DI=0:GOTO 6240
6210 IF OT<15 THEN J=2
6220 IF OT<0 THEN J=1
6230 DI=DM(KD,J,K)
6240 PRINT"DUCT LOSS",,,INT(DI*TQ+.5)
6250 TQ=TQ+TQ*DI
6260 PRINT,"@@@@@@@@@","@@@@@@@@@
6270 PRINT" TOTAL",INT(TA),,INT(TQ)
6280 PRINT:PRINT
6290 PRINT"DESIGN CONDITIONS:"
6300 PRINT"      OUTSIDE DESIGN TEMP";O
      T
6310 PRINT"      INSIDE DESIGN TEMP";I
      T
6320 PRINT"TEMPERATURE DIFFERENCE";D
      T
6330 RETURN
7000 REM FIND SAVINGS USING DEGREE-D
      AYS
7010 DD=2961:DD$="ATLANTA GA"
7012 E1=INT(Q1*DD*24)
7014 E2=INT(Q2*DD*24)
7030 PRINT"{CLEAR}TYPE OF HEATING FU

```

```

      EL USED"
7040 PRINT"      1. ELECTRICITY"
7050 PRINT"      2. NATURAL GAS"
7060 PRINT"      3. FUEL OIL"
7070 INPUT FT:PC=.55
7080 ON FT GOTO 7100,7200,7300
7090 GOTO 7030
7100 REM ELECTRICITY
7110 PRINT"IS HEATING UNIT A HEAT PU
      MP";
7120 INPUT HP$:ER=3413
7130 IF LEFT$(HP$,1)<>"Y" THEN 7150
7140 INPUT"ENTER EER OF HEAT PUMP";E
      R:ER=ER*1000
7150 INPUT"AVERAGE $ COST PER KWH";C
      O:FU$="KWH"
7160 E1=INT(E1/ER+.5)
7165 M1=E1*CO
7170 E2=INT(E2/ER+.5)
7175 M2=E2*CO
7180 MS=M1-M2
7190 GOTO 7400
7200 REM NATURAL GAS
7210 INPUT"AVERAGE $ COST PER THERM ~
      OF NATURAL GAS";CO
7220 E1=INT(E1/(103000*PC)+.5)
7225 M1=E1*CO
7230 E2=INT(E2/(103000*PC)+.5)
7235 M2=E2*CO
7240 MS=M1-M2
7250 FU$="THERMS":GOTO 7400
7300 REM FUEL OIL
7310 INPUT"AVERAGE $ COST PER GALLON
      OF FUEL OIL";CO
7320 E1=INT(E1/(138000*PC)+.5)
7325 M1=E1*CO
7330 E2=INT(E2/(138000*PC)+.5)
7335 M2=E2*CO
7340 MS=M1-M2:FU$="GALLONS"
7400 REM GIVE RESULTS
7410 M1=INT(M1*100)/100
7420 M2=INT(M2*100)/100
7430 MS=INT(MS*100)/100
7440 INPUT"{DOWN}TOTAL $ COST OF YOU
      R IMPROVEMENTS";CI
7450 PB=INT(CI/MS*1000)/1000
7460 REM REPORT SAVINGS AND PAYBACK
7470 PRINT"{CLEAR}","ANALYSIS OF IMP
      ROVEMENTS"
7480 PRINT:PRINT
7490 PRINT,"ENERGY NEEDED"
7500 PRINT"ORIGINAL HOUSE",E1;FU$
7510 PRINT"IMPROVED HOUSE",E2;FU$
7520 PRINT,"@@@@@@@@@
7530 PRINT,"SAVINGS",E1-E2;FU$
7540 PRINT
7550 PRINT,"OPER. COSTS"
7560 PRINT"ORIGINAL HOUSE","$";M1
7570 PRINT"IMPROVED HOUSE","$";M2
7580 PRINT,"@@@@@@@@@
7590 PRINT,"SAVINGS","$";MS
7600 PRINT:PRINT,"PAYBACK",PB;"YEARS
      "
7610 PRINT:PRINT

```

```

7620 PRINT"ABOVE IS BASED ON ONE YEA
      R OF OPERATION"
7630 PRINT"IN ";DD$
7640 RETURN
8000 REM DRAW HOUSE
8010 PRINTCHR$(142):PRINT:PRINT:PRIN
      T
8020 PRINTSPC(8);"
8030 PRINTSPC(8);" {REV} ]"
      OFF}"
8040 PRINTSPC(8);"TiiiiiiiY
8050 PRINTSPC(8);"T<><><><><>*{REV} -
      {OFF}"
8060 PRINTSPC(8);"TiiiiiiiY,"";Y
      "
8070 PRINTSPC(8);"T<><>{REV}!{OFF}!<
      ><>Y{REV}! {OFF}!Y"
8080 PRINTSPC(7);"#####
      ###"
8090 RETURN

```

Program 2:

Make these changes in Program 1 for the Apple II.

```

100 HOME : VTAB 2: PRINT " HOME
      ENERGY PROGRAM"
3510 HOME : PRINT "LIST OF INSUL
      ATION R VALUES, WALLS"
3520 PRINT : PRINT " NO INS
      ULATION
4030 HOME : PRINT "WHAT IS TOTAL
      CEILING AREA
4090 PRINT : PRINT " 1. FIBERGL
      ASS
4130 PRINT : PRINT " 5. U-F FOA
      M": PRINT
5020 HOME : PRINT "WHAT IS TOTAL
      FLOOR AREA?";
5230 PRINT : PRINT "IS YOUR DUCT
      WORK INSULATED?";
5250 PRINT "LOCATION OF HEAT DUC
      TS:
6010 HOME : PRINT "HEAT LOSS EVA
      LUATION"
7440 PRINT : INPUT "TOTAL $ COST
      OF YOUR IMPROVEMENTS";CI
7470 HOME : PRINT "ANALYSIS OF I
      MPROVEMENTS"
7520 PRINT ",, "-----"
7580 PRINT ",, "-----"
8000 RETURN
8010 - 8090"DELETE"

```

Program 3: Atari Version

```

100 POKE 82,0:PRINT "(CLEAR)<2 DOWN>
      HOME ENERGY PROGRAM"
110 PRINT :PRINT
150 GOSUB 8000
170 OPEN #1,4,0,"K:"
180 DIM A(6),Q(6),R(6),RW(4,3),D(4),I
      W(2,3),S(10)
190 DIM RF(3),TC(3),N$(5*10),IC(5),DM
      (2,15),IN(2),A$(1),NL(5)
191 DIM WW$(1),DW$(1),D$(1),DD$(20),H
      P$(1),FU$(10)
200 REM WINDOW R VALUES
210 DATA 1.01,2.22,1.815,3.155
220 DATA .909,1.667,1.437,2.137

```

```

230 DATA .909,2,1.724,2.564
240 REM DOOR R VALUES
250 DATA .41,.75,.95,1.1
260 REM FLOOR R VALUES AND TEMP CORR
270 DATA 3.2,0,3.2,30,1.23,0
280 REM CEILING INSULATION R PER INCH
290 DATA 3.5,3,2.5,4.5,5.5
300 N$(1)="WINDOWS":N$(11)="DOORS":N$
      (21)="WALLS"
310 N$(31)="CEILING":N$(41)="FLOOR $"
315 NL(1)=7:NL(2)=5:NL(3)=5:NL(4)=6:N
      L(5)=7
320 REM DUCT MULTIPLIERS
330 DATA .2,.15,.1,.15,.1,.05,.1,.05,
      .05
340 DATA .2,.15,.1,.1,.1,.05,.05,.05,
      .05
350 REM AIR CHANGES PER FOOT OF CRACK
360 DATA 39,74,52,24,32,33
370 REM READ WINDOW R VALUES
380 FOR F=1 TO 3
390 FOR G=1 TO 4
400 READ TEMP:RW(G,F)=TEMP
410 NEXT G:NEXT F
420 REM READ DOOR R VALUES
430 FOR I=1 TO 4:READ TEMP:D(I)=TEMP:
      NEXT I
440 REM READ FLOOR R VAL AND_TEMP COR
      R
450 FOR I=1 TO 3:READ TEMP:RF(I)=TEMP
      :READ TEMP:TC(I)=TEMP:NEXT I
460 REM READ INSULATION R PER INCH
470 FOR I=1 TO 5:READ TEMP:IC(I)=TEMP
      :NEXT I
480 REM READ DUCT MULTIPLIERS
490 FOR KD=1 TO 2
500 FOR K=1 TO 3
510 FOR J=1 TO 3
520 READ TEMP:DM(KD,J+K*4)=TEMP
530 NEXT J:NEXT K:NEXT KD
540 REM READ AIR CHANGES FOR INFILTRA
      TION
550 FOR I=1 TO 2
560 FOR J=1 TO 3
570 READ TEMP:IW(I,J)=TEMP
580 NEXT J:NEXT I
590 REM INSIDE DESIGN TEMPERATURE
600 IT=75:PK=1
601 ? : ? : ? "Press RETURN to begin:";
605 GET #1,A
610 PRINT "(CLEAR)<DOWN>WINTER OUTSID
      E DESIGN TEMPERATURE";
620 INPUT OT
630 DT=IT-OT
640 GOSUB 1000:REM WINDOWS
650 GOSUB 2000:REM DOORS
660 GOSUB 3000:REM WALLS
670 GOSUB 4000:REM CEILING
680 GOSUB 5000:REM FLOOR
690 GOSUB 5200:REM DUCTS
700 GOSUB 6000:REM REPORT RESULTS
710 Q1=TQ/DT
720 PRINT "DO YOU WISH TO MAKE IMPROV
      EMENTS?:";
730 GET #1,A:A$=CHR$(A)
740 K=2:IF A$="N" THEN 999
750 PRINT "(CLEAR)<2 DOWN>DO YOU WISH
      TO IMPROVE WINDOWS";:INPUT A$
760 IF A$="Y" THEN GOSUB 1000
770 PRINT "(CLEAR)<2 DOWN>DO YOU WISH
      TO IMPROVE DOORS";:INPUT A$
780 IF A$="Y" THEN GOSUB 2000
790 PRINT "(CLEAR)<2 DOWN>DO YOU WISH

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```

      TO IMPROVE WALLS";:INPUT A$
800 IF A$="Y" THEN GOSUB 3000
810 PRINT "(CLEAR)(2 DOWN)DO YOU WISH
      TO IMPROVE CEILING";:INPUT A$
820 IF A$="Y" THEN GOSUB 4000
830 PRINT "(CLEAR)(2 DOWN)DO YOU WISH
      TO IMPROVE FLOOR";:INPUT A$
840 IF A$="Y" THEN GOSUB 5000
850 PRINT "(CLEAR)(2 DOWN)DO YOU WISH
      TO IMPROVE DUCTS";:INPUT A$
860 IF A$="Y" THEN GOSUB 5200
870 GOSUB 6000:REM REPORT RESULTS
880 Q2=TQ/DT
890 PRINT :PRINT "HIT RETURN TO GET S
      AVINGS"
900 GET #1,A
910 GOSUB 7000:REM CALCULATE A YEAR O
      F SAVINGS
999 END
1000 REM WINDOW SUBROUTINE.
1010 I=1:IF PK>1 THEN 1040
1020 PRINT "(CLEAR)(DOWN)HOW MANY DIF
      FERENT TYPES OF WINDOWS";
1030 INPUT NX
1040 IX=1:CW=0:A(I)=0:Q(I)=0
1050 PRINT "(DOWN) ARE WINDOWS WEATHE
      RSTRIPPED";
1060 INPUT WW$
1070 IF WW$="Y" THEN IX=2
1080 FOR J=1 TO NX
1090 PRINT "SIZE ";J:IF PK>1 THEN 116
      0
1100 PRINT "NUMBER OF WINDOWS";
1110 INPUT NW
1120 PRINT "SIZE OF WINDOWS (H,W) FT"
      ;
1130 INPUT H,W
1140 S(J)=H*W*NW
1150 CW=CW+(H+W)*NW
1160 A(I)=A(I)+S(J)
1170 PRINT "TYPE OF WINDOWS"
1180 PRINT "(3 SPACES)1. SINGLE GLASS
      "
1190 PRINT "(3 SPACES)2. SINGLE + STO
      RM"
1200 PRINT "(3 SPACES)3. DOUBLE PANE"
1210 PRINT "(3 SPACES)4. TRIPLE (DOUB
      LE + STORM)"
1220 INPUT G
1230 PRINT "TYPE OF WINDOW FRAME"
1240 PRINT "(3 SPACES)1. WOOD"
1250 PRINT "(3 SPACES)2. METAL OR JAL
      OUSE"
1260 PRINT "(3 SPACES)3. FIXED"
1270 INPUT F
1280 RM=RW(G,F)
1290 Q(I)=Q(I)+S(J)*DT/RM
1300 R(I)=RM
1310 PRINT "(CLEAR)(DOWN)";
1320 NEXT J
1330 IN(I)=0.018*DT*IW(IX,F)*CW
1340 RETURN
2000 REM DOORS SUBROUTINE
2010 I=2:IF PK>1 THEN 2080
2020 PRINT "(CLEAR)(DOWN)NUMBER OF DO
      ORS";
2030 INPUT N
2040 PRINT "SIZE OF DOORS (H,W) FT";
2050 INPUT H,W
2060 A(I)=H*W*N
2070 CD=(H+W)*N
2080 PRINT "(DOWN)TYPE OF DOORS"
2090 PRINT "(3 SPACES)1. WOOD"
2100 PRINT "(3 SPACES)2. WOOD + STORM
      "
2110 PRINT "(3 SPACES)3. METAL URETHA
      NE CORE"
2120 PRINT "(3 SPACES)4. METAL POLYST
      YRENE CORE"
2130 INPUT T
2140 R(I)=D(T)
2150 Q(I)=A(I)*DT/R(I)
2160 DW=138
2170 PRINT "(DOWN)ARE DOORS WEATHERST
      RIPPED";
2180 INPUT DW$
2190 IF DW$="Y" THEN DW=69
2200 IN(I)=0.018*DT*DW*CD
2210 RETURN
3000 REM WALLS SUBROUTINE
3010 I=3:H0=0.17:HI=0.68
3020 PRINT "(CLEAR)(DOWN)TYPE OF WALL
      CONSTRUCTION"
3030 PRINT "(DOWN)(3 SPACES)1. BRICK
      VENEER"
3040 PRINT "(3 SPACES)2. STONE"
3050 PRINT "(3 SPACES)3. WOOD SHINGLE
      S"
3060 PRINT "(3 SPACES)4. STUCCO"
3070 PRINT "(3 SPACES)5. MASONRY BLOC
      K"
3080 PRINT "(3 SPACES)6. LOG"
3090 PRINT "(3 SPACES)7. OTHER:"
3100 PRINT "(6 SPACES)ENTER CALCULATE
      D R VALUE DIRECTLY"
3110 PRINT "(6 SPACES)WHEN ASKED FOR
      INSULATION R VALUE"
3120 INPUT TY
3130 ON TY GOTO 3140,3150,3160,3170,3
      180,3190,3200
3140 RM=0.2*3.5:GOTO 3210:REM BRICK
3150 RM=0.08*5:GOTO 3210:REM STONE
3160 RM=0.87:GOTO 3210:REM WOOD
3170 RM=0.2*2:GOTO 3210:REM STUCCO
3180 RM=2:GOTO 3210:REM MASONRY
3190 RM=1.25*8:GOTO 3210:REM LOG
3200 RM=0:REM OTHER
3210 PRINT "(3 SPACES)FOR LIST OF R V
      ALUES FOR INSULATION"
3220 PRINT "(3 SPACES)ENTER -1 FOR IN
      SULATION R VALUE"
3230 PRINT "INSULATION R VALUE";
3240 INPUT RI
3250 IF RI<0 THEN GOSUB 3500:GOTO 323
      0
3260 R(I)=H0+RM+RI+HI:IF PK>1 THEN 33
      40
3270 PRINT "HOW MANY STORIES IN HOUSE
      ";
3280 INPUT NT
3290 PRINT "WHAT IS THE CEILING HEIGH
      T (FT)";
3300 INPUT CH
3310 PRINT "WHAT IS TOTAL PERIMETER (
      FT)";
3320 INPUT P
3330 A(I)=NT*CH*P-A(1)-A(2)
3340 Q(I)=A(I)*DT/R(I)
3350 RETURN
3500 REM LIST OF INSULATION R VALUES
3510 PRINT "(CLEAR)(DOWN)LIST OF INSU
      LATION R VALUES, WALLS"
3520 PRINT "(DOWN)(8 SPACES)NO INSULA
      TION (AIR) = .94"
3530 PRINT "(4 SPACES)BATT INSULATION
      IN WALL = 11"

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3540 PRINT "(4 SPACES)HALF INCH ASPHA
LT BOARD = 2.4"
3550 PRINT "(3 SPACES)1/2 IN GYPSUM O
R PLASTER = 1.39"
3560 PRINT "(4 SPACES)1/4 IN WOOD FIB
ER BOARD = 1.12"
3570 PRINT "(6 SPACES)FIR OR PINE SHE
ATHING = 1.92"
3580 PRINT "(6 SPACES)3/4 IN PLYWOOD
PANELS = 1.88"
3590 PRINT "(13 SPACES)1/2 IN PLYWOOD
= 1.57"
3600 PRINT :PRINT
3610 RETURN
4000 REM CEILING ROUTINE
4010 I=4
4020 HI=0.61:HO=0.61:IF PK>1 THEN 406
0
4030 PRINT "(CLEAR)(DOWN)WHAT IS TOTA
L CEILING AREA"
4040 PRINT "OF THE HOUSE";
4050 INPUT TEMP:A(I)=TEMP
4060 PRINT "HOW MANY INCHES OF INSULA
TION IN CEILING";
4070 INPUT CI
4080 PRINT "TYPE OF INSULATING MATERI
AL"
4090 PRINT "(DOWN) 1. FIBERGLASS"
4100 PRINT " 2. MINERAL WOOL"
4110 PRINT " 3. VERMICULITE OR PERLI
TE"
4120 PRINT " 4. CELLULOSE FIBER"
4130 PRINT " 5. U-F FOAM(DOWN)"
4140 INPUT T
4150 RM=CI*IC(T)
4160 R(I)=HO+RM+HI
4170 Q(I)=A(I)*DT/R(I)
4180 RETURN
5000 REM FLOOR ROUTINE
5010 I=5:IF PK>1 THEN 5040
5020 PRINT "(CLEAR)(DOWN)WHAT IS TOTA
L FLOOR AREA";
5030 INPUT TEMP:A(I)=TEMP
5040 PRINT "HOW MANY ITEMS OF INSULAT
ION IN FLOOR";
5050 INPUT FI:IF PK>1 THEN 5110
5060 PRINT "TYPE OF FOUNDATION"
5070 PRINT "(3 SPACES)1. OPEN CRAWLSP
ACE"
5080 PRINT "(3 SPACES)2. ENCLOSED CRA
WLSPACE OR BASEMENT"
5090 PRINT "(3 SPACES)3. CONCRETE SLA
B"
5100 INPUT TF
5110 R(I)=HO+FI*3.1+RF(TF)+HI
5120 Q(I)=A(I)*(DT-TC(TF))/R(I)
5130 RETURN
5200 REM DUCTS
5210 DI=0.1
5220 IF TF=3 THEN KD=3:RETURN
5230 PRINT "(DOWN)IS YOUR DUCTWORK IN
SULATED";
5240 INPUT D$:IF PK>1 THEN 5310
5250 PRINT "(DOWN)LOCATION OF HEAT DU
CTS:"
5260 PRINT "(4 SPACES)1. ATTIC OR CRA
WLSPACE"
5270 PRINT "(4 SPACES)2. UNCONDITIONE
D BASEMENT"
5280 PRINT "(4 SPACES)3. IN SLAB FLOO
R"
5290 PRINT "(4 SPACES)4. INSIDE CONDI
TIONED SPACE"
5300 INPUT KD
5310 RETURN
6000 REM WRITE A REPORT
6010 PRINT "(CLEAR)","HEAT LOSS EVALU
ATION"
6020 PRINT :PRINT :TQ=0
6030 PRINT " ITEM"," AREA"," R-VALUE"
,"HEAT LOSS"
6040 PRINT "," SQ.FT.",," BTU/HR":PRIN
T
6050 FOR I=1 TO 5
6060 A(I)=INT(A(I)*100+0.5)/100
6070 R(I)=INT(R(I)*100+0.5)/100
6080 Q(I)=INT(Q(I)+0.5)
6090 PRINT N$(I*10-9,(I-1)*10+NL(I)),
A(I),R(I),Q(I)
6100 TA=TA+A(I):TQ=TQ+Q(I)
6110 NEXT I
6120 REM PRINT INFILTRATION LOSS
6130 PRINT "INFILTRATION",,INT((IN(1)
+IN(2))/2+0.5)
6140 TQ=TQ+(IN(1)+IN(2))/2
6150 REM CALCULATE DUCT LOSS
6160 X=TQ/(A(5)*CH*NT):J=3:K=3
6170 IF X<45 THEN K=2
6180 IF X<35 THEN K=1
6190 DI=0.15+0.05*(3-K)
6200 IF D$="N" AND KD<2 THEN 6240
6205 IF KD>2 THEN DI=0:GOTO 6240
6210 IF OT<15 THEN J=2
6220 IF OT<0 THEN J=1
6230 DI=DM(KD,J+K*4)
6240 PRINT "DUCT LOSS",,,INT(DI*TQ+0.
5)
6250 TQ=TQ+TQ*DI
6260 PRINT ","(8 R)",,"(8 R)":?
6270 PRINT " TOTAL ",INT(TA),,INT(TQ
)
6280 PRINT
6290 PRINT "DESIGN CONDITIONS:"
6300 PRINT "(3 SPACES)OUTSIDE DESIGN
TEMP:":OT
6310 PRINT "(4 SPACES)INSIDE DESIGN T
EMP:":IT
6320 PRINT "TEMPERATURE DIFFERENCE:":
DT
6330 RETURN
7000 REM FIND SAVINGS USING DEGREE-DA
YS
7010 DD=2961:DD$="ATLANTA GA"
7012 E1=INT(Q1*DD*24)
7014 E2=INT(Q2*DD*24)
7030 PRINT "(CLEAR)TYPE OF HEATING FU
EL USED"
7040 PRINT " 1. ELECTRICITY"
7050 PRINT " 2. NATURAL GAS"
7060 PRINT " 3. FUEL OIL"
7070 INPUT FT:PC=0.55
7080 ON FT GOTO 7100,7200,7300
7090 GOTO 7030
7100 REM ELECTRICITY
7110 PRINT "IS HEATING UNIT A HEAT PU
MP";
7120 INPUT HP$:ER=3413
7130 IF HP$<>"Y" THEN 7150
7140 PRINT "ENTER EER OF HEAT PUMP":;
INPUT ER:ER=ER*1000
7150 PRINT "AVERAGE $ COST PER KWH":;
INPUT CO:FU$="KWH"
7160 E1=INT(E1/ER+0.5)
7165 M1=E1*CO
7170 E2=INT(E2/ER+0.5)
7175 M2=E2*CO
7180 MS=M1-M2

```

```

7190 GOTO 7400
7200 REM NATURAL GAS
7210 PRINT "AVERAGE $ COST PER THERM
      OF NATURAL GAS";:INPUT CO
7220 E1=INT(E1/(103000*PC)+0.5)
7225 M1=E1*CO
7230 E2=INT(E2/(103000*PC)+0.5)
7235 M2=E2*CO
7240 MS=M1-M2
7250 FU$="THERMS":GOTO 7400
7300 REM FUEL OIL
7310 PRINT "AVERAGE $ COST PER GALLON
      OF FUEL OIL";:INPUT CO
7320 E1=INT(E1/(138000*PC)+0.5)
7325 M1=E1*CO
7330 E2=INT(E2/(138000*PC)+0.5)
7335 M2=E2*CO
7340 MS=M1-M2:FU$="GALLONS"
7400 REM GIVE RESULTS
7410 M1=INT(M1*100)/100
7420 M2=INT(M2*100)/100
7430 MS=INT(MS*100)/100:IF MS=0 THEN
      MS=1.0E-05
7440 PRINT "(DOWN)TOTAL $ COST OF YOU
      R IMPROVEMENTS";:INPUT CI
7450 PB=INT(CI/MS*1000)/1000
7460 REM REPORT SAVINGS AND PAYBACK
7470 PRINT "(CLEAR)","ANALYSIS OF IMP
      ROVEMENTS"
7480 PRINT :PRINT
7490 PRINT ,,"ENERGY NEEDED"
7500 PRINT "ORIGINAL HOUSE ",E1;" ";F
      U$
7510 PRINT "IMPROVED HOUSE ",E2;" ";F
      U$
7520 PRINT ,,"(9 R)":?
7530 PRINT ,,"SAVINGS",E1-E2;" ";FU$
7540 PRINT
7550 PRINT ,,"OPER. COSTS"
7560 PRINT "ORIGINAL HOUSE","$";M1
7570 PRINT "IMPROVED HOUSE","$";M2
7580 PRINT ,,"(9 R)":?
7590 PRINT ,,"SAVINGS","$";MS
7600 PRINT :PRINT ,,"PAYBACK ",PB;" YE
      ARS"

7610 PRINT :PRINT
7620 PRINT "ABOVE IS BASED ON ONE YEA
      R OF OPERATION"
7630 PRINT "IN ";DD$
7640 RETURN
8000 REM DRAW HOUSE
8010 PRINT :PRINT :PRINT :PRINT
8020 POKE 85,8:? ;"(5 SPACES){I}{O}"
8030 POKE 85,8:? "(H){10 SPACES}{J}"
8040 POKE 85,8:? "(V){I}{O}{I}{O}{I}
      {O}{I}{O}{I}{O}{B}"
8050 POKE 85,8:? "(V){K}{L}{K}{L}{K}
      {L}{K}{L}{K}{L}{B}{5 SPACES}{J}"
8060 POKE 85,8:? "(V){I}{O}{I}{O}{I}
      {O}{I}{O}{I}{O}{B}{I}{4 U}{B}"
8070 POKE 85,8:? "(V){K}{L}{K}{L}{K}{V}
      {Y}{K}{L}{K}{L}{B}{V}{4 SPACES}
      {B}"
8080 POKE 85,7:? "(21 M)"
8090 RETURN

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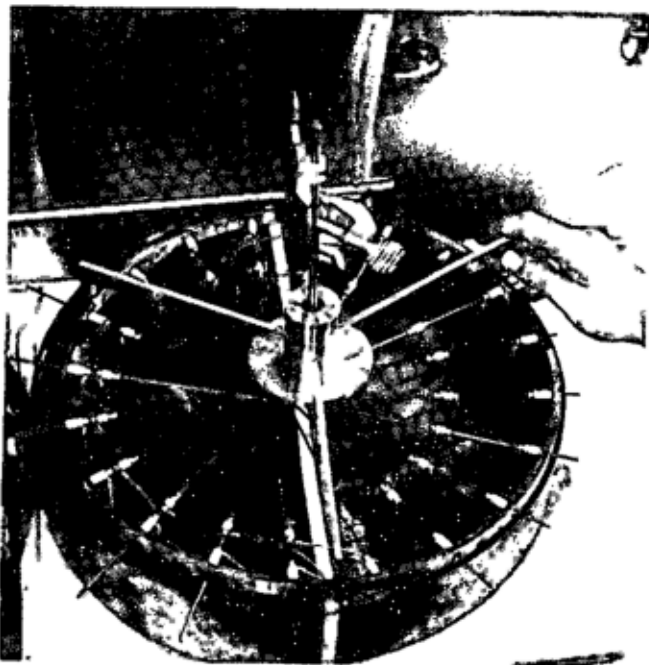
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THE NITINOL HEAT ENGINE

Discussion and Demonstration



The Berkeley Prototype

Certain metallic alloys exhibit a shape memory effect when heated and cooled across a specific temperature threshold. One of the projects under the Solar Energy Program of the Lawrence Berkeley Laboratory's Energy and Environment Division is the development of low temperature heat engines based on this principle. Such engines may make practical the conversion of solar and other forms of low temperature thermal energy to useful mechanical work, such as powering an electric generator.

The shape memory alloy used in the LBL prototype engine is a nickel-titanium intermetallic compound named 55 Nitinol, whose dynamic properties were first observed at the Naval Ordnance Laboratory, Silver Spring, Maryland, in the late 1950's. The name Nitinol is an abbreviation of the elements' symbols Ni and Ti, and the initials of the laboratory. The alloy, composed of nearly equal numbers of nickel and titanium atoms, undergoes a solid state phase transformation (change in crystal structure) on heating and cooling. The thermal threshold

for the transformation is determined by the exact proportions of nickel and titanium in the alloy; variations in the composition will cause the transformation to occur at temperatures which may be below the freezing point of water, or above the boiling point.

During transformation the alloy undergoes abrupt changes in its physical, mechanical and electronic properties. It is primarily the change in elastic properties that made the material interesting for application to heat engines. Above the critical temperature (in the case of the materials used in the prototype engine this is in the neighborhood of hot tap-water) the material is similar to spring steel, but below the threshold it may be as easily deformed it will rapidly and forcefully return, on reheating, to the shape it had before it was deformed.

This is the shape-memory effect on which the design of the original prototype was based. The engine uses a crank system to deform Nitinol wires on cooling, and to take power off as force is exerted on heating. After a period of operation in the engine however, the wire elements developed an increasingly pronounced automatic shape change on cooling. The unanticipated appearance of this "double memory" has contributed to an overall improvement in the performance of the prototype since it was first demonstrated in August 1973. The machine has now made over 21 million revolutions with no deterioration of the original Nitinol working elements.

The LBL Heat Engine Development Project involves the work of an interdisciplinary team of engineers, inventors and scientists whose backgrounds emphasize the areas of materials research, thermodynamics and physical research. The project is supported by the U.S. Energy Research and Development Administration.

by

Ridgway Banks, Technical Associate, Inventor
Lawrence Berkeley Laboratory
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Air Conditioning.

1. The density of dry air at 20°C and 1 atmos is $1.2 \text{ gm/liter} = 1.2 \text{ kg/m}^3$.
2. The specific heat at constant pressure of air is $c_p = (7/2) R$.
3. $R = 8.3 \text{ joules/mole/deg}$ where deg means Kelvin or Centigrade.
4. For air, 1 mole = 28 grams (nitrogen)

$$1 \text{ BTU} = 1054 \text{ joules}$$

$$\text{Thus, } R = .337 \text{ BTU/m}^3/\text{deg} \quad \text{and} \quad c_p = 1.2 \text{ BTU/m}^3/\text{deg}$$

1.2
So, one BTU's cooling power can lower the temperature of 1 m^3 of air by 1°C .

Example: Consider my home office. Volume is $4 \times 4 \times 3 = 50 \text{ m}^3$ roughly. Repeat the above:

$$c_p = 0.7 \text{ BTU/m}^3\text{-}^{\circ}\text{F} \quad \text{where I convert to F.}$$

Suppose in one hours time I want to cool this room 10°F . This would require 350 BTU/hr.

Example: Suppose world were at 90°F and my office were cooled to 70°F . If coobler were turned off, I bet room would rise to 75°F in 5-10 minutes. Suppose 5 minutes. Then in effect heat in-flow rate is 5° in $1/12$ hour or 60°F/hour . To exhaust this heat flow would therefore require 2100 BTU/hor.

Cost estimate: to cool my office might take 2000 BTU/hor typically, so this is a 500 watt machine running half time which is $\frac{1}{2}$ kwh each hour. At 10 hrs/day that is 2.5 kwh/day = 75 kwh/month = \$2.25 added onto electric bill each summer month.

conversion: $1 \text{ BTU/hor} = .3 \text{ watts}$ so $300 \text{ watts} = 1000 \text{ BTU/hour}$

Efficiency: Let $F = \text{heat exhausted/work applied}$. Ideally, this factor is $T_{lo}/(T_{hi}-T_{ko})$. So for an office, $dT = 15^{\circ}\text{K}$ and you could get $300/15 = 20 = F$. But that is only the ideal. Zem quotes $F=5$ as perhaps typical. Of course $F = (\text{cooling watts/elec watts})$ so if the so-called $\text{EER} = \text{BTUH/elec watts}$, then $F = (1/3) \text{EER}$. So a cheap machine with $\text{EER}=6$ has $F = 2$ which is not very impressive. The best machines have $\text{EER}=9$ so $F=3$.

QM

Messiah and the meaning of bras and kets.

1. More than once I have been led back to QM and the question: what is a ket, what is a bra, what is a matrix element. This time I was trying to prove the Wigner Eckart Theorem and came to the conclusion that I did not know what I was doing. I think Messiah understands this stuff.
2. According to Messiah, page 246, a ket is one of those abstract vectors which span the Hilbert space. The space defined by these kets is a linear vector space over $\mathbb{C}$, the complex numbers. This space has additional hot-shot niceness properties which make it a Hilbert Space. A linear operator takes any ket into some other ket. As yet, these a space has no scalar product, no metric, no norm, no nothin.

A bra is defined to be a different kind of vector. A bra is a function whose argument is one of the kets. It is supposed to be a linear function. Each linear function you can think of defines a bra. Any linear combination of such functions is also a linear function, hence, a bra. Thus, the bras span a linear vector space of functions called the dual space. Note that we can speak of a function as distinct from its value at some point (ie, on some ket.) These bras are supposed to map the kets into $\mathbb{C}$.

Next, we are to suppose that to each ket there corresponds a bra in the dual space such that this bra of that ket is a positive real number. This number is called the norm of the ket.

This correspondence between bra and ket is supposed to be antilinear:

("conjugate of a ket " $\equiv$ its "corresponding" bra

$$|v\rangle = a|1\rangle + b|2\rangle \quad \Rightarrow \quad \langle v| = a^* \langle 1| + b^* \langle 2|$$

Note that if you defined a set of kets as an orthonormal basis (in the abstract), the above supposition would lead you to the vector interp. for the ket and the transpose star for the bra. Also, the above guarantees that the norm will be real and positive. Finally, it guarantees that the scalar product will have the star reversal property.

$$\langle 2|v\rangle = b$$

$$\langle v|2\rangle = b^* = \langle 2|v\rangle^*$$

3. The effect of an operator A on a ket is already clear. The operation of A on a bra is supposed to give you a new linear function, a new bra. Indeed

it does with the following definition:

$$\langle n | = \langle x | A \quad \langle n | u \rangle = (\langle x | A) | u \rangle \equiv \langle x | (A | u \rangle)$$

You could prove rigorously that this new bra so defined is a new linear function. Thus, (no parentheses are needed) in this case. Later in volume two page 638 Messiah discusses antilinear operators, in which case the parenthesis does make a difference, but not here.

4. In passing, we should have it clear that complex conjugation is only introduced when you write the corresponding bra to a ket. You do not get it here:

$$(cA) | u \rangle = c (A | u \rangle) \quad \langle x | (cA) = c (\langle x | A)$$

5. What is the hermitian conjugate of an operator A . ? Consider the bra $\langle a | A$. Since this is after all a bra, it must have a conjugate ket. We can hopefully obtain this conjugate ket by operating on $| a \rangle$ with some linear operator, since conjugation of ket to bra or bra to ket is antilinear. This linear operator which generates that conjugate ket is called the hermitian conjugate of A .

$$(\langle a | A = \langle n | \Rightarrow | n \rangle = A^\dagger | a \rangle) \text{ defines } A^\dagger$$

6. By the way, in passing here is a proof that conjugation is an antilinear operation:

$$\langle x | (\lambda_1 | a \rangle + \lambda_2 | b \rangle) = \lambda_1^* \langle a | + \lambda_2^* \langle b | = \lambda_1^* \langle x | (| a \rangle) + \lambda_2^* \langle x | (| b \rangle)$$

Here I use a special notation: $\langle x | [| a \rangle]$ means "the bra that is the conjugate of the ket $| a \rangle$ ". You can see the antilinearity. The conjugation operation is linear except that constants get starred.

7. Now, once we have defined the hermitian conjugate in the above way, it at once follows that:

$$(\langle a | A) | b \rangle = [(\langle b |)_{conj} (\langle a | A)_{conj}]^* = \langle b | (A^\dagger | a \rangle)^*$$

$$\text{i.e., } \langle a | A | b \rangle = \langle b | A^\dagger | a \rangle^*$$

One observes that $(b/A^\dagger$ must be the conjugate of $A/b)$ also. But $(b/$ here was arbitrary. It really is $A^\dagger$ that is a conjugate operator to A , because $A^\dagger$ can generate all the conjugate kets that A can make.

8. Messiah goes on to discuss lucidly all the basics of the QM Math, although I now recall that when I tried to read this once I got lost in his discussion of projectors. I should have just skipped those sections then. Anyway, after a complete and dynamite presentation of the ABC's of matrices, Messiah starts in on matrix representations of bras, kets, and operators.

9. I sure wish that when I was studying this QM in 221 that someone said: Go and read that chapter 7 of Messiah. If I ever teach the course, that chapter will be assigned with projection partially deleted.

Operators in Wavefunction Quantum Mechanics (space only)

① The abstract Hilbert space Q.M. would say

$$\mathbb{P}|p\rangle = p|p\rangle$$

for the momentum eigenvalue equation. Wavefunction mechanics would say instead:

$$\mathbb{P}\phi_p(x) = p\phi_p(x)$$

where $\mathbb{P}$ is a differential operator.

How do we relate $\mathbb{P}$ to $\mathbb{P}$?!

② Choose these state normalizations:

$$\langle x'|x\rangle = \delta(x'-x)$$

$$\langle p'|p\rangle = \delta(p'-p)$$

Then self consistency requires that¹

$$\langle x|p\rangle = e^{ipx}/\sqrt{2\pi} \equiv \phi_p(x)$$

③ Now compute $\langle x|\mathbb{P}|x'\rangle$ as follows:

$$\begin{aligned}\langle x|\mathbb{P}|x'\rangle &= \int dp dp' \langle x|p\rangle \langle p|\mathbb{P}|p'\rangle \langle p'|x'\rangle \\ &= \int dp dp' e^{ipx} p \delta(p-p') e^{-ip'x'} (2\pi)^{-1} \\ &= \int dp \cdot p \cdot \exp i(p(x-x')) \cdot (2\pi)^{-1} \\ &= \int dp \cdot [-i\partial_x] e^{ip(x-x')}/2\pi \\ &= -i\partial_x [\delta(x-x')]\end{aligned}$$

④ Now, let $\psi(x) = \langle x|\psi\rangle$ be some normalizable wave function, so that $\psi(x) \rightarrow 0$ at $+\infty$ and $-\infty$. Then we can say:

$$\begin{aligned}\langle x|\mathbb{P}|\psi\rangle &= \int dx' \langle x|\mathbb{P}|x'\rangle \langle x'|\psi\rangle = \int dx' [-i\partial_x \delta(x-x')] \psi(x') \\ &= \int dx' [+i\partial_{x'} \delta(x-x')] \psi(x') = \text{parts} - \int dx' \delta(x-x') i\partial_{x'} \psi(x') \\ &= -i \frac{d}{dx} \psi(x) \\ &\equiv \mathbb{P}\psi(x)\end{aligned}$$

⑤ In general, we may define the (script) wavefunction operator as:

$$\langle x|\mathbb{A}|\psi\rangle = \mathcal{A}\psi(x)$$

¹ How did we know that $\langle x|p\rangle \propto e^{ipx}$? In my view, this is one of the fundamental postulates of quantum mechanics. From this assumption we derive the fact that

$$\langle x|\mathbb{P}_\mu|\psi\rangle = +i\partial_\mu \psi(x) \equiv \mathbb{P}_\mu \psi(x)$$

and this + classical mechanics yields the Schrodinger equation.

⑥ Now let's look at angular momentum as a second example of A.
The order of doing things is this:

a) write the "angular momentum algebra" of SU_2 : find the representations $|lm\rangle$ and deduce that:

$$L^2 |lm\rangle = l(l+1) |lm\rangle$$

$$L^+ |lm\rangle = \sqrt{l(l+1)-m(m+1)} |l, m+1\rangle$$

$$L_3 |lm\rangle = m |lm\rangle$$

$$L^- |lm\rangle = \sqrt{l(l+1)-m(m-1)} |l, m-1\rangle$$

$$\langle l'm' | lm \rangle = \delta_{ll'} \delta_{mm'}$$

$$1 = \sum_{lm} |lm\rangle \langle lm|$$

b) Define (script) wavefunction operators as:

$$\langle \Theta \phi | L^2 | lm \rangle = L^2 \langle \Theta \phi | lm \rangle \quad \text{etc.}$$

c) Identify these script operators with angular momentum, the crucial step:

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{p} = -i \vec{\nabla}$$

d) Now solve the equations of (a) to find that

$$\langle \Theta \phi | lm \rangle = Y_{lm}(\Theta, \phi)$$

$$\langle \Theta' \phi' | \Theta \phi \rangle = \delta(\Omega - \Omega')$$

e) Note in passing that these functions are solutions to

- 1) part of the Schrod. equation when potential is sph. sym.
- 2) Laplace's equ. of electrostatics, sph. sym. BC's.

f) Also, these $Y_{lm}(\Theta, \phi)$ are convenient basis functions to "span" the Θ - ϕ space.

⑦ Having said all this, we now ask:

(1) how are L_3 and L^2 related

(2) what is $\langle \vec{r} | L_3 | \vec{r}' \rangle$?

⑧ First, let's answer (2). We want

$$1 = \int d^3x |\vec{r}\rangle \langle \vec{r}| \quad \text{where} \quad |\vec{r}\rangle = |l\Theta\phi\rangle = |l\rangle \otimes |\Theta\phi\rangle.$$

$$\text{Since,} \quad \langle \Theta' \phi' | \Theta \phi \rangle = \delta(\Omega - \Omega')$$

$$\text{we need,} \quad \langle l' | l \rangle = \delta_{ll'} \delta(\Omega - \Omega')$$

Thus,

$$\begin{aligned} \langle \vec{r} | L_3 | \vec{r}' \rangle &= \langle l | l' \rangle \langle \Theta \phi | L_3 | lm \rangle \langle lm | \Theta' \phi' \rangle \\ &= \langle l | l' \rangle m Y_{lm}(\Theta, \phi) Y_{lm}^*(\Theta', \phi') \\ &= \langle l | l' \rangle [-i \partial_\phi] \cdot \sum_{lm} Y_{lm}(\Theta, \phi) Y_{lm}^*(\Theta', \phi') \\ &= \langle l | l' \rangle [-i \partial_\phi] \delta(\Omega - \Omega') \\ &= -i \partial_\phi \langle \vec{r} | \vec{r}' \rangle \end{aligned}$$

⑨ Then we can say:

$$\begin{aligned} \langle \vec{r} | L_3 | \psi \rangle &= \int d^3x' \langle \vec{r} | L_3 | \vec{r}' \rangle \langle \vec{r}' | \psi \rangle \\ &= \int d^3x' [-i \partial_\phi] \langle \vec{r} | \vec{r}' \rangle \psi(\vec{r}') \\ &= -i \partial_\phi \psi(\vec{r}) \end{aligned}$$

Clearing up the problem of active and passive transformations.

Consider the following:

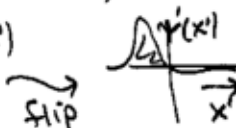
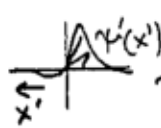
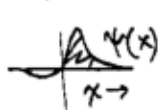


$$\phi(x) \equiv M^a \psi(x) = \psi(-x)$$

Now consider:

$$x' \equiv -x$$

$$\psi'(x') \equiv M^p \psi(x) = \psi(x)$$



$$\psi'(s) = \psi(-s) \neq \psi(s).$$

Here, there is no mention of x' . You could add a definition that $x' = -x$. Notice that in no way is M^a a unit operator. The system stays fixed and the function changes. This is the "active" way to do transformations. This is the way I think Tinokham does it (the "countours move, coord system stays fixed.")

Here we have the passive method of doing transformations. The function stays right where it is (and in that sense M^p here is a unit operator) but the coord system moves. $f'(x')$ is a function as seen in the new system. As a function of its argument, f' is a different function from f .

My contradiction developed as follows: start with $f(x)$ and do a passive transformation. In the original system nothing has happened so you get $M_p f(x) = f(x)$ as shown above. Now start with $f(x)$ and do an active transformation so that you have $M_a f(x) = f(-x)$. Then I equated the two transformations and concluded that $f(x) = f(-x)$ which is a lie.

Comment: Once you have chosen a transformation "method" you really have to stick with it or you will screw up. When an author uses P for parity, he can mean two very different things. Bjorken and Drell seem to consistently use the passive point-of-view: for example on page 71:

$$\psi(x', t') = P \psi(x, t) = \gamma^0 \psi(x, t) \quad x' = -x$$

$$\phi(x', t') = P \phi(x, t) = \phi(x, t) \quad [\text{this would be their claim for scalar fields}]$$

$$P \ell(z') = P \ell(z) = \ell(z) \quad \text{where } z' = -z$$

This last line shows how BD would say a legendre function transforms. However, lots of books would tell you:

$$P \ell(z) = \ell(-z) = (-1)^l \ell(z)$$

That is the active method again. Enough said.

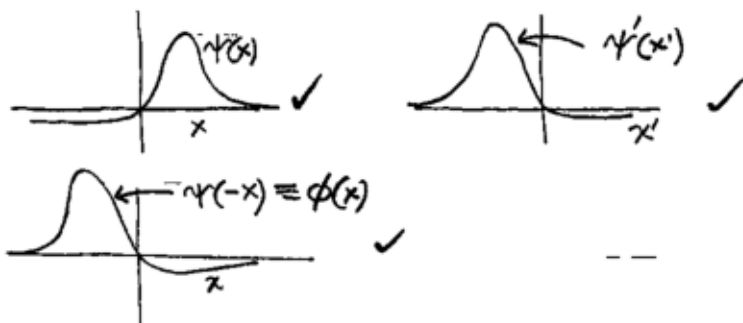
The meaning of a transformation.

- This sheet chronicles an error that cost me 8 hrs.

Somhow I have developed a mental block against understanding something that I know is quite trivial. The context is the PCT wavefunction and knowing what it means in B+D, but the trouble has nothing to do with diracias.

Consider the following one-dim mirror "transformation":

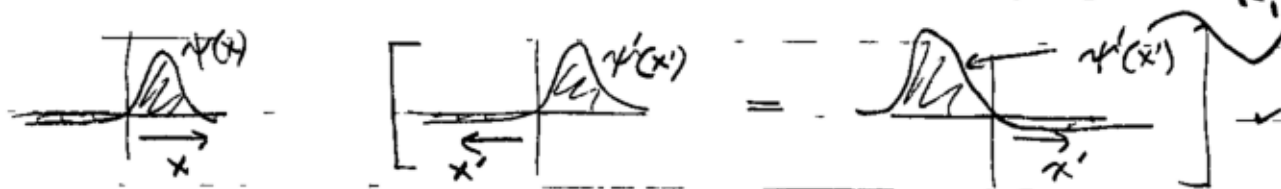
$$M_x \psi(x) = \psi'(x') \quad \checkmark \quad x' = -x \quad \checkmark$$



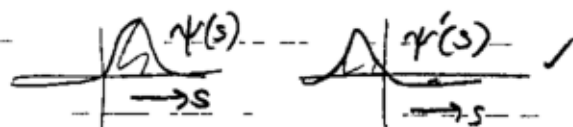
Most intuitively, I think of M_x as something which reflects the function ~~psi(x)~~ $\psi(x)$ "in the mirror" to give the new function $\phi(x)$ as shown. This is the active transformation where you hold tight to your reference system and let the "thing" do the moving.

$$\phi(x) = \overset{\text{act.}}{M_x} \psi(x) = \psi(-x) \quad \checkmark$$

The passive interpretation is different. In this case, you leave the "thing" fixed where it is but you hop into a different "system". The picture you then draw depends in whose frame you draw it. Drawn in the same frame as the start frame you get:



On the right I have drawn it as it would look from the new system (sort of body axes). If we go to some dummy variable s we have:



Clearly, $Y(s)$ and $Y'(s)$ are not the same function. $Y(1) = 3$ but $Y'(1) = -.3$.

We have shown that in this particular case the following is true:

$$\psi'(x') = \psi(-x) = \psi(x') \quad \text{No!} \quad \psi'(x') = \overset{\text{pass}}{M_x} \psi(x) [= \psi(x)] = \overset{\text{act}}{M_x} \psi(x) = \psi(-x)$$

The question that remains is this: WHY DO THE BOOKS PUT THAT PRIME ON Y??? The prime doesn't seem to make any difference.

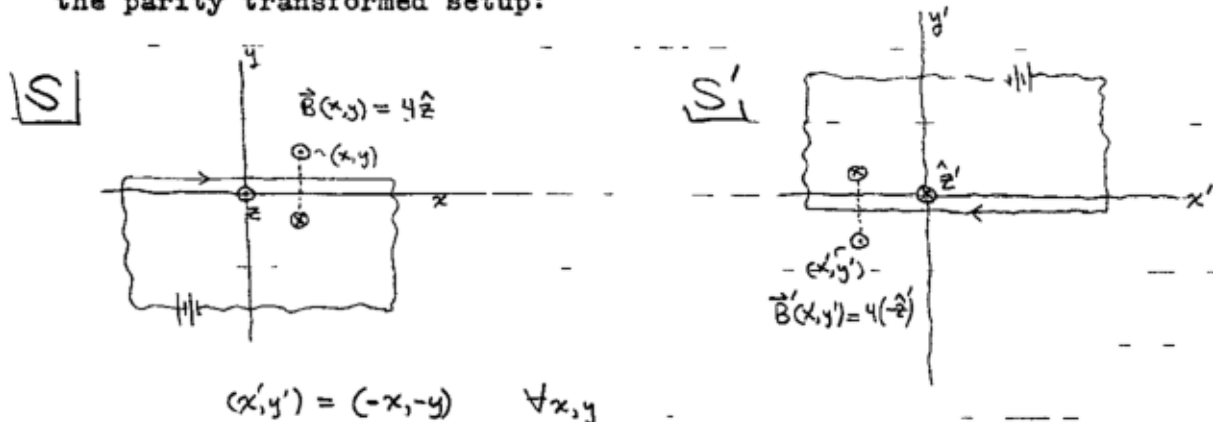
But consider the example on the next sheet:

↑
conhabitation
caused by assuming
 $M^p = M^a$

passive B+D type transform.

Illustration of the statement: "B transforms as a pseudovector"

Start with a current loop and a RHCS denoted (x,y,z) . Next to it, draw the parity transformed setup:



In the transformed frame we drew the loop first, then the current. All electron velocities changed sign, as did all coordinates. Then in the new frame I used the RH rule to find the direction of B at those corresponding points. (This RH rule is coordinate system independent).

After drawing in those B directions, we notice that the direction of B at the parity-related point is the same, which suggests that B is a pseudovector. Notice that the new system (x',y',z') is a left-handed one.

Now we can practice with the transformation statement. The statement that B is a pseudovector under parity is written:

$$\vec{B}'(x',t') = P \vec{B}(x,t) = + \vec{B}(x,t) \quad \text{where} \quad x' = -x, \quad t' = +t$$

To check this out for the one point we have picked out:

$$\begin{aligned} \vec{B}'(x',t') &= 4(-\hat{z}) \\ + \vec{B}(x,t) &= 4(\hat{z}) \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{These are equal since } \hat{z}' = -\hat{z}$$

Here, $B'(x',t')$ is the magnetic field observed in the frame S' at S' location x' . We must realize that B' and B are two completely different functions of two variables (x,y) here. For example, it

$$B(s,t) = 10 - 3s + 7st$$

$$\text{Then } B'(s,t) = 10 + 3s + 7st \quad \text{since } B'(s,t) = B(-s,-t)$$

In terms of the above pictures, let $(x,y) = (2,1)$. Then

$$\vec{B}(2,1) = 4\hat{z} \quad (\text{meaning of } \vec{B}(x,y) \text{ or } \vec{B}(x',y'))$$

$$\vec{B}'(-2,-1) = 4\hat{z} \quad (\text{meaning of } \vec{B}'(x',y') \text{ or } \vec{B}'(x,y))$$

$$\vec{B}'(2,1) = \text{unknown}$$

Notice that the parity statement does NOT say:

$$\vec{B}'(x',t') = P \vec{B}(x,t) = \vec{B}(-x,t)$$

erroneous!!



solution of the two-dimension harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}kr^2$$

$$\vec{p} = -i\hbar \vec{\nabla}$$

$$H = -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2}kr^2$$

Go into cylindrical coords: $\nabla^2 = \frac{1}{r} \partial_r (r \partial_r) + \frac{1}{r^2} \partial_\theta^2$

The SE in $d=2$ with $V(r)$ potential is $H\psi = E\psi$, by

$\psi = R(r) S(\theta)$ then

$$\left[-\frac{\hbar^2}{2m} \left(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 \right) + V(r) - E \right] R(r) S(\theta) = 0$$

$$\left[\left(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 \right) + \frac{2m}{\hbar^2} (E - V(r)) \right] RS = 0$$

$$\ddot{R}S + \frac{1}{r} \dot{R}S + \frac{1}{r^2} R\ddot{S} + \frac{2m}{\hbar^2} (E - V(r)) RS = 0$$

$$\ddot{R} + \frac{1}{r} \dot{R} + \frac{2m}{\hbar^2} (E - V) R = -\frac{1}{r^2} R \frac{\ddot{S}}{S}$$

$$\left\{ \frac{\ddot{R}}{R} + \frac{1}{r} \frac{\dot{R}}{R} + \frac{2m}{\hbar^2} (E - V(r)) \right\} = -\frac{1}{r^2} \frac{\ddot{S}}{S}$$

$$r^2 \ddot{R}/R + r \dot{R}/R + \frac{2mr^2}{\hbar^2} (E - V(r)) = -\ddot{S}/S \quad \text{ie } f(r) = g(\theta).$$

Both sides must be constants. Let it be k :

$$-\frac{\ddot{S}}{S} = k \Rightarrow \ddot{S} + kS = 0 \Rightarrow S(\theta) = A e^{i\sqrt{k}\theta} + B e^{-i\sqrt{k}\theta}$$

Try $S(\theta) = e^{i\sqrt{k}\theta}$. We know that $J_3 = (-i) (x \partial_y - y \partial_x) = (-i) (\partial_\theta)$

So $J_3 = L_3 = -i \partial_\theta$. Then want $L_3 \psi = l \psi = -i \partial_\theta e^{i\sqrt{k}\theta}$

So $(-i)(i\sqrt{k}) = l \Rightarrow k = l^2$, not $l(l+1)$. So:

$$S(\theta) = e^{i\sqrt{k}\theta} \quad k = l^2 \quad \text{Thus: } l = 0, \pm 1, \pm 2, \text{ etc}$$

$$r^2 \frac{\ddot{R}}{R} + r \frac{\dot{R}}{R} + \frac{2mr^2}{\hbar^2} [E - V(r)] = l^2$$

Radial Equation is now:

$$\left\{ r^2 \frac{d^2}{dr^2} + r \frac{d}{dr} + \frac{2m}{\hbar^2} r^2 \left[E - V(r) - \frac{\hbar^2}{2m} \frac{l^2}{r^2} \right] \right\} R = 0.$$

or: $\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{2m}{\hbar^2} \left(E - V(r) - \frac{\hbar^2}{2m} \frac{l^2}{r^2} \right) \right] R = 0$

$\uparrow$ notice $\frac{1}{r}$ $\underbrace{\left(E - V(r) - \frac{\hbar^2}{2m} \frac{l^2}{r^2} \right)}$ effective potential?

Now set $\hbar^2/2m = a$. Then:

$$\left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \right) + \frac{1}{a} \left(E - V(r) - a \frac{l^2}{r^2} \right) \right] R = 0.$$

This is the radial equation for a particle in central potential $V(r)$ in two dimensions.

How do you solve it? Try $R = u/r$. Then

$$\begin{aligned} R &= \tilde{r}' u & \dot{R} &= -\tilde{r}^2 \ddot{u} + \tilde{r}' \dot{u} \\ \ddot{R} &= 2\tilde{r}^3 \ddot{u} - \tilde{r}^2 \ddot{u} + \tilde{r}' \ddot{u} - \tilde{r}^2 \ddot{u} &= 2\tilde{r}^3 \ddot{u} - 2\tilde{r}^2 \ddot{u} + \tilde{r}' \ddot{u} \end{aligned}$$

So $\ddot{R} + \frac{1}{r} \dot{R} + \frac{1}{a} \left(E - V - \frac{a l^2}{r} \right) R = 0.$

$$\Rightarrow 2\tilde{r}^3 \ddot{u} - 2\tilde{r}^2 \ddot{u} + \tilde{r}' \ddot{u} + \tilde{r}^2 \ddot{u} - \tilde{r}^3 \ddot{u} + \frac{1}{a} \left(E - V - \frac{a l^2}{r} \right) \frac{u}{r} = 0.$$

Then, the is does not cancel, so $R = u/r$ is not good.

Try $\sim$ with no potential, this is Bessel's Equation $r = z$.

the 2-D H.O., version 4

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2 = -\frac{\hbar^2}{2m} \nabla^2 + V(r) \quad \nabla^2 = \partial_r^2 + \frac{1}{r} \partial_r - \frac{l^2}{r^2}$$

$$\text{So: } \left[-\frac{\hbar^2}{2m} \left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{l^2}{r^2} \right) + \frac{1}{2} m \omega^2 r^2 \right] \psi(r) = E \psi(r)$$

$$\text{or, } \left[\partial_r^2 + \frac{1}{r} \partial_r - \frac{l^2}{r^2} + \frac{(-) m \omega^2 r^2}{\hbar^2} \right] \psi = -\frac{2mE}{\hbar^2} \psi$$

$$\text{or } \left(\partial_r^2 + \frac{1}{r} \partial_r + \left[\left(\frac{2mE}{\hbar^2} \right) - \left(\frac{m\omega}{\hbar^2} \right) r^2 - \frac{l^2}{r^2} \right] \right) \psi = 0$$

So, here are two versions: define $\epsilon = \frac{2mE}{\hbar^2}$ $a = \frac{m\omega}{\hbar}$

Then:

$$\left[\partial_r^2 + \frac{1}{r} \partial_r + \left(\epsilon - a^2 r^2 - \frac{l^2}{r^2} \right) \right] \psi(r) = 0$$

$$\psi'' + \frac{1}{r} \psi' + (\epsilon - a^2 r^2 - \frac{l^2}{r^2}) \psi = 0$$

$$\epsilon = 2mE/\hbar^2 \quad a = m\omega/\hbar$$

later i show solution to be $\psi = e^{-ar^2/2} r^l \ln(ar^2)$

where $E = \hbar\omega(l+2n+1)$

For large r , we have $\psi'' - a^2 r^2 \psi \approx 0 \Rightarrow \psi \approx e^{-ar^2/2}$

$$\psi' \approx -ar \psi$$

$$\psi'' \approx (-ar)' \psi = a^2 r \psi$$

So we should try

$$\psi = e^{-ar^2/2} f$$

Find an equation for f . Let $e^{-ar^2/2} = h$

$$h = e^{-ar^2/2}$$

$$h' = (-ar)h$$

$$h'' = (-ar)'h - a h^2 = (a^2 r^2 - a)h' = ah(a^2 r^2 - 1) = a(a^2 r^2 - 1)h$$

$$\psi = hf \quad \psi' = hf' + fh' \quad \psi'' = hf'' + 2f'h' + fh''$$

$$\psi' = hf' + f(-ar)h = (f' - arf)h$$

$$\psi'' = hf'' + 2f'(-ar)h + f a(a^2 r^2 - 1)h = (f'' - 2arf' + a(a^2 r^2 - 1)f)h$$

So we arrive at:

$\psi = h f$ ✓	$h = e^{-ar^2/2}$ ✓
$h = h$ ✓	
$h' = -arh$ ✓	
$h'' = a(ar^2 - 1)h$ ✓	
$\psi = fh$ ✓	
$\psi' = (f' - arf)h$ ✓	
$\psi'' = (f'' - 2arf' + a[ar^2 - 1]f)h$ ✓	

Insert into our ODE:

$$(f'' - 2arf' + a[ar^2 - 1]f) + \frac{1}{r}(f' - arf) + (\epsilon - ar^2 - l^2 r^2)f = 0$$

$$f'' + f' \left[\frac{1}{r} - 2ar \right] + f \left[\cancel{a^2 r^2} - a - a + \epsilon - \cancel{a^2 r^2} - l^2 r^2 \right] = 0$$

$$f'' + (\tilde{r}' - 2ar)f' + (\epsilon - 2a - l^2 \tilde{r}^2)f = 0$$

So f must satisfy:

$$f'' + (\tilde{r}' - 2ar)f' + (\epsilon - 2a - l^2 \tilde{r}^2)f = 0$$

find equation solved by $f(x) = x^a L_n(\beta x^\gamma)$.

This is B = NBS = GR Laguerre, so:

$$x L_n'' + (x+1) L_n' + n L_n = 0$$

Let $x^a = h$ $h' = a x^{a-1} = a x^{a-1}$
 $h'' = a(a-1) x^{a-2} = a(a-1) x^{a-2} h$.

Let $y = \beta x^\gamma$ $y' = \beta \gamma x^{\gamma-1} = \beta \gamma x^{\gamma-1} y = \gamma x^{\gamma-1} y$

$y'' = \beta \gamma (\gamma-1) x^{\gamma-2} = \gamma (\gamma-1) x^{\gamma-2} y = \gamma (\gamma-1) x^{\gamma-2} y$

So: let $L = L_n$ then:

$f = h L(y)$ $f' = h' L + h L' y$

$f'' = h'' L + h' L' y' + h' L' y' + h y' L' y' + h L' y''$

So $f' = a x^{a-1} h L + h L' \gamma x^{\gamma-1} y = h x^{a-1} (a L + \gamma y L')$

$f'' = a(a-1) x^{a-2} h L + 2 a x^{a-1} h \gamma x^{\gamma-1} y L' + h \gamma^2 x^{a-2} y^2 L'' + h L' \gamma (\gamma-1) x^{\gamma-2} y$

$= h x^{a-2} [a(a-1) L + 2 a \gamma y L' + \gamma^2 y^2 L'' + \gamma (\gamma-1) y L']$

$= h x^{a-2} [\gamma^2 y^2 L'' + (2 a \gamma y + \gamma^2 y - \gamma y) L' + a(a-1) L]$

Sum:

$h = x^a$ ✓	$y = \beta x^\gamma$ ✓
$h' = a x^{a-1}$ ✓	$y' = \gamma x^{\gamma-1} y$ ✓
$h'' = a(a-1) x^{a-2} h$ ✓	$y'' = \gamma (\gamma-1) x^{\gamma-2} y$ ✓
$f = h L(y)$ ✓	
$f' = h x^{a-1} (a L + \gamma y L')$ ✓	
$f'' = h x^{a-2} (\gamma^2 y^2 L'' + [2 a \gamma y + \gamma^2 y - \gamma y] L' + a(a-1) L)$ ✓	
$f = h L(y)$	

Now solve:

$$h^2 f = L$$

$$x h^4 f' = \delta_y L' + a L$$

$$x^2 h^4 f'' = \delta_y^2 L'' + (2a + \delta - 1) \delta_y L' + a(a-1)L$$

$$\delta_y L' = x h^4 f' - a h^4 f = h^4 (x f' - a f)$$

$$\Rightarrow L' = \frac{1}{\delta_y h} (x f' - a f)$$

$$\text{Then } h \delta_y^2 L'' = x^2 \cancel{h^4} f'' - (2a + \delta - 1) \cancel{\delta_y} \frac{1}{\cancel{\delta_y} h} (x f' - a f) - a(a-1) \cancel{h^4} f$$

$$L'' = \frac{1}{h \delta_y^2} [x^2 f'' - (2a + \delta - 1)(x f' - a f) - a(a-1)f]$$

$$[] = x^2 f'' - x(2a + \delta - 1)f' + \underbrace{[a(2a + \delta - 1) - a(a-1)]}_{2a^2 + \delta a - a^2 + a} f$$

$$\frac{2a^2 + \delta a - a^2 + a}{a(a + \delta)}$$

$$L'' = \frac{1}{h \delta_y^2} [x^2 f'' - x(2a + \delta - 1)f' + a(a + \delta)f]$$

So in sum:

$$L = f/h \checkmark$$

$$L' = (x f' - a f)/h \delta_y \checkmark$$

$$L'' = [x^2 f'' - x(2a + \delta - 1)f' + a(a + \delta)f]/h \delta_y^2 \checkmark$$

$$(h \delta_y^2 L'') + x^2 \delta_y (\alpha + 1 - x) (h \delta_y L') + x^2 \delta_y^2 w(hL) = 0$$

$$yL'' + (\alpha + 1 - y)L' + \alpha L = 0$$

$$\Rightarrow (h\delta^2 y^2 L'') + \delta(\alpha + 1 - y)(h\delta y L') + \delta^2 y \alpha (hL) = 0 \checkmark$$

$$\text{So } x^2 f'' - x(2\alpha + \delta - 1)f' + \alpha(\alpha + \delta)f$$

$$+ \delta(\alpha + 1 - y)(xf' - \alpha f) + \delta^2 y \alpha f = 0 \checkmark$$

$$\Rightarrow x^2 f'' + xf'(-2\alpha + \delta - 1) + f(\alpha^2 - \alpha\delta + \alpha\delta y + \delta^2 y \alpha) = 0 \checkmark$$

$$\Rightarrow \left[f'' + \left[(\delta\alpha - 2\alpha) \bar{x}^{-1} - \delta\beta \bar{x}^{\delta-1} \right] f' + \left[\alpha(\alpha - \delta\alpha) \bar{x}^{-2} + \delta\beta(\alpha + n\delta) \bar{x}^{\delta-2} \right] f \right] = 0$$

solved by $f(x) = X^\alpha L_n(\beta X^\delta)$

Once more:

$$f'' + p(x)f' + q(x)f = 0$$

$$p(x) = (1 + \delta\alpha - 2\alpha) \bar{x}^{-1} - \delta\beta \bar{x}^{\delta-1}$$

$$q(x) = \alpha(\alpha - \delta\alpha) \bar{x}^{-2} + \delta\beta(\alpha + n\delta) \bar{x}^{\delta-2}$$

$$\text{solution: } f(x) = X^\alpha L_n(\beta X^\delta)$$

where L = that of $B = GR = NBS$

fit the 2D HO to Laguerre

① The HO has:

$$p(x) = \tilde{x}' - 2Ax$$

$$q(x) = (\epsilon - 2A) - 2\tilde{x}^2$$

where $A = m\omega/\hbar$ $\epsilon = 2mE/\hbar^2$

$$e^{-Ar^2/2} \propto x \cdot L_n(\beta x^2)$$

② My Laguerre solution has

$$p(x) = (1 + \gamma\alpha - 2\alpha)\tilde{x}' - \gamma\beta\tilde{x}^{\gamma-1}$$

$$q(x) = \gamma\beta(\alpha + n\gamma)\tilde{x}^{\gamma-2} + \alpha(\alpha - \gamma\alpha)\tilde{x}^{-2}$$

Try $\gamma=2$ and $\beta=A$ then

$$p(x) = (1 + 2\alpha - 2\alpha)\tilde{x}' - 2Ax$$

$$q(x) = 2A(\alpha + n2) + \alpha(\alpha - 2\alpha)\tilde{x}^{-2}$$

Now want $\alpha = a$, so

$$p(x) = \tilde{x}' - 2Ax$$

$$q(x) = 2A(\alpha + 2n) - a^2\tilde{x}^{-2}$$

Next, set $a=1$; then $2A(\alpha + 2n)$

$$p(x) = \tilde{x}' - 2Ax$$

$$q(x) = 2A(1 + 2n) - 2\tilde{x}^{-2}$$

$$\frac{2mE}{\hbar^2} = \frac{2m\omega}{\hbar} (1 + 2n + 1)$$

Finally, $2A(1 + 2n) = \epsilon - 2A \Rightarrow$

$$1 + 2n = \epsilon/2A - 1 \Rightarrow \epsilon = 2A(1 + 2n + 1)$$

$n = (\epsilon/2A - 1 - 1)/2$	$\beta = A$	$a = 1$
$\alpha = 1$	$\gamma = 2$	

$$E = \hbar\omega(1 + 2n + 1)$$

So, given $f'' + (r' - 2ar)f' + (\epsilon - 2a - l^2 r^{-2})f = 0$

$$\epsilon = 2mE/\hbar^2 \quad a = m\omega/\hbar$$

We now know the solution to be:

$$f(r) = r^l L_n^l(ar^2)$$

where n obtains from $\epsilon = \hbar\omega(l+2n+1)$.

Therefore, the full HO radial is:

$$\psi(r) = N \cdot e^{-ar^2/2} r^l L_n^l(ar^2)$$

where $a = \frac{m\omega}{\hbar} \quad n = \frac{1}{2} \left(\frac{\epsilon}{\hbar\omega} - 1 - l \right)$

③ Normalizable states labeled by $n = 0, 1, 2, \dots$
also $l = 0, 1, 2, \dots$

④ Check orthogonality?

$$(\psi_n^l, \psi_{n'}^l) \propto \int r dr \cdot e^{-ar^2} r^{2l} L_n^l(ar^2) L_{n'}^l(ar^2)$$

$$x = ar^2 \Rightarrow \propto \int dx \cdot e^{-x} x^l L_n^l(x) L_{n'}^l(x)$$

$$= \propto \delta_{n,n'} \quad \checkmark$$

power series solution of 2-D H.O.

$$f = \sum_{n=0}^{\infty} a_n r^n$$

$$f' = \sum_{n=0}^{\infty} a_n n r^{n-1} = \sum_{n \rightarrow n+1} a_{n+1} (n+1) r^n$$

$$f'' = \sum_{n=0}^{\infty} a_{n+1} (n+1) n r^{n-1} = \sum_{n \rightarrow n+2} a_{n+2} (n+2)(n+1) r^n$$

So we require:

$$\sum \left[a_{n+2} (n+2)(n+1) r^n + (r^2 - 2ar) a_{n+1} (n+1) r^n + (\epsilon - 2a) a_n r^n - l^2 r^{-2} a_n r^n \right] = 0$$

$a_{n+2} (n+2)(n+1) r^n$	$n \rightarrow n$	$a_{n+2} (n+2)(n+1) r^n$
$a_{n+1} (n+1) r^{n-1}$	$n \rightarrow n+1$	$a_{n+1} (n+1) r^n$
$-2a a_{n+1} (n+1) r^{n+1}$	$n \rightarrow n-1$	$-2a a_n (n) r^n$
$(\epsilon - 2a) a_n r^n$	$n \rightarrow n$	$(\epsilon - 2a) a_n r^n$
$-l^2 a_n r^{n-2}$	$n \rightarrow n+2$	$-l^2 a_{n+2} r^n$

$$\text{So: } a_{n+2} [(n+2)(n+1) - l^2] + a_n [-2an + \epsilon - 2a] = 0$$

$$a_{n+2} [(n+2)^2 - l^2] + a_n [-2an + \epsilon - 2a] = 0$$

This seems reasonably simple

$$\left| \frac{a_{n+2}}{a_n} = \frac{2a(n+1) - \epsilon}{(n+2)^2 - l^2} \right|$$

Notice the truncation condition: $2a(n+1) = \epsilon$

$$\text{h} \frac{2a(n+1)}{h} = \frac{2m\epsilon}{h} \Rightarrow \underline{E = \hbar \omega(n+1)} \quad \checkmark$$

given recursion, how do you get a function?

$$a_2 = \left(\frac{2a - \epsilon}{2^2 - l^2} \right) a_0$$

$$a_4 = \left(\frac{2a \cdot 3 - \epsilon}{4^2 - l^2} \right) \left(\frac{2a - \epsilon}{2^2 - l^2} \right) a_0$$

$$a_6 = \left(\frac{2a \cdot 5 - \epsilon}{6^2 - l^2} \right) \left(\frac{2a \cdot 3 - \epsilon}{4^2 - l^2} \right) \left(\frac{2a - \epsilon}{2^2 - l^2} \right) a_0$$

$$a_{2n} = \prod_{m=1}^n \left(\frac{2a \cdot (2m-1) - \epsilon}{(2m)^2 - l^2} \right) a_0$$

So: The even parity solutions are:

$$f(r) = \sum_{n=0}^{\infty} a_{2n} r^{2n} = \sum_{n=0}^{\infty} \prod_{m=1}^n \left(\frac{2a \cdot (2m-1) - \epsilon}{(2m)^2 - l^2} \right) a_0 r^{2n}$$

What is:

$$\prod_{m=1}^n \left[\frac{2a \cdot (2m-1) - \epsilon}{4m^2 - l^2} \right]$$

What is denom:

$$(2^2 - l^2)(4^2 - l^2)(6^2 - l^2) \dots (2n^2 - l^2) =$$

Too hard.

The HO in 3D.

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2 = \frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} m \omega^2 r^2$$

but now, $\nabla_r^2 = \frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{l(l+1)}{r^2}$ /p 27 Schiff

$$\nabla_r^2 = \partial_r^2 + \frac{2}{r} \partial_r + \frac{l(l+1)}{r^2}$$

so equation is

$$\psi'' + \frac{2}{r} \psi' + (\epsilon - a^2 r^2 - l(l+1) r^{-2}) \psi = 0$$

Very similar. Again, try $\psi = e^{-ar^2/2} f$. Again

$$\psi = h f$$

$$\psi' = (f' - arf) h$$

$$\psi'' = (f'' - 2arf' + a(ar^2 - 1)f) h$$

So $f'' - 2arf' + a(ar^2 - 1)f + \frac{2}{r} (f' - arf) - l(l+1) r^{-2} f = 0$
 $+ (\epsilon - a^2 r^2) f$

And: $\left[f'' + \left(\frac{2}{r} - 2ar \right) f' + \left(\epsilon - 3a - l(l+1) r^{-2} \right) f = 0 \right]$

There are three slight changes

② Now solve: $p(x) = 2\tilde{x}' - 2ax$
 $q(x) = \epsilon - 3a - l(l+1) \tilde{x}^{-2}$

Set $\delta = 2, \beta = a = A$ as before

Last: $2a(l+2n) = \epsilon - 3a$

$$p(x) = (1 + 2a - 2a) \tilde{x}' - 2ax$$

$$l + 2n = \epsilon/2a - 3/2$$

$$q(x) = a(a - 2a) \tilde{x}^{-2} + 2a(a + 2n)$$

$$= \epsilon/2a - 3/2$$

Now set $2a - 2a = 1 \Rightarrow a = a + \frac{1}{2}$

Set $a(a - 2a) = -l(l+1) = a(a - 2a - 1) = a(-a - 1) = -a(a+1)$

$$\Rightarrow a = l \quad \alpha = l + \frac{1}{2}$$

$$\epsilon = \hbar \omega (l + 2n + 3/2)$$

To summarize:

$$n = \frac{1}{2} \left(\epsilon/2A - \frac{3}{2} - l \right)$$

$$\beta = A$$

$$a = l$$

$$l = l + \frac{1}{2}$$

$$\gamma = 2$$

Thus, solution is

$$f(x) = x^l L_n^{l+\frac{1}{2}}(Ax^2)$$

③ Therefore, radial wavefunction for 3D HO is

$$\psi(r) = C e^{-ar^2/2} r^{2l} L_n^{l+\frac{1}{2}}(ar^2)$$

$$a \equiv \frac{m\omega}{\hbar}$$

$$n = \frac{1}{2} \left(\frac{E}{\hbar\omega} - \frac{3}{2} - l \right)$$

④ Check orthog:

$$\int r^2 dr e^{-ar^2/2} r^{2l} L_n^{l+\frac{1}{2}}(ar^2) L_{n'}^{l+\frac{1}{2}}(ar^2)$$

$$x \sim ar^2$$

$$dx \equiv 2ar dr$$

$$r^2 dr \equiv x^{\frac{1}{2}} dx$$

$$\sim \int x^{\frac{1}{2}} dx e^{-x} x^{2l} L_n^{l+\frac{1}{2}}(x)$$

$$\sim \int dx e^{-x} x^{l+\frac{1}{2}} L_n^{l+\frac{1}{2}}(x) \quad \text{note match!}$$

⑤ Ground state: $L_0^{l+\frac{1}{2}}(ar^2) = 1$

Quantum Mechanics.

In quantum mechanics, certain quantities are quantized:

1. ~~(a)~~ In the analysis of a "particle in a box", (infinitely high potential walls), what causes the energy to be quantized?
2. ~~(b)~~ Consider the more complicated case of the hydrogen atom, ~~or any "central potential" problem.~~ As you know, the wavefunction may be factorized into the form $\Psi(r, \theta, \phi) = f_{E, l}(r) Y_{l, m}(\theta, \phi)$ where Y is a spherical harmonic, and f is the radial wavefunction. The angular momentum l is quantized so that $l = 0, 1, 2, \dots$ (in units of $\hbar$), and the energy is also quantized, the eigenvalues being given by:

$$E_n = -\frac{e^2}{2 a_0 n^2} \quad \text{where } n = 1, 2, 3, \dots \quad \text{and} \quad a_0 = \frac{\hbar^2}{m e^2}$$

- (2) In this problem, what is it which causes the ~~quantized~~ angular momentum and energy to be quantized? (The following two parts of this problem are hints for this part.)
- (b) The radial equation for the hydrogen atom contains the energy E . For any value (but li of E this differential equation certainly has solutions. If E is not an eigenvalue, what could you say about the radial wavefunction $f(r)$?
- (c) When $m=0$, $Y_{l, m}(\theta, \phi) = P_l(\cos\theta)$, a Legendre polynomial. It is possible to extend the meaning of the function $P_l(z)$ to all real values (in fact complex values) of the parameter l . If l is a real number but is not an integer, what would you guess would be true about the following integral: (and why.)

$$I = \int_{-1}^1 [P_l(z)]^2 dz$$

- (3) ~~(a)~~ Consider the earth and sun as a ~~quantum~~ quantum mechanical system like the hydrogen atom.

Is the energy of the earth quantized? Is the angular momentum of the earth quantized?

~~(a)~~ Compute the "Bohr radius" for the earth-sun problem

(in cgs units)

Please make the following computations only to within 10 orders of magnitude: ~~(in cgs units)~~

(a) Compute the "Bohr radius" of the earth-sun problem.

(b) What is the energy of the ground state of the earth-sun system (the "Rhydberg")

(c) What in fact is the energy of the earth (of course ignoring all other celestial objects and setting $E=0$ for infinite separation).

(d) What is the principal quantum number ~~fix the xxxxxx~~ n of the earth?

(e) Comment on the role of qm in celestial mechanics.

Relativity

Relativistic Mechanics and Massless Particles

① First, deal with massive particle. We define as usual,

$$(ds)^2 = -\eta_{\alpha\beta} dx^\alpha dx^\beta \Rightarrow \frac{dx}{ds} \cdot \frac{dx}{ds} = -1 \quad (1.1)$$

Then define:

$$p^\mu \equiv m \frac{dx^\mu}{ds} = m\gamma \frac{dx^\mu}{dt} \quad (1.2)$$

Newton's 2nd Law is now: $f^\mu = \frac{dp^\mu}{ds} = m \frac{d^2 x^\mu}{ds^2}$

② Now watch what things do as $m \rightarrow 0$. From the def. of p^μ and (1.1) we know that

$$m^2 + |\vec{p}|^2 = E^2 \quad (2.1)$$

From (1.2) we know: $\vec{p} = m\gamma \vec{v}$, $E = m\gamma$, $\vec{v} = \vec{p}/E$.

Also from (1.1) we know: (2.2) (2.3)

$$\gamma ds = dt. \quad (2.4)$$

③ Now, for fixed E and for some fixed dt , what happens as $m \rightarrow 0$? From (2.3), $\gamma \rightarrow \infty$. From (2.4), $ds \rightarrow 0$. From def γ , $|\vec{v}| \rightarrow 1$ and $|\vec{p}| \rightarrow E$. Thus, $m\gamma$ holds at E even as $m \rightarrow 0$.

④ We want laws without ds because $ds = 0$. So, replace (1.2) generally with:

$$p^\mu \equiv E \frac{dx^\mu}{dt}$$

And replace nt law with:

$$m f^\mu = m\gamma \frac{dp^\mu}{dt} = E \frac{dp^\mu}{dt}$$

As $m \rightarrow 0$ for fixed E , $m f^\mu \rightarrow 0$ and we get

$$0 = E \frac{dp^\mu}{dt} \Rightarrow \frac{dp^\mu}{dt} = 0 \Rightarrow \frac{d\vec{p}}{dt} = \frac{dE}{dt} = 0.$$

No finite force can act on a massless particle? $\rightarrow$

⑤ So in general we have $mf^\mu = E \frac{dp^\mu}{dt}$. If f^μ is finite, then as $m \rightarrow 0$,

$$\frac{dp^\mu}{dt} = 0$$

This same equation obtains for a massive with no force.

⑥ Now use p^μ from ④:

$$0 = \frac{d}{dt}(p^\mu) = \frac{d}{dt}\left(E \frac{dx^\mu}{dt}\right) = E \frac{d^2 x^\mu}{dt^2} \quad \text{since } \frac{dE}{dt} = 0$$

Therefore,

$$\frac{d^2 x^\mu}{dt^2} = 0$$

in two cases:

- 1) massless
- 2) massive w no force.

⑦ Actually, the definition $p^\mu = E \frac{dx^\mu}{dt}$ is correct for massless, so still OK on claim of ④? Well, when you write

$$mf^\mu = E \frac{dp^\mu}{dt}$$

There is no meaning to f^μ because $\nexists$ no rest frame.

The accelerating rocket business.

- ① Here is the setup: one frame stays put on this piece of paper. It is the inertial frame S . The other inertial frame S' travels at velocity v and thus coincides (in velocity) with the rocket which is also going v . So in frame S' , the rocket is at rest, but is instantaneously accelerating by some constant amount which we call g .

- ② We write Newton's law as:

$$m \frac{d^2 x^\mu}{d\tau^2} = f^\mu.$$

In frame S' this reads: $\frac{d^2 x'^\mu}{d\tau^2} = a'^\mu = f'^\mu/m$

where $f'^\mu = (0, 0, 0, g)$. We know that $a'^0 = 0$ because $dt' = d\tau$. That is, comoving time is always proper time.

- ③ We also know that, in general $\frac{dx^\mu}{d\tau} = v^\mu$, and in particular,

$$\frac{dx'^\mu}{d\tau} = 0^\mu$$

- ④ Now what do these same equations look like back in frame S ? These frames are related by γ, β, σ, v and any other thing you choose. Thus, right off the bat we can say.

$$\frac{d^2 x^\mu}{d\tau^2} = g^\mu = (\pm g \sinh \sigma, 0, 0, g \cosh \sigma)$$

- ⑤ But all we care about are these two equations:

$$\frac{d^2 z}{d\tau^2} = g \cosh \sigma$$

$$\frac{dz}{d\tau} = \frac{dz}{dt} \frac{dt}{d\tau} = \gamma \cdot \dot{z} = \beta \dot{z} = \sinh \sigma$$

→ over.

⑥ Of course we all know that $dt = \gamma d\tau = \cosh d\tau$. Now we want to solve these equations:

$$\left. \begin{aligned} (a) \quad \frac{dz}{d\tau} &= g \cosh \\ (b) \quad \frac{dz}{d\tau} &= \sinh \end{aligned} \right\} \text{ where } \sigma = \sigma(\tau) = \text{as yet unknown fcn.}$$

(a)+(b) says: $\cosh \cdot \frac{d\sigma}{d\tau} = g \cdot \sinh$

$$\Rightarrow \frac{d\sigma}{d\tau} = g \Rightarrow d\sigma = g d\tau$$

If we choose $\tau=0$ when $\sigma=0$, [well, not a choice? OK, S' is "same" as S when $\tau=0$, so rocket started accel from rest in S at time $\tau=0$], we get

$\sigma = g\tau$ $\leftarrow$ This is the full solution to the problem.

⑦ Duplications?

(a) $\tilde{v}(s) = \gamma \cosh = \gamma \cosh(g\tau) \rightarrow 1$ note: $\tilde{v} = \frac{dx}{dt}$ here.

(b) $\gamma(\tau) = \cosh = \cosh(g\tau) \rightarrow \infty$ $\tilde{v} = \frac{dx}{d\tau}$

(c) $dt = \cosh \cdot d\tau = \cosh(g\tau) d\tau$
 $\Rightarrow gt = \sinh(g\tau) = \gamma \beta$ $\tilde{v} = \tilde{v} \cdot \gamma = \tilde{v} \cosh(g\tau)$

This relates the two times $\Rightarrow \tilde{v} = \sinh(g\tau)$

(d) $\frac{dx}{d\tau} = \tilde{v}(\tau) \Rightarrow dx = \sinh(g\tau) d\tau$
 $\Rightarrow x = \frac{1}{g} \cosh(g\tau) = x(\tau)$

~~$x(\tau) = \frac{1}{g} [\cosh(g\tau) - 1]$~~ $\downarrow$ $x(\tau) = \frac{1}{g} [\cosh(g\tau) - 1]$
 ~~$= \frac{1}{g} [1 + (gt)^2]$~~

note: ~~sp~~ must be there so $x(t) \rightarrow \frac{1}{2} gt^2$ for small t .

(accel rocket)

(e) We now have $v(z)$ and $x(z)$. What are $v(t)$ and $x(t)$?

$$v(t) = v(z(t)) = \tanh(gz(t)) = \tanh\{\text{sh}^{-1}(gt)\} = \frac{gt}{\sqrt{(gt)^2 + 1}}$$

$$x(t) = \frac{1}{g} [\cosh\{\text{sh}^{-1}(gt)\} - 1] = \frac{1}{g} [\sqrt{(gt)^2 + 1} - 1]$$

(f) Consistency check:

$$\frac{dx(t)}{dt} = \frac{1}{g} \cdot \frac{1}{\sqrt{\dots}} \cdot gt \cdot g = v(t) \checkmark$$

(g) What acceleration does S see?

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = g \cdot \frac{1}{[1+(gt)^2]^{3/2}} = g \frac{1}{(\cosh)^3} \\ = g/\gamma^3$$

It seems to have a decreasing accel, of course.

⑧ First problem: how long (t and z) does it take to go some distance x at one g ? Go to 9 at onceThe equations with c are:

$$x(z) = \frac{c^2}{g} [\cosh(gz/c) - 1] \quad t = \frac{c}{g} \text{sh}(gz/c)$$

Define a new unit of distance — the light-second or lisee.
Then $c = 1 \text{ lisee/sec}$, so

$$x = \frac{1}{g} [\cosh(gz) - 1] \text{ and } t = \frac{1}{g} \text{sh}(gz)$$

$$\text{Here, } g = \frac{9.8 \text{ m}}{\text{sec}} \left[\frac{\text{lisee}}{3 \times 10^8 \text{ m}} \right] = 3.27 \times 10^{-8} \text{ lisee/sec}$$

This is seen to be a small accel in the big picture.
Now

$$1 \text{ yr} = 3.15 \times 10^8 \text{ sec}$$

⑨ Best system of units is ly and yr for x, t

Then

$$1 \text{ ly} = a \cdot b \text{ m}$$

$$1 \text{ yr} = b \text{ sec}$$

$$a = 3 \times 10^8$$

$$b = 3.15 \times 10^7$$

$$\frac{b}{a} = .105$$

$$\begin{aligned} \text{Then } g &= \frac{9.8 \text{ m}}{\text{sec}^2} \left[\frac{1 \text{ ly}}{ab \text{ m}} \right] \left[\frac{b \text{ sec}}{1 \text{ yr}} \right]^2 = 9.8 \frac{b}{a} \text{ ly/yr}^2 \\ &= 1.03 \text{ ly/yr}^2 \end{aligned}$$

In this system, g is very close to unity, and $c = 1$.

⑩ Then we have:

$$\begin{aligned} gt &= \text{sh}(gs) \\ gx &= [\text{ch}(gs) - 1] \end{aligned}$$

Assume $g = 1$ exactly. Then:

$$t = \text{sh}(s)$$

$$x = \text{ch}(s) - 1$$

(a) Table:

s	t	x
1	1.17 1.17	1.54
2	3.6	3.8
3	10.0	10.1

⑪ So, here is "the trip". One guy accelerates out at g for some comoving time $3/4$. Then he decelerates + stops. Then he returns. The equations do not care if he accelerates or decelerates. So if twin B makes a trip whose quarter leg is 3 years, he will go 20 ly away + return in 12 years age. Twin A will be 40 years older.

⑫ Traveller must go time $gs \gg 1$ to get the relativistic effect.

⑬ Trip to galactic center at $1g$? Quarter leg = 15,000 ly.

$$\begin{aligned} x+1 &= \text{ch}(s) \Rightarrow s = \text{ch}^{-1}[x+1] \approx \text{ch}^{-1}(x) = \ln[x + \sqrt{x^2 + 1}] \\ &\approx \ln[2x] \approx \ln[30,000] \approx 10.3 \text{ yrs} \end{aligned}$$

So quarter leg = 10.3 years = s . Meanwhile, $t = \text{sh}(s) \approx \frac{1}{2} e^s$

$t \approx 15,016$ years.

(accel. rocket)

3.

- ⑭ So, the round trip to our galactic center would take about 40 years comoving time. The earth would age about 60,000 years. There is simply no way to beat this 60,000 yr. (mine)

- ⑮ Andromeda? Nearest galaxy, 2×10^6 ly. Sirius (brightest star) at 8.6 ly. Nearest star at 4.3 ly.

Andromeda at 1g is: $3 = \text{ch}^{-1}(x+1) \approx \ln[2x]$

Well, suppose...

$$3_{1/4} = \ln[2x_{1/2}] = \ln(x_{\text{tot}}).$$

So $3_{\text{tot}} = 4 \ln(x_{\text{tot}})$. Let $x_{\text{tot}} = a \times 10^n$ ly.
Then

$$3_{1/4} = \ln[a \cdot 10^n] = \ln a + 2.3n$$

But $\ln a$ is between 0, 2.3, so

$$3_{1/4} = \ln(\underbrace{x \cdot x}_a) + 2.3n.$$

So, For Andromeda, $a = 2, n = 6 \Rightarrow 3_{1/4} = .69 + 13.8 = 14.5$ yrs.
Of course $t_{1/4} \geq 10^6$ years.

- ⑯ Practical trip to Alpha Centauri? 4.3 ly.

$$3 = \text{ch}^{-1}[x+1] = \text{ch}^{-1}[5.3] = 2.36 \times$$

$$3_{1/4} = \text{ch}^{-1}[x_{1/2}+1] = \text{ch}^{-1}(3.2) = 1.9 \text{ yrs.}$$

$$t_{1/4} = \text{sh}(3_{1/4}) = \text{sh}(1.9) = 3.26$$

Round trip

$$\begin{array}{l} 3 = 2.5 \text{ years} \\ t = ? \\ t = 13.0 \text{ years.} \end{array}$$

- ⑰ Twin paradox? Since rocket twin's rest frame is non-inertial, rules do not apply, no paradox.

Electromagnetism and Special Relativity

① The general $F=ma$ law is

$$m \frac{d^2 x^\mu}{d\tau^2} = f^\mu = e F^\mu{}_\nu \frac{dx^\nu}{d\tau}$$

obviously only for charged (hence massive) particles. Still we could use $d\tau = m dp$ to get

$$\frac{d^2 x^\mu}{dp^2} = e F^\mu{}_\nu \frac{dx^\nu}{dp} \quad \text{OR} \quad \frac{dp^\mu}{dp} = e F^\mu{}_\nu p^\nu$$

Alternatively:

$$m \frac{dp^\mu}{d\tau} = e F^\mu{}_\nu p^\nu$$

② Think of a particle at motion in a lab, so it has a world line $x^\mu(\tau)$ and $\frac{dx^0}{d\tau} = \frac{dt}{d\tau} = \gamma$. So let $x^\mu = \text{lab coords}$.

③ In Weinberg's metric, we have

$$F^{0i} = F^0{}_i = E_i = -F^{i0} = +F^i{}_0 \quad \text{/be careful with mixed tensors!}$$

$$F^{ij} = F^i{}_j = \epsilon^{ijk} B_k$$

Thus, we conclude:

$$1) \quad m \frac{dp^0}{d\tau} = e F^0{}_i p^i = e E_i p^i$$

$$2) \quad m \frac{dp^i}{d\tau} = e F^i{}_0 p^0 + e F^i{}_j p^j$$

④ In 3-vector form, we say:

$$= +e E_i p^0 + e \epsilon^{ijk} B_k p^j$$

$$m \frac{dE}{d\tau} = e \vec{E} \cdot \vec{p}$$

$$m \frac{d\vec{p}}{d\tau} = +e E \vec{E} + e \vec{p} \times \vec{B}$$

⑤ Since $x^\mu = x^\mu(\tau)$, $p^\mu = p^\mu(\tau)$, and $\frac{dp^\mu}{d\tau} = \frac{dt}{d\tau} \frac{dp^\mu}{dt} = \gamma \frac{dp^\mu}{dt}$

$$\Rightarrow E \frac{dE}{dt} = e \vec{E} \cdot \vec{p} \quad c^2$$

$$\frac{d\vec{p}}{dt} = +e \vec{E} + e \vec{v} \times \vec{B} / c$$

(em in special rel)

⑥ So, the true equations of motion are:

$$\frac{dE}{dt} = e \vec{v} \cdot \vec{E}$$

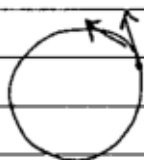
$$\frac{d\vec{p}}{dt} = e \left[\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right]$$

Of course the first one can be derived from the second via the mass shell condition.

⑦ Conclusion: The Lorentz force equation expressed in this form is completely correct. However, one must remember that:

$$\vec{p} = \gamma m \vec{v} \quad E = \gamma m c^2$$

⑧ Application: particle accelerator: we say



$$\begin{aligned} \frac{d\vec{p}}{dt} &= \frac{d}{dt}(\rho \hat{p}) = \rho \frac{d\hat{p}}{dt} = \rho \omega \hat{n} \\ &= e \frac{\vec{v} \times \vec{B}}{c} = e \frac{v B \hat{n}}{c} \end{aligned}$$

$$\Rightarrow \rho \omega = \frac{e v B}{c}$$

Classically, we'd say $p = mv$ so LHS = mv^2/r . Now we must use $p = \gamma mv$, so

$$\gamma m v^2 / r = \frac{e v B}{c}$$

$$\Rightarrow B = \frac{m}{e} \cdot \frac{\gamma v}{rc} = \boxed{\frac{E v}{e R c} = B}$$

⑨ Example: at PEP, $R = 220 \text{ m} = 2.2 \times 10^4 \text{ cgs}$. $v = c = 3 \times 10^{10} \text{ cgs}$.
Then for:

$$B = \frac{E(\text{cgs})}{(4.8 \times 10^{-10}) R(\text{cgs})} = (20/4.8) \times 10^{19} \frac{E(\text{cgs})}{R(\text{cgs})}$$

$$\left. \begin{aligned} \text{But } E(\text{cgs}) &= 1.6 \times 10^{-3} E(\text{GeV}) \\ R(\text{cgs}) &= 10^5 R(\text{km}) \end{aligned} \right\} \Rightarrow$$

$$\boxed{B(\text{gauss}) = 33 \frac{E(\text{GeV})}{R(\text{km})}}$$

With $R = .2 \text{ km}$

$$E = 15 \text{ GeV} \Rightarrow B = 5,000 \text{ gauss}$$

$$E = 200 \text{ GeV} \Rightarrow B = 33,000 \text{ gauss} \leftarrow \text{need superconductors.}$$

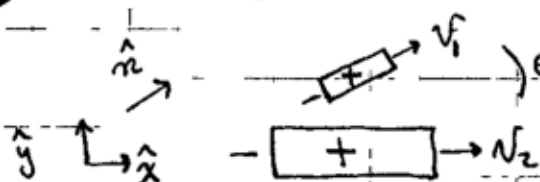
a major limitation on accelerators

Doc Thomas' Kinematic Rotation

or "train and car"

1. -25

- ① First, a galilean discussion. Here are a train and a car as seen from an overhead land based observer:

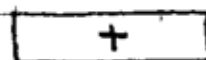


On each, we photocopy a little coordinate grid. These grids + are parallel.

Assume that $v_1 \cos \theta = v_2$, so when we hop onto the train, train observer sees:



$$v_3 = v_1 \sin \theta$$



The point is this: The car (and its little grid) do not appear rotated. The train observer sees the grids exactly aligned; he measures the same angle θ as the land based observer. Of course the car's velocity is in a new direction.

Mathematically speaking, the car's grid (at rest on the car) must go through two boosts to be seen by the train observer $[-v_1, \text{ then } +v_2]$. In the galilean group, boosts commute, so the net result is just a boost (not a boost + a rotation.)

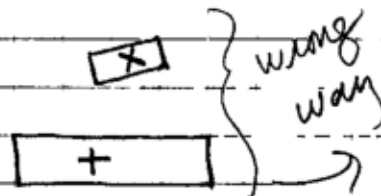
a) Why do galilean boosts commute?

$$\begin{aligned} \vec{x}' &= \vec{x} + \vec{v}_1 t & \vec{x}'' &= \vec{x} + \vec{v}_2 t \\ \underbrace{t' = t}_{\text{boost 1}} & & \underbrace{t'' = t}_{\text{boost 2}} \end{aligned}$$

$$B_1 B_2 x^\mu = B_1 \begin{bmatrix} t \\ \vec{x} + \vec{v}_2 t \end{bmatrix} = \begin{bmatrix} t \\ \vec{x} + (\vec{v}_2 + \vec{v}_1) t \end{bmatrix} = B_2 B_1 x^\mu$$

The boost generators of the G.G. commute and that's all there is to it.

- ② Poincare: if velocities were large, poincare group says the train observer will see the car and its grid rotated!



③ If the angle θ is small, we can compute the amount of Thomas rotation as follows:

Let K_c = rest frame of car. In frame K_c , someone observes a 4-vector V_c with certain components.

How does observer in lab = K_L see this vector V ? i.e., what is V_L ? One must do a boost of some sort.

$$\text{i.e., } V_L = B_{\hat{n}}^{A,P?}(\pm v_i?) V_c$$

How do we get all these signs right? If V_c were a rest momentum vector $V_c = (m, \vec{0})$, we want V_L to be $(E, |\vec{p}| \hat{n})$. Eg, let V_c be the car's momentum!

So, this looks like a standard active boost, so:

$$V_L = B_{\hat{n}}^A(+v_i) V_c = B_{\hat{n}}^A(v_i) V_c$$

Similarly, we can conclude that

$$V_L = B_{\hat{n}}^A(v_i) V_T \leftarrow \text{train.}$$

Therefore,

$$V_T = B_{\hat{n}}^A(-v_i) B_{\hat{n}}^A(v_i) V_c = \mathcal{L} \cdot V_c$$

I have no doubt about these signs.

④ Now let's let $v_1 = v_2 = v$ [$\alpha, \gamma, \sigma, \beta, \dots$].
From our table, we look up

$$\times \left(\mathcal{L} = \begin{bmatrix} ch & -sh & -sh\theta & 0 \\ -sh & ch & (\gamma-1)\theta & 0 \\ -sh\theta & (\gamma-1)\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \text{ error.}$$

$$\mathcal{L} = \begin{bmatrix} ch & -sh & 0 & 0 \\ -sh & ch & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ch & sh & \theta sh & 0 \\ sh & ch & \alpha\theta & 0 \\ \theta sh & \alpha\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B_{\hat{n}}^A(-v) B_{\hat{n}}^A(+v)$$

Here we have used

$$\begin{aligned} ch &= \cosh \sigma & \Theta &= \sin \theta & \gamma &\equiv (\gamma-1) \\ sh &= \sinh \sigma & 1 &= \cos \theta \end{aligned}$$

Doing the multiplication, we get:

$$\begin{aligned} L &= \begin{bmatrix} 1 & 0 & \Theta \cdot sh & 0 \\ 0 & 1 & -\alpha \Theta & 0 \\ \Theta \cdot sh & \alpha \Theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \Theta \cdot sh \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \Theta \alpha \begin{bmatrix} 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= 1 - \Theta \cdot sh M^{20} - \Theta \alpha M^{12} \\ &\approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &\approx B_{zy}^A(\Theta sh) \cdot R_{xz}^A(\Theta \alpha) \end{aligned}$$

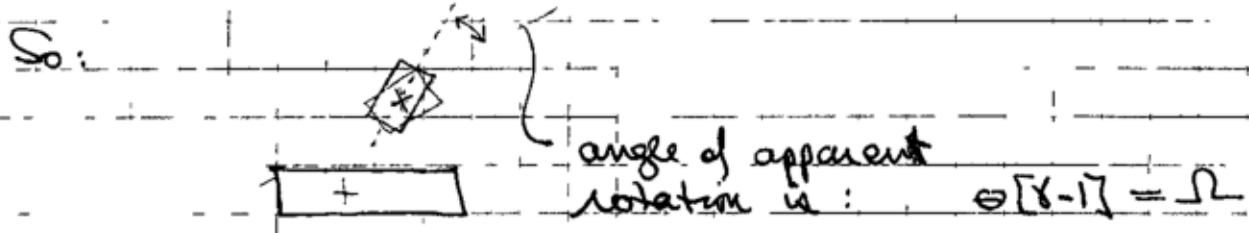
Now be careful with signs:

$$\ominus [-\Theta \alpha M^{12}] = \ominus [+\Theta \alpha M^{31}] = R_3^P(-\Theta \alpha) = R_3^A(\Theta \alpha).$$

So,

$$L = B_{zy}^A(\Theta sh) \cdot R_{xz}^A(\Theta [\gamma-1])$$

This says that a spatial vector on the car gets (appears) rotated counterclockwise to train observer.

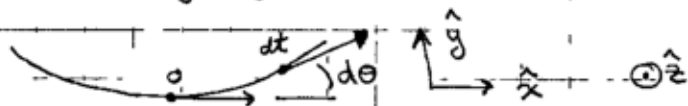


↙ This effect is not just contraction. (as long as infinitesimal).

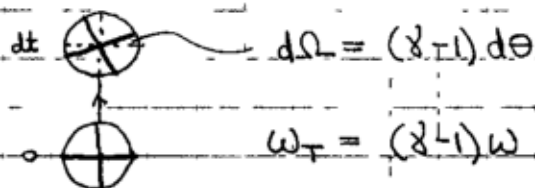
Thus, train observer sees part of the car that ground cannot see!! This merits further consideration some other time.

Thomas precession and the $\frac{1}{2}$ factor

- ① Electron is going in a circle around a proton.

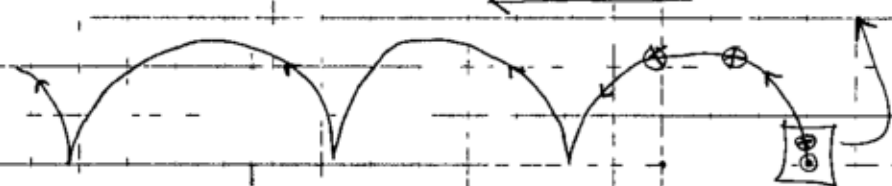


Define frame T as an inertial frame moving in $\hat{x}$ at v relative to this atom. According to previous discussion, here is what an observer in T sees:



$$\omega_T = (\gamma + 1) \omega \approx \frac{1}{2} \gamma^2 \omega \quad \text{NR}$$

- ② The full show for observer in T is fairly complicated: $\hat{\omega}_T = \hat{z}$



Whether or not observer in T sees uniform precession at ω_T , (or just locally is it ω_T) is still an open question in my mind; The answer is not needed to continue, however.

- ③ In frame T , observer sees a magnetic field due to the proton shooting off to the left.

$$\vec{B} = \hat{z} \left[\frac{qv}{r^2} \right]$$

- ④ Equations (from lab frame) relating various variables are:

$$v = r\omega; \quad \frac{q^2}{r^2} = \frac{mv^2}{r} \Rightarrow \quad q^2 = mv^2 r \quad \text{NR}$$

- ⑤ The electron has a magnetic moment $\vec{\mu}$ and a spin angular momentum $\vec{S}$. They are related by:

$$\vec{\mu} = \frac{e}{m} \vec{S}$$

$$e < 0$$

where e = charge of electron.

Thomas precession

⑥ In inertial frame T , the spin precesses due to presence of $\vec{B}$:

$$\left. \frac{d\vec{S}}{dt} \right|_T = \vec{\mu} \times \vec{B} \quad \text{or,} \quad \frac{d\vec{S}}{dt} = \vec{N}$$

⑦ Let electron be the "body" or "rotating frame", = "electron rest frame" $\neq$ inertial frame.

According to usual Goldstein p133,

$$\left. \frac{d\vec{S}}{dt} \right|_{\text{e.r.f.}} = \underbrace{\left[\frac{d\vec{S}}{dt} \right]_T}_{\text{non rot}} - \vec{\omega}_B \times \vec{S}$$

where $\vec{\omega}_B = \omega$ of the body. But here, $\vec{\omega}_B = \vec{\omega}_T$, the Thomas precession!

⑧ So in the electron rest frame, equation of motion is:

$$\begin{aligned} \frac{d\vec{S}}{dt} &= \vec{\mu} \times \vec{B} - \vec{\omega}_T \times \vec{S} \\ &= \vec{\mu} \times \left[\vec{B} + \frac{m}{e} \vec{\omega}_T \right] \quad e < 0 \end{aligned}$$

⑨ It is "as if" the electron were in an effective magnetic field:

$$\vec{B}_{\text{eff}} = \vec{B} + \frac{e}{m} \vec{\omega}_T = \frac{1}{\hbar} \left[\frac{\hbar^2 \vec{v}}{r^2} + \frac{\hbar^2 (e)}{m} \frac{1}{2} \vec{v} \omega \right]$$

$$|\vec{B}_{\text{eff}}| = \frac{21}{m} \left[\frac{qmv\hbar}{r^3} \right] - \frac{1}{2} [\text{same}] \quad \left(= \frac{m}{e} \cdot \frac{1}{2} \frac{v^3}{r^2} = \frac{mv}{e} \cdot \frac{1}{2} \frac{v}{r^2} \cdot \left[\frac{q\hbar}{m} \right] = \frac{-q\hbar}{2m} \right)$$

But, $(mvr)\hbar = \vec{L}$, the orbital angular momentum, so

$$\vec{B}_{\text{eff}} = \frac{1}{2} \frac{q}{r^3} \vec{L}$$

(This is the Thomas correction.
Hence, the " $\frac{1}{2}$ " below)

The interaction term you would therefore add to the Hamiltonian for this electron would be:

$$\mathcal{H}' = -\vec{\mu} \cdot \vec{B}_{\text{eff}} = + \frac{q}{m} \vec{S} \cdot \frac{1}{2} \frac{q}{r^3} \vec{L} = \boxed{+ \frac{1}{2} \frac{q^2}{m r^3} \vec{S} \cdot \vec{L} = \mathcal{H}'}$$

Usually this is written:

$$\mathcal{H}' = \frac{1}{2m} \frac{1}{r} \frac{dV}{dr} \cdot \vec{S} \cdot \vec{L}$$

The "spin-orbit coupling"

where $V = -\sum \frac{q^2}{r}$
note;

$f(r)$

Thomas precession)

- ⑩ How would one measure this thing? Look at 2p excited state of hydrogen.

$$\begin{aligned} E(\uparrow) - E(\downarrow) &= g(1) \langle \uparrow | \vec{S}_p \cdot \vec{L}_p | \uparrow \rangle - \text{same in } |\downarrow\rangle \\ &= g(1) \cdot \frac{1}{2} \cdot 1 - g(1) \cdot -\frac{1}{2} \cdot 1 \\ &= g(1) \quad \text{where } 1 = \langle L \rangle_p \end{aligned}$$

Experiment would easily tell you that the Thomas $\frac{1}{2}$ factor is there.

- ⑪ References: L.H. Thomas, Nature 117, 514 (1926)
Jackson, 1st Ed. "EQ", p 364 (1962)

- ⑫ Note that this Thomas $\frac{1}{2}$ does not affect the Zeeman term:

$$\begin{aligned} H_{\text{Zeeman}} &= -\vec{\mu}_s \cdot \vec{B}_{\text{ext}} - \vec{\mu}_{\text{orb}} \cdot \vec{B}_{\text{ext}} \\ &= -[\vec{\mu}_{\text{spin}} + \vec{\mu}_{\text{orb}}] \cdot \vec{B}_{\text{ext}} \end{aligned}$$

Here, $\vec{\mu}_{\text{spin}} = -\frac{e\hbar}{m} \vec{S}$ as before, and

$$|\vec{\mu}_{\text{orb}}| = (\pi r^2) \cdot \left(e \frac{\omega}{2\pi} \right) = \frac{1}{2} e \omega r^2 = \frac{1}{2} e v r = \frac{e}{2m} (mvr) = \frac{e\hbar}{2m}$$

$$\vec{\mu}_{\text{orb}} = -\frac{e\hbar}{2m} \vec{L} \quad \mu_B = \frac{e\hbar}{2mc}$$

Thus: $H_{\text{Zeeman}} = +\frac{e\hbar}{2m} [\vec{L} + 2\vec{S}] \cdot \vec{B}_{\text{ext}} \quad \text{ie, } \vec{L} \cdot \vec{B} / \hbar = m_L B_z / \hbar$

- ⑬ For a given state $|l, s\rangle$, the splitting is clear [Paschen-Back].

- ⑭ For weak B_{ext} , use $|J, M, S\rangle$ where $\vec{J} = \vec{L} + \vec{S}$, so

$$H_{\text{Zeeman}} = \mu_B [\vec{J} + \vec{S}] \cdot \vec{B}_{\text{ext}}$$

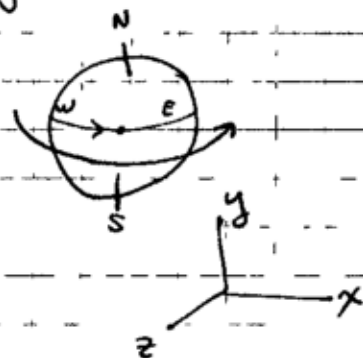
After some Clebsch stuff, this yields

$$H_{\text{Zeeman}} = g_{\text{Landé}} \mu_B |B_{\text{ext}}| \quad \begin{array}{l} \text{"Landé"} \\ g \\ \text{factor"} \end{array}$$

Do not confuse $g_{\text{Landé}}$ with "gyromagnetic ratio" g

$$g_{\text{Landé}} \equiv \frac{\mu}{\mu_B} = 2, \text{ for electron spin} \\ \mu_B = 1, \text{ for electron orbit.}$$

Mechanics problem: you are looking down in the north pole of the earth, in the earth's frame.

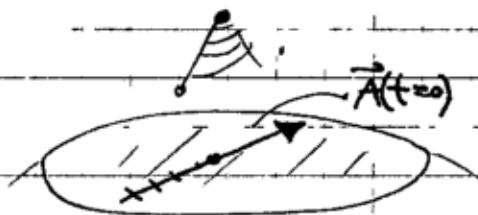


The earth rotates at $\vec{\omega}_e = |\omega_e| \hat{y}$.

At $t \approx 0$, a Foucault pendulum draws an arrow $\vec{A}(t=0)$

Earth observer will see this arrow $\vec{A}$ precess clockwise, obviously.

We know that $\left. \frac{d\vec{A}}{dt} \right|_{\text{space}} = 0$



According to Goldstein, the body observer sees:

$$\left. \frac{d\vec{A}}{dt} \right|_{\text{body}} = \left. \frac{d\vec{A}}{dt} \right|_{\text{sp}} - \vec{\omega} \times \vec{A} = -\vec{\omega} \times \vec{A}$$

$$\Rightarrow \delta \vec{A} = \vec{A} \times \vec{\omega} \delta t$$

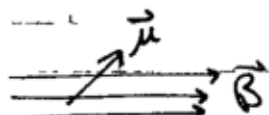
Hey, it works!

To demonstrate:

- 1) $\vec{\omega} \times$ correction
- 2) magnetic precession

Magnetism/Mechanics Problem

- ① Magnetic dipole sits in uniform $\vec{B}$ field:



We know that it will feel a torque $\vec{\mu} \times \vec{B}$ which tends to line it up with $\vec{B}$. If $\vec{B}$ is suddenly turned on, then $\vec{\mu}$ is released, it will oscillate in a plane

$$\frac{d^2\theta}{dt^2} = -\mu B \sin\theta \approx -\mu B \theta$$

$$\text{So } \ddot{\theta} + \left(\frac{\mu B}{I}\right)\theta = 0 \Rightarrow \omega_{\text{osc}}^{\text{SM}} = \sqrt{\frac{\mu B}{I}}$$

General stuff is: $\vec{N} = d\vec{L} = \frac{d\vec{L}}{dt}$

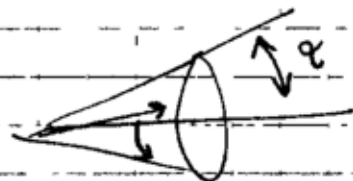
And all vectors are out of the paper here.

- ② Now repeat the problem, but assume the $\vec{\mu}$ object has a spin $\vec{S}$ on its axis. Then what happens when you let it go!

Technically, we have: $\vec{N} = \vec{\mu} \times \vec{B} = \frac{d}{dt} [\vec{S} + \vec{L}]$

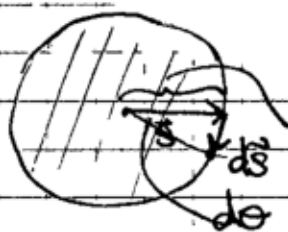
where $\vec{L}$ is the part of $\vec{J}$ due to whatever the thing does [like $\vec{S}$ in ①]. But, if we assume at the start that $L \ll S$, we get

$$\frac{d\vec{S}}{dt} = \vec{\mu} \times \vec{B}$$



This says that the thing precesses on a cone.

How long to go around once? Look in $\vec{J}$ -space:



$$d\theta \times (S \sin\alpha) = dS = \mu B \sin\alpha dt$$

$$\Rightarrow \omega_{\text{pre}} = \frac{d\theta}{dt} = \boxed{\frac{\mu B}{S} = \omega_p}$$

(more on precession of spinning mag. dipole).

③ How much is L which we neglected?

$$L \approx m r^2 \omega_p \Rightarrow m R^2 \mu B \ll S^2$$

where r is length of dipole.

④ For an electron, perhaps $r^2 = \lambda_{com}^2 = 8 \times 10^{-30} \text{ m}^2$
 $m = 10^{-30} \text{ kg}$.

$$\text{Need } B \ll \frac{S^2}{m R^2 \mu} \text{ etc. (Later.)}$$

⑤ Now, put this experiment on the north pole of the earth, i.e. observe it from the earth.
Then:

$$\left. \frac{d\vec{S}}{dt} \right|_{\text{space}} = \vec{\mu} \times \vec{B} \quad \text{still}$$

But, earth observer sees:

$$\left. \frac{d\vec{S}}{dt} \right|_{\text{body}} = \left. \frac{d\vec{S}}{dt} \right|_{\text{space}} - \vec{\omega}_e \times \vec{S}$$

i.e., earth guy says $\vec{S}$ described by:

$$\frac{d\vec{S}}{dt} = \vec{\mu} \times \vec{B} - \vec{\omega}_e \times \vec{S}$$

Now suppose $\vec{\mu} = a \vec{S}$, then:

$$\frac{d\vec{S}}{dt} = \vec{\mu} \times \left[\vec{B} + \frac{1}{a} \vec{\omega}_e \right]$$

It's just as if there were an extra effective $\vec{B}$ field.

The potential energy stored in $\vec{B}$ will be

$$U(\text{earth obs}) = -\vec{\mu} \cdot \left[\vec{B} + \frac{1}{a} \vec{\omega}_e \right].$$

This is the
Thomas
calculation!

Particle

History data.

Gell-Mann/Levy α -model	1960	
Gell-Mann/Neuman SU(3)	1964	
Sabido	1963	
π -meson discovery	1947	Lattes et al
$K^0 - \bar{K}^0$ mix system	1955	Gellmann-Pais
parity violation	1956	Ch ⁶⁰ Wu Lee Yang
strangeness 1953	Gell-Mann-Nishijima	
SSB, Goldstone	1960	
Higgs	1964	
Goldberger-Treiman relation	1958	
CVC theory 1958	Gell-Mann/Feynman	
Ward Identity	1950	

Want to show why the expression in (12b) is in fact an eigenstate of J_{op}^2 . First, we must recognize that we are dealing with a mixed notation for "state". Usually we think of a state being something like $|lm\rangle$, i.e., a "ket". So in (12b) it looks like $U_{S,V} = |S,V\rangle$ is the only state around, and $\langle \sim | Y^T | \sim \rangle$ is just a "function". Well, it turns out that $\langle \sim | \sim \rangle$ contains the continuum or l-part of the "state" (12b). It is in the same sense that $\langle \phi | lm \rangle$ is a "state". (it is a "wavefunction", or a "state closed into a particular representation). You can still operate on these "closed" states or wavefunctions, but you must use an operator that is expressed in the same representation onto which the state is closed. In other words, $L_z \langle \phi | lm \rangle$ means $\pm i \partial/\partial \phi$ on $Y_{lm}(\theta, \phi)$. The connection of course is $L_z \langle \phi | lm \rangle = \langle \phi | L_z | lm \rangle = m \langle \phi | lm \rangle$.

So, going back to (12a) and (12b), if we wanted explicitly to calculate $J_{op}^2 = (L_{op}^2 + S_{op}^2 + 2\vec{L}_{op} \cdot \vec{S}_{op})$ on the state (12b), we would think of S_{op} as acting on the $U_{S,V}$, whereas L_{op} acts on the Y_{lm} wavefunction hidden within (12b), as shown in (12a).

For example, $S_{op} U_{S,V} = V U_{S,V}$, whereas $L_{op} Y_{lm} = m Y_{lm}$.

The $U_{S,V}$ states are unclosed $[|S,V\rangle]$ whereas the Y_{lm} states are closed $\langle \phi | lm \rangle$. Thus, we use that natural sense of S_{op} , but

The coordinate rep. sense of $\hat{L}_z$.

However, it turns out that we don't need to do anything explicit. Rewrite (12b) as:

$$(12b) = \sum_{m, M} Y_{\ell}^{m*}(\hat{R}) \langle \ell S m \rangle | J M \rangle \times \left[\sum_{m', v'} Y_{\ell'}^{m'}(\hat{n}) \langle \ell' S' m' v' | J M \rangle U_{S' v'} \right]$$

Now realize that $\vec{J} = \vec{L}' + \vec{S}'$ so J^2 written in this way (the final J if you like) acts only on the part in brackets $[]$. This is because S' and L' do not appear in the first portion. Now it turns out that $[]$ is an eigenstate of J because, from usual argument (see, e.g., Eq. 2.40 p 35 of Williams),

$$\sum_{m', v'} | \ell' m' S' v' \rangle \langle \ell' m' S' v' | J M \rangle = | J \ell' S' M \rangle$$

where here I am using $Y_{\ell'}^{m'}(\hat{n}) U_{S' v'} = | \ell' m' S' v' \rangle$. More accurately, $Y_{\ell'}^{m'}(\hat{n}) U_{S' v'} = \langle \hat{n} | \ell' m' S' v' \rangle$, a closed state, so:

$$\sum_{m', v'} \langle \hat{n} | \ell' m' S' v' \rangle \langle \ell' m' S' v' | J M \rangle = \langle \hat{n} | J M \ell' S' \rangle = \langle \hat{n} | J M \rangle$$

Thus, it is clear that $J^2 (12b) = J(J+1) (12b)$ where

J^2 is the coord rep of J^2 .

① DEFINE F_π WITH $\dim(F_\pi) = E$ BY:

$$M = \langle 0 | A_\mu^i(x) | \pi^j(p) \rangle = i F_\pi p_\mu \cdot \delta_{ij} \cdot \frac{e^{-ip \cdot x}}{\sqrt{(2\pi)^3 2E}} = i p_\mu F_\pi \langle 0 | \phi^i(x) | \pi^j \rangle$$

$\nwarrow$ weak, hadronic, axial vector current

② USE DEF. $\Box \phi^i = j^i$ WITH $\Box = \partial_\mu \partial^\mu = -p^2$ TO SAY

$$M = (-i p_\mu) \cdot \frac{F_\pi}{p^2} \cdot \langle 0 | j^i(x) | \pi^j(p) \rangle = + \frac{F_\pi}{p^2} \langle 0 | \partial_\mu j^i(x) | \pi^j(p) \rangle$$

③ MAKE THE PCAC GENERALIZATION:

$$\langle b | A_\mu^i(x) | a \rangle = \frac{F_\pi}{p^2} \langle b | \partial_\mu j^i(x) | a \rangle. \quad p = p_a - p_b$$

$$\text{OR } \langle b | \partial^\mu A_\mu^i(x) | a \rangle = -F_\pi \langle b | j^i(x) | a \rangle = + m_\pi^2 F_\pi \langle b | \phi^i(x) | a \rangle$$

④ IN USUAL π -N THEORY WE KNOW:

$$A_\mu^i = -g_A \bar{\Psi} \gamma_5 \frac{1}{2} \tau^i \gamma_\mu \Psi \quad \nwarrow \text{usual pionic currents} \quad j^i = -ig_0 \bar{\Psi} \gamma_5 \tau^i \Psi$$

THUS, 3 BECOMES:

$$ig_A \cdot \frac{1}{2} \langle b | \bar{\Psi} \gamma_5 \tau^i \Psi | a \rangle = + i F_\pi g_0 \langle b | j^i | a \rangle$$

AND AT ONCE WE GET THE GTR:

$$\boxed{g_A M_N = F_\pi g_0}$$

⑤ Numerics:

$$F_\pi = \left(\frac{g_A}{g_0} \right) M_N = \frac{1.25}{13.5} (938) = 86 \text{ MeV.}$$

I think Thz decay says $F_\pi = 93 \text{ MeV}$, exper

PCAC, wesen 5.5

$$\langle 0 | A_\mu^i(x) | \pi j(p) \rangle = \frac{F_\pi}{p^2} \langle 0 | \partial_\mu j^i(x) | \pi j(p) \rangle = -\frac{i F_\pi p^\mu}{p^2} \langle 0 | j^i(x) | \pi j \rangle$$

$$p^2 \langle 0 | A_\mu^i | \pi j \rangle = -i F_\pi p^\mu \langle 0 | j^i | \pi j \rangle$$

$$j^i = -g_0 \bar{\Psi} i \gamma_5 \partial^i \Psi$$

$$\begin{aligned} p^2 \langle N_\alpha | A_\mu^i | N_\beta \rangle &= -i F_\pi p^\mu \langle N_\alpha | j^i | N_\beta \rangle \\ &= -F_\pi g_0 p^\mu \langle N_\alpha | \bar{\Psi} \gamma_5 \partial^i \Psi | N_\beta \rangle \end{aligned}$$

$$A_\mu^i = \frac{1}{2} g_A \bar{\Psi} \gamma_5 \partial_\mu \Psi$$

$$\Rightarrow -\frac{1}{2} g_A p^2 \langle N_\alpha | \bar{\Psi} \gamma_5 \partial_\mu \Psi | N_\beta \rangle = -F_\pi g_0 p^\mu \langle N_\alpha | \bar{\Psi} \gamma_5 \partial^i \Psi | N_\beta \rangle$$

$$\sim p^\mu \quad -\frac{1}{2} g_A p^2 \cdot 2M \cdot \langle N_\alpha | \bar{\Psi} \gamma_5 \partial^i \Psi | N_\beta \rangle = -F_\pi g_0 p^2 \cdot \text{same.}$$

$$\Rightarrow \boxed{g_A \cdot M = F_\pi \cdot g_0} = F_\pi$$

PCAC GTR w.h.

Playing with the chiral Lagrangian. PCAC, version 5.

- ① First, let's go review the origin of the G-T relation, for the n^{th} time. I have recorded one path in the Obers notes. First, we must define clearly the constant $F_\pi = 93 \text{ MeV}$. Here is Obers p35 definition:

$$\langle 0 | A^i(x) | \pi^j(p) \rangle = \frac{i F_\pi \bar{e}^{-ipx}}{\sqrt{(2\pi)^3 2E}} p_\mu \delta_{ij}$$

How do the dimensions work here? Since $|\pi^j(p)\rangle$ means $a_j^\dagger(p)|0\rangle$, we need a norm on these a 's. In BD, since $[a, a^\dagger] = \delta^3(k-k')$, you conclude that $\dim(a^2) = E^{-3}$ so $\dim(a) = E^{-3/2}$. Then with BD (2.7) this would make $\dim(\phi) = E$, right. Now, what about A^i ? If $\mathcal{L} = j_\mu j^\mu$ and $j_\mu \sim A_\mu$, and $\dim(\mathcal{L}) = E^4$, then expect $\dim(A_\mu) = E^2$. This would make $\dim(\text{LHS}) = E^{1/2}$ but $\dim(\text{RHS}) = E^{3/2}$, no go. Correction: $\mathcal{L} \sim G j_\mu j^\mu$ and $\dim(G) = E^{-2}$ so $\dim(A^\mu) = E^3$. Then $\text{LHS} = E^3 \cdot E^{-3/2} = E^{3/2}$, now agrees. So.

Conclusion: 1) $[a, a^\dagger] = \delta()$ sets $\dim[|\pi^j(p)\rangle] = E^{-3/2}$ as in BD.

2) $\mathcal{L} \sim G A_\mu A^\mu$ sets $\dim[A] = E^3$

3) Thus, $\dim(F_\pi) = E$ as desired.

So let's take this as the all standard def of π .

- ② Now, how could you rewrite the RHS using the pion fields? Its easy, + shows why they used the entire wavefunction:

$$= i F_\pi \langle 0 | \phi^i(x) | \pi^j(p) \rangle$$

- ③ Next, use: $\partial_\mu \uparrow = -p^2$ when acting on $\uparrow = e^{-ipx}$, so

$$= (-i p_\mu) \cdot \frac{F_\pi}{p^2} \cdot \langle 0 | j^i(x) | \pi^j(p) \rangle \text{ where } [\phi^i(x) = j^i(x).$$

- ④ Finally, rewrite $-i p_\mu$ as $+\partial_\mu$ to get:

$$\langle 0 | A_\mu^i(x) | \pi^j(p) \rangle = \frac{F_\pi}{p^2} \langle 0 | \partial_\mu j^i(x) | \pi^j(p) \rangle$$

This is supposed to show that the A- π has the pion pole.

To relate A^μ, V^μ to the generator currents

On p 280 Goe, he relates the octet state $\phi_{\pi^+}^+$ to the generators of SU_3 :



$$\phi_{\pi^+}^+ \longleftrightarrow \frac{1}{\sqrt{2}} [\phi_1 + i\phi_2]$$

That is, the generators get linearly combined before they appear as eigenstates of t, t_3, y . Here, $|1, 1, 0\rangle$.

Next, we look at the axial weak current octet and guess a connection to the pseudoscalar mesons.

$$(A_{S=0}^\mu) = (F_1^5)^\mu + i(F_2^5)^\mu$$

The connection is then:

$$\partial^\mu A_\mu = \partial^\mu (F_1^5 + iF_2^5) = \text{const} \times \phi_{\pi^+}^+$$

Why do we choose the π -like current. Why not associate the "axial hadronic $S=0$ weak current" with some other pseudo scalar meson? All we need is $S=0, Y=0$. Only other choice is the η^0 . I think the reason is that there is $I=1$ character to this piece. If the η^0 , then would have $A_\mu = A_\mu^+$. So what? $\Delta I = 1$ is observed in decays, eg, π in decay!! Thus it is true: but how do you know that some η^0 isn't mixed in!!! In the vector case, $\Delta I = 1$, never 0, so no admixture. The third piece is $\Sigma + \Lambda$.

$$A_{S=0}^\mu = (F_1^5 + iF_2^5)^\mu + \text{nothing else.}$$

To show that $f_\pi = c$

Starting M.G.F. statement is true:

$$\sqrt{2\omega} \langle \pi^+(q) | A^\mu(0) | 0 \rangle = (-i) q^\mu \cdot \frac{f_\pi}{m_\pi^2}$$

Use translation to find:

$$\langle q | A^\mu(0) | 0 \rangle = \langle q | e^{-iP \cdot x} A^\mu(x) e^{+iP \cdot x} | 0 \rangle = e^{-iq \cdot x} \langle q | A^\mu(x) | 0 \rangle$$

Therefore:

$$\sqrt{2\omega} \langle q | A^\mu(x) | 0 \rangle = e^{iq \cdot x} (-iq^\mu) \cdot \frac{f_\pi}{m_\pi^2}$$

$$\sqrt{2\omega} \langle q | \partial_\mu A^\mu(x) | 0 \rangle = e^{iq \cdot x} \cdot f_\pi$$

$$\sqrt{2\omega} \langle q | \partial_\mu A^\mu(0) | 0 \rangle = f_\pi$$

Now use PCAC theorem:

$$\partial_\mu A^\mu(x) = b \cdot m_\pi^2 \cdot \phi_{\pi^+}^\dagger$$

Above element is:

$$\begin{aligned} \sqrt{2\omega} \langle q | \partial_\mu A^\mu(0) | 0 \rangle &= \sqrt{2\omega} \langle q | b m_\pi^2 \phi_{\pi^+}^\dagger(0) | 0 \rangle \\ &= \sqrt{2\omega} \langle 0 | a_{\pi^+}(q) \cdot b m_\pi^2 \cdot \int d^3k \frac{1}{\sqrt{2\omega}} a_{\pi^+}^\dagger(k) | 0 \rangle \\ &= b m_\pi^2 \end{aligned}$$

Therefore, the inescapable conclusion is thus:

$$b m_\pi^2 = f_\pi$$

And PCAC may be written:

$$\partial_\mu A^\mu(x) = f_\pi \phi_{\pi^+}^\dagger(x)$$

Gelberger-Trieman

Look at the emission of a π^+ :



$$\begin{aligned}
 \langle n\pi^+ | p \rangle_{in} &= \frac{i}{\sqrt{E}} \int d^4x \langle n | \phi_\pi^\dagger(x) | p \rangle \bar{K}_x \frac{1}{\sqrt{2\omega_\pi}} e^{+iq \cdot x} \\
 &= \frac{i}{\sqrt{E}} (m_\pi^2 - q^2) \int d^4x \frac{1}{\sqrt{2\omega_\pi}} e^{+iq \cdot x} \underbrace{\langle n | \phi_\pi^\dagger(x) | p \rangle}_{e^{i p_\pi \cdot x} \phi_\pi^\dagger(0) e^{-i p \cdot x}} \\
 &= \frac{i}{\sqrt{E}} (m_\pi^2 - q^2) \frac{1}{\sqrt{2\omega_\pi}} (2\pi)^4 \delta^4(x) \langle n | \phi_\pi^\dagger(0) | p \rangle
 \end{aligned}$$

Thus, the M -matrix element is:

$$\begin{aligned}
 M_{fi} &= + \frac{1}{\sqrt{E}} (m_\pi^2 - q^2) \frac{1}{\sqrt{2\omega_\pi}} \cdot \langle n | \phi_\pi^\dagger(0) | p \rangle \times \sqrt{2\omega_\pi} \sqrt{2E_p} \sqrt{2E_n} \\
 &= + \frac{1}{\sqrt{E}} (m_\pi^2 - q^2) \langle n | \phi_\pi^\dagger(0) | p \rangle \sqrt{2p_0} \sqrt{2E_n}
 \end{aligned}$$

Use PCAC to get rid of $\phi_\pi^\dagger$ + to conclude that:

$$\langle n | \partial_\mu A^\mu(0) | p \rangle = \frac{(+\sqrt{E}) f_\pi}{\sqrt{2p_0} \cdot \sqrt{2E_n} \cdot (m_\pi^2 - q^2)} \times M_{fi}$$

Now calculate M_{fi} in π -N field theory to get:

$$M_{fi} = \sqrt{2} g \bar{u}(n) \gamma_5 u(p)$$

Thus, we know that

$$(1) \quad \langle n | \partial_\mu A^\mu(0) | p \rangle = \frac{+ f_\pi \sqrt{2} g \bar{u}(n) \gamma_5 u(p)}{\sqrt{2p_0} \cdot \sqrt{2E_n} \cdot (m_\pi^2 - q^2)}$$

Now let's take a whole different approach to getting this matrix element.

The general form is:

$$\langle n | A^\mu(0) | p \rangle = \frac{1}{\sqrt{2p_0} \sqrt{2m_0}} \bar{u}(n) \left[g_A(q^2) \gamma^\mu \gamma_5 + q^\mu h_A(q^2) \gamma_5 \right] u(p)$$

Why is this the most general form?! Why no $q_\nu \sigma^{\mu\nu}$ term, and why are these functions real? Time reversal makes them real, and the demand that all terms have the same G-parity rules out $q_\nu \sigma^{\mu\nu}$ (no second class currents). But since C is violated, seems dubious.

Accepting the above form? Well wait. even if $q_\nu \sigma^{\mu\nu}$ were here, $\partial_\mu \rightarrow \partial_\mu$ so $q_\mu q_\nu \sigma^{\mu\nu} = 0$!

$$\langle n | A^\mu(0) | p \rangle = \frac{1}{\sqrt{2p_0} \sqrt{2m_0}} \bar{u}(n) \left[g_A(q^2) \gamma^\mu \gamma_5 + q^\mu h_A(q^2) \gamma_5 + h_A(q^2) q_\nu \sigma^{\mu\nu} \gamma_5 \right] u(p)$$

Now do your divergence $= \times i q_\mu$ to get:

$$\langle n | \partial_\mu A^\mu(0) | p \rangle = \frac{1}{\sqrt{2p_0} \sqrt{2m_0}} \bar{u}(n) \left[g_A 2M \gamma_5 + q^2 h_A \gamma_5 + 0 \right] u(p)$$

$$(2) = \frac{1}{\sqrt{2p_0} \sqrt{2m_0}} (g_A 2M + q^2 h_A) \bar{u}(n) \gamma_5 u(p)$$

Now leave the pion off shell to get, comparing the two equations (1) and (2):

$$(3) \quad \frac{f_\pi \sqrt{2} g(q^2)}{m_\pi^2 - q^2} = 2M_N g_A(q^2) + q^2 h_A(q^2)$$

Finally, look at $q^2 = 0$. Note: $g = g(m_\pi^2) \neq g(0)$.

$$\frac{f_\pi \sqrt{2} g(0)}{m_\pi^2} = 2M_N g_A \quad g_A = g_A(0) \neq g_A(m_\pi^2)$$

Now use smoothness to say $g(0) \approx g(m_\pi^2) = g$. Then:

$$\boxed{f_\pi = \frac{\sqrt{2} M_N m_\pi^2 g_A}{g}}$$

∴ Actually calculation of f_π :

$$\sqrt{2}$$

$$M_N = 938$$

$$m_\pi = .135$$

$$g_A = 1.18$$

$$g = \sqrt{4\pi \times 14.6}$$

$$f_\pi = 2.11 \times 10^{-3} (\text{GeV})^3 \times \left[\frac{m_\pi}{.135 \text{ GeV}} \right]^3$$
$$= .856 m_\pi^3$$

Experiment gives $f_\pi = .96 m_\pi^3$!

Comments: PCAC connects something weak (A^μ) with something strong (Φ_π). Thus, the G.T.R. relates the strong constant g to the weak constants g_A and f_π . Is this relation truth, or is it total coincidence...

Addendum: if PCAC were CAC, then the pion could not weak decay; i.e., $\partial_\mu A^\mu = 0 \Rightarrow f_\pi = 0$. Also, we could conclude that, looking at (3),

This says that $\chi_A(q^2)$ has a pole at $q^2 = m_\pi^2$

$$\chi_A(q^2) = -\frac{2M_N}{q^2} g_A(q^2)$$
$$\approx \frac{-2M_N}{q^2 - m_\pi^2} g_A(q^2).$$

This leads to Nambu's GTR derivation.

PCAC: Goldberger Trieman Relation

This business is fairly complicated and there are loads of conventions to throw you off what you're doing. The basic idea is this:

- 1) we know the width of the weak pion decay. We also know that the matrix element of the hadronic axial weak current between the single pion state and the vacuum has a very simple form involving a constant, the pion decay constant.
- 2) if we make the assumption that $d_\mu A^\mu = c \not{\partial} \pi$, then it is easy to show that the constant c ~~is~~ the pion decay constant. Notice that this constant is defined differently by Commins and Adler.
- 3) Now consider the NNPI vertex for an off shell pion. From the usual reduction formulas we can write the S matrix element in terms of the pion field sandwiched between the n and p states. But we also know this same matrix element from lowest order field theory in terms of the pion coupling constant g_π . Again, a most general form argument is used here.
- 4) 4) Now use PCAC in this matrix element and use the most general form again to bring in the form factors g_A and g_P . Then we get a fancy relation relating g_A , g_P , g_π , f_π , and some masses.
- 5) finally, you look at $q = 0$ and get almost the GTR. The last step is to assume a smoothness in the behavior of g_π near small q^2 . This gives the relation.

Chiral: show that $\bar{\Psi}\Psi$ is 4th component of an $O(4)$ vector.

- ① How does an $O(4)$ vector $(\vec{A}, A_4)$ transform under the $SU(2) \times SU(2)$ generators?*

$$[Q_\alpha, A_\beta] = i \epsilon_{\alpha\beta\gamma} A_\gamma$$

$$[\bar{Q}_\alpha, A_\beta] = -i \delta_{\alpha\beta} A_4$$

$$[Q_\alpha, A_4] = 0$$

$$[\bar{Q}_\alpha, A_4] = i A_\alpha$$

- ② How does Ψ transform?

Under Q_α , Ψ goes as a spinor.

$$[Q_\alpha, \Psi] = -\frac{1}{2} \sigma_\alpha \Psi$$

Under $\bar{Q}_\alpha$, Ψ goes as $\gamma_5 \times$ spinor.

$$[\bar{Q}_\alpha, \Psi] = -\frac{1}{2} \gamma_5 \sigma_\alpha \Psi$$

- ③ How does $\bar{\Psi}$ transform?

Keeping indices in order, get:

$$[Q_\alpha, \bar{\Psi}] = +\frac{1}{2} \bar{\Psi} \sigma_\alpha$$

$$[\bar{Q}_\alpha, \bar{\Psi}] = -\frac{1}{2} \bar{\Psi} \gamma_5 \sigma_\alpha$$

- ④ Now act on $\bar{\Psi}\Psi$.

In detail I have shown:

$$[Q_\alpha, \bar{\Psi}\Psi] = 0$$

$$[\bar{Q}_\alpha, \bar{\Psi}\Psi] = i [i \bar{\Psi} \gamma_5 \sigma_\alpha \Psi]$$

Thus: $A^\mu = (i \bar{\Psi} \gamma_5 \sigma_\alpha \Psi, \bar{\Psi}\Psi) =$ chiral four-vector.

I have also verified that $[Q_\alpha, i \bar{\Psi} \gamma_5 \sigma_\beta \Psi] = i \epsilon_{\alpha\beta\gamma} [i \bar{\Psi} \gamma_5 \sigma_\gamma \Psi]$.

- *⑤ Note: Q_α and $\bar{Q}_\alpha$ are really L_α and A_α , the $O(4)$ generators. To get the $SU(2) \times SU(2)$ generators, go with:

$$Q_L^\alpha = \frac{1}{2} (Q_\alpha + \bar{Q}_\alpha)$$

$$Q_R^\alpha = \frac{1}{2} (Q_\alpha - \bar{Q}_\alpha)$$

$$SU(2)_L \otimes SU(2)_R$$

$$\uparrow Q_L^\alpha \quad \uparrow Q_R^\alpha$$

The g-2 experiment, ...

① First, we have to establish a certain fact about $\pi^+ \rightarrow \mu^+ \bar{\nu}_\mu$ decay. We know that

$$M = \frac{G}{\sqrt{2}} \bar{\nu}_\mu \gamma^\mu (1 + \gamma^5) \nu_\mu \cdot \langle 0 | \bar{\nu}_\mu | \pi^+ \rangle$$

$$= i f_\pi q_\mu$$

← pion decay constant



$$\text{Thus, } |M|^2 = \frac{G^2}{2} f_\pi^2 \bar{u}(1) \gamma^\mu (1 + \gamma^5) v(2) q_\mu \cdot \bar{v}(2) \gamma_\mu (1 + \gamma^5) u(1) q_\mu$$

② Pause to think about $\frac{(1 + \gamma^5)}{2} u(p) = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ \frac{\sigma \cdot p}{E+m} u \end{pmatrix} = \frac{1}{2} \sqrt{E+m} \begin{bmatrix} (1 - \frac{\sigma \cdot p}{E+m}) u \\ -(1 - \frac{\sigma \cdot p}{E+m}) u \end{bmatrix}$

Suppose massless then $m=0, E=p$, so

$$\frac{1}{2}(1 + \gamma^5) u(p) = \frac{1}{2} \sqrt{p} \begin{bmatrix} + (1 - h) u \\ - (1 - h) u \end{bmatrix} \text{ where } h = \vec{\sigma} \cdot \hat{p}$$

Thus, this gives 0 if $u(p)$ is spin-up. Yes. Recall that $\frac{1 + \gamma^5}{2} =$ helicity projector. Thus, in V-A theory, you ~~are~~ amplitude always favors one helicity over the other.

③ So consider: $\bar{\nu}_\mu \leftarrow \pi^+ \rightarrow \mu^+$
 ← neutrino has helicity +1

Thus, muon is totally polarized??

④ Well, $|M|^2 = \frac{G^2}{2} f_\pi^2 \text{trace} (u(1) \bar{u}(1) \not{q} (1 + \gamma^5) v(2) \bar{v}(2) \not{q} (1 + \gamma^5))$

Now: $u(1) \bar{u}(1) = (\not{p}_1 + m) \frac{1}{2} (1 - \gamma^5 \not{S})$
 $v(2) \bar{v}(2) = \not{p}_2 \frac{1}{2} (1 - \gamma^5 \not{S})$

So $|M|^2 = \frac{G^2}{2} f_\pi^2 \text{trace} \{ (\not{p}_1 + m) \frac{1}{2} (1 - \gamma^5 \not{S}) \not{q} (1 + \gamma^5) \not{p}_2 \frac{1}{2} (1 - \gamma^5 \not{S}) \not{q} (1 + \gamma^5) \}$

⑤ OK, OK, the muon is completely polarized based on trivial spin arg. It is polarized "against its will" however which is why:

$$B(\pi^+ \rightarrow \mu^+ \nu_\mu) = 100\%$$

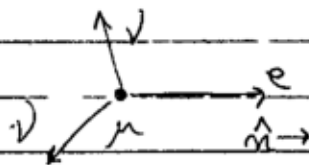
$$B(\pi^+ \rightarrow e^+ \nu_e) \approx 1\%$$

① So, here are the basic facts:

(A) $\pi^\pm$ decays into $\mu^\pm \sim 100\%$.

(B) The $\mu^\pm$ are 100% polarized "against will", meaning $h(\mu^\pm) = -1$.

② Now consider muon decay:

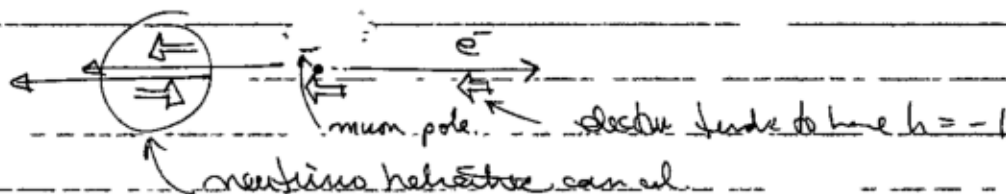


The electron can have energy ranging from 0 to $m_\mu/2$. If we sum over electron spins, then:

$$\text{rate} \approx n(\epsilon) \cdot \left[1 + \left(\frac{1-2\epsilon}{3-2\epsilon} \right) \cos\theta \right] \cdot d\Omega d\epsilon.$$

Now, fast electrons have $\epsilon = 1$, so $\text{rate} \sim (1 - \cos\theta)$ meaning most electrons with $E = m_\mu/2$ come off the back.

So:



So easy to see why fast electrons prefer to come off the back.

③ So we add:

(C) $\mu^\pm$ decays into $e^\pm$ 100%

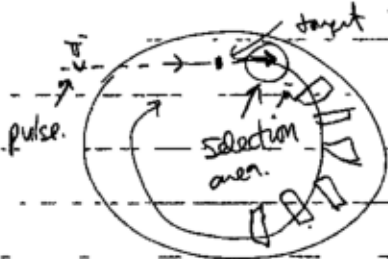
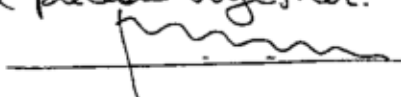
(D) This decay is strongly directional as $(1 \pm \cos\theta)$ for hot electrons in spectrum. Eg, $\bar{e}$ always want to come off the back end of spinning muon

④ So here is the g-2: certain muons only are selected, say those which come straight forward.

These have $h(\mu) = +1$. All of them have spin pointing to the right at selection area. Put hot electron counters all around μ . Hook all in parallel. Shoot in a pulse of μ 's.

Then pulse of muons circulates — all precess together.

Total count rate muon oscillates:



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DRAFT

Cosmological Hilbert Space

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Abstract

A Fock space of ‘cosmological preons’—quantum-theoretic universe constituents—associates to a ‘Milne spacetime’ based on the Lorentz group and to Gelfand-Naimark unitary group representations. Lorentz invariance of Milne-universe ‘age’ accommodates ‘total relativity’. Two ‘extra’ dimensions of the 6-dimensional space occupied by preons at certain exceptional ages of the universe define self-adjoint single-preon operators that include a canonically-conjugate pair representing preon energy and *local time*. Global spacetime divides into ‘slices’ of fixed macroscopic width in age, with ‘cosmological rays’ defined on slice *boundaries*. Self-adjoint-operator expectations at such a boundary prescribe throughout the subsequent slice a non-fluctuating ‘mundane reality’—current densities of conserved electric charge and energy-momentum together with electromagnetic and gravitational potentials. The ray at the lower boundary of a slice is propagated to the upper boundary by cosmological *branched* Feynman paths across the slice that carry (divergence-free) potential-depending action. A macroscopically-stable positive-energy single-preon wave function identifies either with a Standard-Model elementary particle or with a graviton. (Unstable negative-energy preon wave functions remain to be interpreted.) Special relativity--Poincaré invariance--although inexact, is accurate for low-density regions of the universe at spacetime scales which are far below that of Hubble and far above that of Planck.

Introduction

Nonexistence of unitary finite-dimensional Lorentz-group representations has long been supposed to preclude, for dynamically-changing numbers of particles, a Dirac-type quantum theory that represents individual-particle properties such as location, momentum and spin by self-adjoint operators on a rigged Hilbert space. ⁽¹⁾ The Standard Model, which employs as foundation not particles but quantum fields associated to finite-dimensional Lorentz-group representations and which represents action by ill-defined local-field-product operators with perturbative renormalization procedures to manage consequent divergences, has failed to accommodate gravity. Problematic, furthermore, is the Standard-Model description of bound states ('condensed matter'), perturbation theory being unsuited to macroscopically-stationary composite wave functions.

The discrete quantum cosmology (DQC) of Reference (2) and the present paper introduces a 'cosmological-preon' Fock space via unitary (infinite-dimensional) representations of the complex Lorentz-group. Any cosmological preon (henceforth throughout this paper simply called 'preon') is 'lightlike'—in the sense of having velocity c and a polarization transverse to velocity direction—and carries momentum, angular momentum and energy. However, only very-special preon wave functions exhibit, through expectations of self-adjoint operators, the relation between energy, momentum, spin and spacetime location that characterizes 'ordinary matter'. A DQC Fock-space ray, representing the entire universe, comprises sums of products of preon wave functions whose discrete quantum numbers allow certain macroscopically-stable positive-energy preon states to be interpreted as lepton, quark, weak boson, photon or graviton. (A cosmological meaning is given below for the adjective 'macroscopic'.) Reference (2) specifies 'creation-annihilation' Feynman paths whose action includes gravity as well as electromagnetism and weak-strong interaction.

The DQC Fock space is built from Pauli-symmetrized superpositions of products of invariantly-normed single-preon functions. Displayed in this paper is a single-preon basis that is '6-labeled' by preon energy, 'momentum magnitude', direction of momentum (2 angles) and a *pair* of invariant helicities. One of the latter—here called 'velocity helicity'—is angular momentum with respect to velocity direction while the other is the (usual) angular momentum in the direction of momentum. Preon energy is the component of its momentum in the direction of its velocity. The term 'momentum magnitude' has a continuous significance, through Lorentz-group Casimirs, that parallels the significance in nonrelativistic quantum theory of a particle's discrete 'angular-momentum magnitude'.

Commutability of the complete set of 6 corresponding self-adjoint operators derives from commutability of right Lorentz transformations with left transformations. Our usage of the adjectives 'right' and 'left' will be explained. DQC Hilbert space unitarily represents a 12-parameter group—the product of right and left Lorentz groups. Action is *right*-Lorentz invariant, DQC right transformations being those employed by Milne to define a spacetime ⁽³⁾ and which we call 'Milne transformations' to avoid confusion with the Einstein-Poincaré meaning for a Lorentz transformation. The 6 self-adjoint-operator generators of Milne transformations, which do *not* commute with each other, represent preon momentum and angular momentum. DQC path action conserves momentum and angular momentum but not energy—which associates to one of the *left*

spin?

mass
spin

Lorentz generators. Velocity helicity associates to another left generator, one which commutes with preon energy as well as with momentum and angular momentum.

To the best of our knowledge DQC left transformations have no 6-parameter-group precedent in natural philosophy but a 1-parameter left subgroup associates to *local-time* translation (not translation of global time--which we call 'age'). The generator of this left subgroup represents preon (total) energy. Another U(1) left subgroup, generated by velocity helicity, comprises rotations about the velocity direction. In the DQC algebra of self-adjoint operators (which, because DQC accords no *a priori* meaning to 'measurement', we avoid calling 'observables') the two left Lorentz-group Casimirs are *equal* to the two right Casimirs (commuting with all 12 group generators).

Reference (2) addresses the DQC action of branched Feynman paths. The present paper is complementary--ignoring path action while addressing various 6-labeled bases for single-preon Hilbert space. Each basis corresponds to a complete set of 6 commuting self-adjoint operators (a 6-csco). Unitary Hilbert-space *regular* representation of the product of right and left Lorentz groups provides a DQC path-contactable basis that parallels the Feynman-path-contacting *coordinate* basis for Dirac's nonrelativistic quantum theory. ⁽¹⁾ An analog of Dirac's *momentum* basis associates to the unitary irreducible $SL(2, \mathbb{C})$ Gelfand-Naimark (G-N) representation ('unirrep'). ⁽⁴⁾

The algebra of preon self adjoint operators represents in Dirac sense preon spatial location, velocity, polarization, energy, momentum and angular momentum, as well as velocity-helicity and momentum-helicity. Of course not all these operators commute with each other. Each of the DQC bases discussed here associates to a different 6-csco.

In the 'path basis' each preon is 'classically specified' by (a product of) the 6 continuous coordinates of a manifold traversed by Feynman paths comprising straight lightlike 'arcs' which may be created or annihilated as a path progresses. (The local time along any arc has *unit* derivative with respect to global age.) A path propagates some 'ray'--a fixed-age cosmological wave function whose norm lacks significance through probability or otherwise--to the subsequent ray. Each preon of a 'starting' ray contacts exactly one starting arc of a Feynman path. Before reaching the age of the subsequent ray, any path arc may be annihilated at a cubic vertex--an 'event'--where new arcs are created.

One of two 'extra' velocity-associated manifold dimensions, by defining a self-adjoint operator associated to individual-preon local time, allows reality in the 'near-future' of a ray to be prescribed by self-adjoint-operator *expectations* over that ray. The meaning of 'near future' attaches to a DQC 'macroscopic slicing' of Milne spacetime that will be discussed below. Cosmological rays are defined only on slice boundaries.

DQC Fock space comprises sums of Pauli-symmetrized products of normed single-preon functions. At the risk of obscuring total relativity, the present paper chooses to emphasize the unfamiliar labels on which depends a function belonging to an *individual* preon. Special such functions that represent elementary particles (quarks, leptons and weak bosons, together with photons and gravitons) will be exposed.

What we call 'total relativity' recognizes time arrow and absoluteness of motion while respecting Mach's principle in the sense discussed by Wilczek. ⁽⁵⁾ Einstein-Poincaré special relativity ignores time arrow and motion absoluteness; special relativity further disregards Mach. Application of DQC to *physics* requires scale-based approximation; DQC addresses an expanding universe that lacks meaning for

reproducible measurement. The general meaning of 'physics' and of special relativity in particular is confined to spacetime scales tiny compared to that of Hubble while huge compared to that of Planck. Reference (6) presents a Euclidean-group-based physics-scale gravity-less approximation to DQC that may be described as a (Higgsless) 'sliced-spacetime Standard Model' (ssSM).

The DQC (global) *age*, invariant under both right and left transformations, plays a discrete role paralleling that of continuous time in nonrelativistic quantum theory. DQC Feynman paths connect successive macroscopically-spaced *exceptional* ages at each of which is defined a cosmological Fock-space ray—in a sense recalling S-matrix theory. DQC spacetime divides into 'slices' of macroscopic width whose boundaries locate at the exceptional ages. Path branching—path-arc creation or annihilation—is forbidden at slice boundaries where a ray is defined—occurring at ages *interior* to a slice where rays are not defined. DQC dynamics prescribes quantum propagation in the discrete S-matrix sense of an "*in state*" leading to a subsequent "*out state*" without any wave function being defined between *in* and *out*.

Although not discussed in the present paper, the aggregation of DQC Hilbert space, path rules and initial condition "spontaneously" breaks *C*, *P* and *CP* symmetries. Because the DQC Hilbert space represents a group isomorphic to the complex Lorentz group (as does analytic S-matrix theory), it is plausible that some cosmological counterpart to 'CPT symmetry' will eventually become recognized.

The here-examined infinite-dimensional single-preon Hilbert space comprises normed functions of the continuous coordinates (path-basis labels) that 'locate' an individual preon within a 6-dimensional manifold which is at once a right and a left group manifold (*common* Haar measure). The single-preon Hilbert space has as a factor a *finite*-dimensional Hilbert subspace of *discrete* labels, invariant under both right and left continuous transformations, labels that are largely ignored by the present paper. Discrete labels carried by both path and ray distinguish different preon 'sectors' (e.g., electron, up-quark, photon, graviton) by specifying electric charge, color, generation, etc. ⁽⁷⁾

We *shall* here attend to an invariant 2-valued parity-related "handedness" carried both by path arcs (between path-branching points) and by preons—in contrast to preon helicities that are meaningless for a path arc. DQC Hilbert space correlates handedness to sign of helicity in assigning to each preon sector a unique velocity helicity that coincides with momentum helicity.

G-N discussed two different bases for a Hilbert space that represents unitarily the group $SL(2, c)$, ⁽⁴⁾ without attempting for either a natural-philosophical interpretation. The vectors of one basis—analog to the coordinate basis of nonrelativistic Dirac theory—are normed functions over the 6-dimensional (continuous, left-right) manifold. We call this the path basis because DQC Feynman paths traverse this 6-space. G-N's second basis—that we here call 'G-N unirrep'—parallels the Dirac-Fourier-Wigner momentum-spin basis of nonrelativistic quantum theory that unitarily and irreducibly represents the Euclidean group. ⁽⁸⁾ (The 6-parameter compound Euclidean group is a contraction of the 6-parameter semisimple Lorentz group.) Although the transformation connecting the two G-N bases is not entirely of Fourier-Wigner form, wave-function norm is preserved; the transformation is unitary.

We modify the G-N unirrep basis by (unitary) Fourier transformations of wave-function dependence on a pair of complex *directional* labels, so as to diagonalize

simultaneously preon energy—the component of its momentum in its velocity direction—and components of momentum and angular momentum in some arbitrarily-specified direction. A continuous Casimir label, carried over undisturbed from the G-N unirrep basis, we call ‘magnitude of momentum’. Two discrete labels are helicity interpretable—components of angular momentum in velocity and momentum directions. The altered basis, which facilitates meaning for ‘preon parity reflection’, we call the ‘energy-unirrep’ basis.

The universe spacetime identified by Milne in the nineteen thirties ⁽³⁾—an open forward-lightcone interior whose boundary allows ‘big-bang’ interpretation—endows ‘Lorentz transformation’ with a cosmological time-arrowed meaning different from the Einstein-Poincaré special-relativistic physics meaning (ignoring time arrow) that augments Lorentz invariance by spacetime-displacement invariance. (A sufficiently large spacelike or negative-timelike displacement may move a point within Milne spacetime outside that spacetime—i.e., outside the universe.) In contrast to an Einstein boost between different ‘rest frames’ that each assigns a different set of velocities to massive entities within some ‘laboratory’ spacetime-localized region, Milne boosts relate to each other different ‘local frames’ that each associates to a different spatial location. Milne boosts--right DQC transformations--are spatial displacements at fixed universe age in a curved (hyperbolic) 3-space.

Although Milne-spacetime flatness--manifested in DQC by straight lightlike arcs within Feynman paths--might seem incompatible with general relativity’s association of gravity to spacetime curvature, DQC gravitational action at a distance plus creation and annihilation of *soft-gravitonic* arcs enables *discretized curvature* via ‘gentle’ branchings of stationary-action classical paths. ⁽²⁾ Any DQC path, as prescribed in Reference (2), is an ‘event graph’—a set of spacetime-located cubic vertices connected by arcs of positive-lightlike 4-velocity that carry energy as well as discrete attributes. By disregarding Planck’s constant, general relativity ignores gravitons and approximates by a spacetime-curving trajectory (e.g., an electron trajectory) an arc-sector-maintaining stationary-action *sequence* of DQC straight ‘hard’ arcs that are separated at gentle events by ‘soft’ gravitonic-arc absorption or emission.

The adjectives ‘hard’ and ‘soft’ refer to the energy scale set by Planck’s constant times the *inverse* of the age width of a spacetime slice—the time interval that *defines* cosmologically the adjective ‘macroscopic’. Both the Standard Model and the Reference (6) ssSM revision thereof attend to the Planck constant, while making a $G \rightarrow 0$ approximation and achieving flat 3-space through disregard of the Hubble constant--suppressing redshift by regarding universe age as infinite. The ssSM physics approximation to DQC—differs from the Standard Model by its recognition of macroscopic spacetime slicing. The revision accommodates soft photons and thereby maintains capacity (even while ignoring gravity and redshift) to provide an *electromagnetic* theory of physical measurement.

Milne Spacetime

The open interior of a forward lightcone—what we call ‘Milne spacetime’--is the product of a lower-bounded one-dimensional ‘age space’ with an unbounded 3-dimensional ‘boost space’. The spacetime displacement from the forward-lightcone vertex (whose spacetime location is meaningless) to any spacetime point is a positive-

timelike 4-vector $(t, \mathbf{x})$. Defining the “age” τ of a spacetime point to be its Minkowski distance from lightcone vertex—i.e., the Lorentz-invariant modulus $(t^2 - \mathbf{x}^2 c^{-2})^{1/2}$ of its spacetime-location 4-vector—the set of points sharing some common age occupies a 3-dimensional (global) hyperboloid. Any point within such a hyperboloid may be reached from any other by a 3-vector *boost*. Once an origin within boost space is designated, an arbitrary spacetime point is specified by $(\tau, \boldsymbol{\beta})$, where $\boldsymbol{\beta}$ is the 3-vector boost-space displacement from the selected origin to the point. Writing $\boldsymbol{\beta} = \beta \mathbf{n}$, where $\mathbf{n}$ is a unit 3-vector and β is positive,

$$t = \tau \cosh \beta, \quad \mathbf{x} = c\tau \mathbf{n} \sinh \beta. \quad (1)$$

The spatial-location label $\boldsymbol{\beta}$ will in the following section and in Appendix A be identified within the path basis for DQC Hilbert space. Compatibility of age discretization with Milne’s meaning for Lorentz invariance allows DQC’s spacetime to be temporally discrete for its Feynman-path quantum dynamics even though spatially continuous. Discretization occurs at two *different* fundamental scales: (1) Any DQC Feynman path traverses an age-discretized ‘macroscopic slice’ of Milne spacetime—a slice bounded above in age as well as below. (2) Within each slice any (straight and lightlike) arc proceeds in Planck-scale age steps, whose precise value is established in Reference (2) from action quantization.

Although consistency requires slice width to be an integral multiple of arc step, the huge-integer ratio will not be addressed by the present paper—which ignores arc steps. Before attending at all to path arcs this paper chooses to address the single-preon Hilbert-space path basis. Nevertheless the termination of path arcs at exceptional (ray-age) hyperboloids might be taken as *defining* the path basis of preon Fock space.

To each point of boost space associates a “local” Lorentz frame in which $\beta = 0$ —i.e., a frame defined up to a rotation by the point’s location 4-vector being purely timelike in that frame. Age change and time change are equal in local frame. In local frame an infinitesimal spatial displacement $d\mathbf{x}$ at age τ relates to an infinitesimal boost-space displacement $d\boldsymbol{\beta}$ by $d\mathbf{x} = c\tau d\boldsymbol{\beta}$. (A phenomenological meaning for ‘local frame’ resides in the approximate isotropy of cosmic background radiation observed in that frame. This meaning parallels that of standard cosmology’s ‘co-moving coordinates’.) The rotational ambiguity of local-frame meaning is reduced below through an origin of a 6-dimensional space that (arbitrarily) designates not only the origin’s boost-space location but also an attached orthogonal and handed (1, 2, 3) set of 3 reference axes which may be parallel transported along a boost-space geodesic from the origin to any other boost-space location.

The (3-parameter) global orientation ambiguity is accommodated by total relativity—a DQC Fock-space restriction that requires rays to be *globally* rotationally invariant—unchanged when a common Milne rotation is applied to *all* preons. Ray *expectations* of Milne-boost generators are also globally invariant. Total relativity might be said to mean that both the total momentum and the total angular momentum of the universe are zero (6 conditions). The DQC universe is not only rotationally invariant but, associating right-boost generators with infinitesimal spatial displacements at fixed age, the universe is also ‘classically-invariant’ under (non-abelian) boost-space displacements.