

Phil Lucht's old
Math I Binder

Integrals

Misc

Fiber Bundles

Fredholm Theory

Asymptotic Methods

non-Euclidean Geometry

Misc

Miscellaneous Math Notes.

1. jacobians
2. a special kind of jacobian
3. the Poisson distribution
4. practice Stanley integration
5. facts on coordinates (eg, x^u is not really a vector)
6. relation between gradient and normal to a surface
7. a dispersion relation exercise
8. a theorem on exponentiated differential operators (see Hermann?)
9. standard Wick rotation
10. on integrating a scalar function in E^n
11. a peculiar double integral (Hassam)
12. the Greens Theorem and applications
13. generalized fourier integral theorem
14. restricted diagonalization theorem and chains.
15. unitarity, hermiticity, etc.
16. Hilbert Spaces and all that.
17. pseudofunction distributions
18. gaussian quadrature and application to series sums (miklos) ← moved to NUMBER

19. derivation of $\delta(x-y) = -\frac{1}{2\pi i} \oint d\lambda g(x/y, \lambda)$

20. regge integrals.

21. Legendre Transform trick

22. Selected Green's Functions

23. Euclidean $O(4)$ integrals $d\Omega^3 = \dots$ (see Field Oh)

24. About arcsin function.

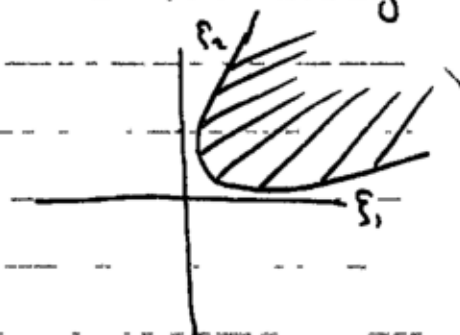
25. Fractionation

26. Gen. binomial theorem.

Jacobians

Suppose you transform variables from (q_1, q_2) to (ξ_1, ξ_2) . The initial integration was over the entire (q_1, q_2) plane.

fact: the domain of the integration in the (ξ_1, ξ_2) plane may be restricted by the requirement that the jacobian be real.



The boundary of the new domain is then determined by: jacobian goes imaginary. In common cases it turns out that

$$\text{jacobian} = \sqrt{H}$$

so the boundary is now $H=0$; valid region is $H>0$. Thus, $\Theta(H)$ is introduced.

lemma: the determinant of a product of two matrices = the det of the product = product of dets.

lemma: $\det(S) = \det(\tilde{S})$.

example: it is easy to see that, if

$$\xi_1 = (p+q)^2$$

$$\xi_2 = (p-q)^2$$

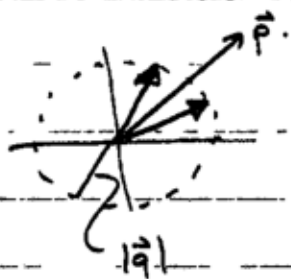
$$d\xi_1 d\xi_2 = J(\xi_1, \xi_2) dq_1 dq_2$$

However, our mapping is not 1-to-1:

$$\frac{\xi_1 + \xi_2}{2} = p^2 + q^2$$

$$\frac{\xi_1 - \xi_2}{2} = 2p \cdot q$$

Given ξ_1, ξ_2 , there are 2 vectors $\vec{q}$ which work. Namely



So we have a 2-to-1 map:

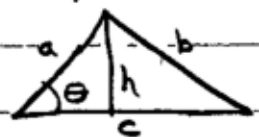


Thus, if we do the $d\xi_1 d\xi_2$ integral, we count all points in the $q_1 q_2$ domain twice, so must correct: when you do the $\xi_1 \xi_2$ integral, $\div 2$: Then:

$$\int_{\text{all space}} dq_1 dq_2 \dots = \int d\xi_1 d\xi_2 \left[\frac{\Theta(-\Delta)}{2\sqrt{-\Delta}} \right]$$

$$\text{or, } \boxed{\tilde{J}(\xi_1, \xi_2) = \frac{\Theta(-\Delta)}{2\sqrt{-\Delta}} = \int dq_1 dq_2 \delta(\xi_1 - \dots) \delta(\xi_2 - \dots)}$$

interpretation of $-\Delta(\xi_1, \xi_2, \xi_3)$ "triangle function"



$$(\text{area})^2 = \frac{1}{4} h^2 c^2 = \frac{1}{4} a^2 c^2 \sin^2 \theta$$

but $\cos \theta = \frac{a^2 + c^2 - b^2}{2ac}$ so

$$\boxed{\text{area} = \frac{1}{4} \sqrt{-\Delta}}$$

$$(\text{area})^2 = \frac{1}{4} a^2 c^2 \left[1 - \left(\frac{a^2 + c^2 - b^2}{2ac} \right)^2 \right] = -\frac{1}{16} \Delta(a^2, b^2, c^2)$$

Jacobians and Delta Functions

Let (x_1, x_2) and (q_1, q_2) be two representations of one point.

Then:

$$dx_1 dx_2 = \left| \frac{\partial(x_1, x_2)}{\partial(q_1, q_2)} \right| dq_1 dq_2$$

Other notations for the jacobian are:

$$J(x_1, x_2 / q_1, q_2) = J(x_1, x_2) = \frac{\partial(x_1, x_2)}{\partial(q_1, q_2)} = J(x_i, q_i)$$

Regardless, the definition is:

$$J(x_i, q_i) = \frac{\partial(x_1, x_2)}{\partial(q_1, q_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} \end{vmatrix}$$

As an example, let's do spherical's to cartesian spatial's:

$$\begin{array}{lll} x_1 = x & q_1 = r & \frac{\partial(x, y, z)}{\partial(r, \cos\theta, \phi)} = \begin{vmatrix} \dots \end{vmatrix} = r^2 \\ x_2 = y & q_2 = \cos\theta & \\ x_3 = z & q_3 = \phi & \end{array}$$

$$dx dy dz = r^2 dr d(\cos\theta) d\phi$$

Delta function products transform as follows:

$$\delta(x_1 - x'_1) \delta(x_2 - x'_2) \delta(x_3 - x'_3) = \frac{1}{|J(x_i, q_i)|} \delta(q_1 - q'_1) \delta(q_2 - q'_2) \delta(q_3 - q'_3)$$

They go just backwards from the volumes, obviously to achieve the desired cancellation so that:

$$\delta(x_1 - x'_1) \delta(x_2 - x'_2) \delta(x_3 - x'_3) dx_1 dx_2 dx_3 =$$

$$\delta(q_1 - q'_1) \delta(q_2 - q'_2) \delta(q_3 - q'_3) dq_1 dq_2 dq_3$$

A special kind of jacobians

- ① Suppose we have four Lorentz scalar functions of a four vector P^μ :

$$\begin{array}{lll} a = a(P) & A_\mu \equiv \frac{da}{dP^\mu} = \partial_\mu a & C_\mu = \dots \\ b = b(P) & & \\ c = c(P) & B_\mu \equiv \frac{db}{dP^\mu} = \partial_\mu b & D_\mu = \dots \\ d = d(P) & & \end{array}$$

Then we may compute the jacobian from P^μ to a, b, c, d as follows:

$$da db dc dd = J \cdot d^4 P$$

$$J = \frac{\partial(a, b, c, d)}{\partial(P^\mu)} = \begin{vmatrix} A_\mu & B_\mu & C_\mu & D_\mu \end{vmatrix} = \left| \begin{pmatrix} 4 \times 4 \text{ matrix} \end{pmatrix} \right|$$

- ② Define a pseudotensor $T_{\mu\nu} \equiv T_{\mu 0} = A_\mu, T_{\mu 1} = B_\mu, \text{ etc.}$
Then:

$$J = |T|$$

- ③ The metric tensor has $|g| = -1$, so we can say:

$$|T| = (-1) |g| |T| = -|gT|$$

OR: $|T_{\mu\nu}| = -|T^\mu{}_\nu|$ since $g^{\mu\nu} T_{\mu\nu} = T^\mu{}_\mu$

Note that we never mess with the (nontensor) index ν .

- ④ Define $\tilde{T}_{\mu\nu} = \begin{bmatrix} A_\mu \\ B_\mu \\ C_\mu \\ D_\mu \end{bmatrix} = \text{transpose.}$

$$\begin{aligned} \text{Now: } J = |T_{\mu\nu}| &= \sqrt{|T_{\mu\nu}| |\tilde{T}_{\mu\nu}|} = \sqrt{|\tilde{T}_{\mu\nu}| |T_{\mu\nu}|} \\ &= \sqrt{|\tilde{T}_{\mu\nu}| (-1) |T^\mu{}_\nu|} = \sqrt{-|\tilde{T}_{\mu\nu}| |T^\mu{}_\nu|} \end{aligned}$$

Therefore: $J = \sqrt{-G(A, B, C, D)}$

This last step in more detail is:

$$\begin{aligned}
 \mathcal{J} &= \int -|\tilde{T}_{\mu\nu}| |T^{\mu}{}_{\nu}| = \left[-|\tilde{T}_{\mu\nu} T^{\mu}{}_{\nu}| \right]^{1/2} \\
 &= \left[-\left| \begin{pmatrix} A_{\mu} \\ B_{\mu} \\ C_{\mu} \\ D_{\mu} \end{pmatrix} (A^{\mu} B^{\mu} C^{\mu} D^{\mu}) \right| \right]^{1/2} = \left[-\begin{vmatrix} A^2 & A \cdot B & A \cdot C & A \cdot D \\ B \cdot A & B^2 & B \cdot C & B \cdot D \\ C \cdot A & C \cdot B & C^2 & C \cdot D \\ D \cdot A & D \cdot B & D \cdot C & D^2 \end{vmatrix} \right]^{1/2} \\
 &= \int -G(A, B, C, D)
 \end{aligned}$$

③ Given the four invariant's, then, one has:

$$\frac{\mathcal{Z}(a, b, c, d)}{\mathcal{Z}(P^{\mu})} = \int -G \left(\begin{matrix} D_a, D_b, D_c, D_d \\ D_a, D_b, D_c, D_d \end{matrix} \right)$$

where $(D_a)_{\mu} = \partial_{\mu} a$

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}$$

Exercise: compute:

$$D = \begin{vmatrix} (p-q)^2 & (q-p) \cdot (q+k) & (q-p) \cdot q \\ (q-p) \cdot (q+k) & (q+k)^2 & (q+k) \cdot q \\ (q-p) \cdot q & (q+k) \cdot q & q^2 \end{vmatrix}$$

Try (-1) third col, add to 1, 2:

$$\begin{cases} (p-q)^2 - (q-p) \cdot q = (q-p) \cdot [q-p-q] = p \cdot (p-q) \\ (q-p) \cdot (q+k) - q(q+k) = (q+k) \cdot [q-p-q] = -p \cdot (q+k) \\ (q-p) \cdot q - q^2 = -p \cdot q \end{cases}$$

$$\begin{aligned} (q-p)[q+k-q] &= k(q-p) \\ (q+k)[q+k-q] &= k(q+k) \end{aligned}$$

$$D = \begin{vmatrix} -p \cdot (p-q) & k \cdot (q-p) & q \cdot (q-p) \\ -p \cdot (q+k) & k \cdot (q+k) & q \cdot (q+k) \\ -p \cdot q & k \cdot q & q^2 \end{vmatrix}$$

$$= (-1) \begin{vmatrix} p \cdot (q-p) & k \cdot (q-p) & q \cdot (q-p) \\ p \cdot (q+k) & k \cdot (q+k) & q \cdot (q+k) \\ p \cdot q & k \cdot q & q^2 \end{vmatrix}$$

Now take (-1) third row, add to 1st, 5th

$$= (-1) \begin{vmatrix} -p^2 & -k \cdot p & -q \cdot p \\ p \cdot k & k^2 & q \cdot k \\ p \cdot q & k \cdot q & q^2 \end{vmatrix} = \begin{vmatrix} p^2 & p \cdot k & p \cdot q \\ k \cdot p & k^2 & k \cdot q \\ q \cdot p & q \cdot k & q^2 \end{vmatrix}$$

$$D = G(p, k, q) !$$

Could convert this to Cayley:

$$\begin{aligned} S_{23} &= (p+k)^2 - s \\ &= (k+q)^2 = s_2 \end{aligned}$$

$$S_{12} = (p+k+q)^2$$

$$= (p+k+q)^2 = s + u' - 2p'(p+k)$$

$$= \begin{matrix} \text{ } \\ \text{ } \\ \text{ } \end{matrix} \begin{matrix} p \\ p' \\ k \end{matrix} \begin{matrix} p \\ p' \\ k \end{matrix} = s_1 + u' + 2k(p-p')$$

0				
1	0			
1	$\sqrt{}$	0		
1	s_2	u'	0	
1	u	s	?	0

Exercise: compute:

$$\begin{aligned}
 D &= \begin{vmatrix} p^2 & p \cdot k & p \cdot q \\ k \cdot p & k^2 & k \cdot q \\ q \cdot p & q \cdot k & q^2 \end{vmatrix} \\
 &= p^2 \begin{vmatrix} k^2 & k \cdot q \\ k \cdot q & q^2 \end{vmatrix} - (k \cdot p) \begin{vmatrix} p \cdot k & p \cdot q \\ q \cdot k & q^2 \end{vmatrix} + (q \cdot p) \begin{vmatrix} p \cdot k & p \cdot q \\ k^2 & k \cdot q \end{vmatrix} \\
 &= p^2 k^2 q^2 - p^2 (k \cdot q)^2 - (p \cdot k)^2 q^2 + (k \cdot p)(p \cdot q)(q \cdot k) + (q \cdot p)(p \cdot k)(k \cdot q) - (q \cdot p)^2 k^2 \\
 &= 2(k \cdot p)(p \cdot q)(q \cdot k) + p^2 k^2 q^2 - p^2 (k \cdot q)^2 - q^2 (p \cdot k)^2 - k^2 (q \cdot p)^2.
 \end{aligned}$$

Now, we can define:

$$\begin{aligned}
 s_1 &= (p+q)^2 = p^2 + q^2 + 2p \cdot q & p^2 &\equiv u & q^2 &\equiv u' \\
 s_2 &= (k-q)^2 = k^2 + q^2 - 2k \cdot q & k^2 &\equiv v
 \end{aligned}$$

$$\begin{aligned}
 s_1 - u - u' &= 2p \cdot q \\
 s_2 - v - u' &= -2k \cdot q \\
 s - u - v &= 2p \cdot k
 \end{aligned}$$

So:

$$\begin{aligned}
 4D &= -(s_1 - u - u')(s_2 - v - u')(s - u - v) + u v u' - u(s_2 - v - u')^2 \\
 &\quad - u'(s - u - v)^2 - v(s_1 - u - u')^2
 \end{aligned}$$

Thus, boundary is Kibble's!

Exercise: special determinant:

$$\textcircled{1} \begin{vmatrix} a & b & c \\ b & a & d \\ c & d & a \end{vmatrix} = a(a^2 - d^2) - b(ba - cd) + c(bd - ac)$$

$$= a^3 - ad^2 - b^2a + bcd + bcd - ac^2$$

$$= a^3 + 2bcd - a(b^2 + c^2 + d^2)$$

~~Exercise: special determinant:~~

The Poisson Distribution

- ① First, let's work in terms of Reif's one-dim. drunk. There, the question was: after N steps, what is the probability that drunk is m steps away from start $[-N \leq m \leq N]$. The answer was:

$$P_N(m) = \frac{N!}{[(N+m)/2]! [(N-m)/2]!} p^{(N+m)/2} (1-p)^{(N-m)/2}$$

$$= \frac{N!}{n_1! n_2!} p^{n_1} q^{n_2}$$

- ② Alt, we could ask: what is the prob. that, in all, he took n_1 steps to the right. That would be:

$$P_N(n_1) = \frac{N!}{(n_1)! (N-n_1)!} p^{n_1} (1-p)^{N-n_1}$$

- ③ Now, if drunk is on a steep hill + n_1 = steps up the hill, maybe p will be small, like $1/100$. In this case $P_N(0)$ will be larger than $P_N(1)$, etc. In fact:

$$P_N(n_1) = \binom{N}{n_1} p^{n_1} (1-p)^{N-n_1}$$

$$P_N(0) = 1 \cdot \left(\frac{9}{10}\right)^N$$

$$P_N(1) = N \cdot \left(\frac{9}{10}\right)^{N-1} \cdot \left(\frac{1}{10}\right) \approx N \cdot \left(\frac{1}{10}\right) \cdot P_N(0)$$

$$P_N(2) = \frac{N(N-1)}{2} \cdot \left(\frac{9}{10}\right)^{N-2} \cdot \left(\frac{1}{10}\right)^2$$

So in this case where $p \ll 1$, the binomial distribution shifts to the left. If $Np > 1$, it starts to rise. Note:

$$P_N(m) = \frac{N!}{m! (N-m)!} p^m q^{N-m}$$

$$P_N(m+1) = \frac{N!}{(m+1)! (N-m-1)!} p^{m+1} q^{N-m-1}$$

So, if $pN < 1$ to start with, you drop from start!

Thus:

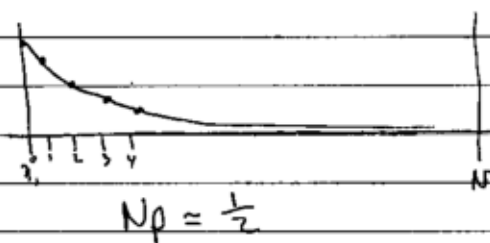
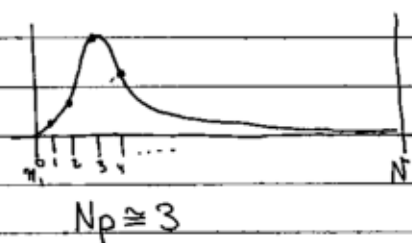
$$\frac{P_N(m+1)}{P_N(m)} = \frac{N!}{m! (N-m)!} \frac{p^{m+1} q^{N-m-1}}{p^m q^{N-m}} \cdot \frac{m! (N-m)!}{(m+1)! (N-m-1)!} = \left(\frac{p}{q}\right) \cdot \frac{N-m}{m+1}$$

$$\approx \left(\frac{p}{q}\right) \cdot \frac{N}{m+1}$$

$$\approx \frac{pN}{m+1}$$

Eventually, as m increases, this ratio must drop below one. That happens when $pN \approx m+1$. $[N \gg m, p \ll 1]$.

- ④ To summarize, we have seen that if $p \ll 1$ and if $n \ll N$, the distribution has this form:



- ⑤ Now the mathematical way to do this is: [prob. 1.9, p 41 Leib]

$$W(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

We have: $(1-p)^{N-n} = X$

$$\frac{(N-n) \ln(1-p)}{\approx N \approx (-p)} = \ln X = -Np \Rightarrow X = e^{-Np}$$

Also: $\frac{N!}{(N-n)!} = \frac{N(N-1)\dots(N-n+1)}{\approx N \cdot N \dots N \approx N^n}$

Thus: $W(n) \approx \frac{N^n}{n!} p^n e^{-Np} \approx \boxed{\frac{(Np)^n}{n!} e^{-Np}}$

This is Poisson. Now

$$\frac{W(n+1)}{W(n)} = \frac{(Np)^{n+1}}{(n+1)!} \frac{n!}{(Np)^n} \approx \frac{Np}{n+1}, \text{ as before.}$$

So, $= 1$ when $Np = n+1$, if ever. This "Poisson distrib" causes graphs as drawn above:

- ⑥ What is Np ? It is the mean # of steps uphill. In fact,
 $\langle n \rangle = \sum n P_n = ?$

In the exact binomial distrib, $\bar{n}_1 = Np$ by calc. Here it should be the same:

$$\langle n \rangle = \sum_{n=0}^{\infty} n \frac{(Np)^n}{n!} e^{-Np} = e^{-Np} \cdot Np \cdot \sum_{n=0}^{\infty} \frac{(Np)^{n-1}}{(n-1)!}$$

Well, $(Np)^0 =$

So you could say:

$$\boxed{W(n) = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle}}$$

⑦ How can we describe Poisson in general terms? You have an ensemble of total events (walks); ^{or set} Each total event is made up of a "sequence" of smaller events. These smaller events are supposed to be uncorrelated! It is the assumption of zero event correlation which leads to binomial or Poisson. (ie, constant prob.)

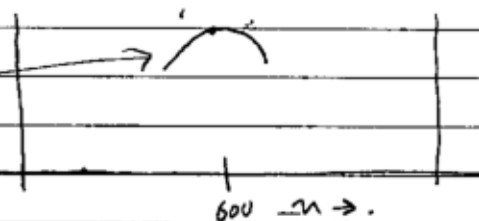
⑧ The typo example (Ref). There are 600 typos in 600 pages. Here the smaller event is "typing a letter". Assume each page has 1000 letters. Then

$$\frac{600}{1000 \cdot 600} = 10^{-3} = p = \text{probability of making a typo.}$$

Here, $N = 1000 \cdot 600$, $Np = 600 = \text{mean \# of typos in ensemble.}$

Now, given the mean # of typos in a book, + given typo de correlation, + given $p \ll 1$, $N \gg \langle n \rangle$, say, we can deduce the error distrib of books!

Shape is $\frac{600^n}{n!} e^{-600} = P(n)$.



→ Eg, what is prob. that given page has 3 typos.

⑨ Now, total event = typing one page. Smaller event = 1 letter. $N = 1000$, letters on one page. Still, $p = 10^{-3}$; $Np = 1 = \langle n \rangle = \text{average errors per page. } n = \text{typos per page.}$

Now, $\frac{1}{n!} e^{-1} = \frac{1}{e} \frac{1}{n!} = W_n = \text{prob distrib of typos.}$

$W(0) = \frac{1}{e} \approx .30 = \text{prob that there are no typos on given page.}$

$W(1) = \frac{1}{e} \approx .30 = \text{prop of one type.}$

$W(2) = .15 = \text{prob of 2 typos.}$

$W(3) = .05 = \text{prob of 3 typos.}$

$W(4) = .01 = \text{prob of 4 typos. etc.}$

Note: $\sum_{n=0}^{\infty} \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle} = 1$ for any $\langle n \rangle$, so don't worry.

⑩ Now, how do we relate Poisson to particle production processes?!

Here is the particle idea: assume you are at a very large S ~~SB~~ and that every event ("large event") produces a spray of 100 particles, mostly pions. This event ~~corresponds~~ corresponds to a page of the preceding example.

Now, assume that the mean number of $\bar{p}$ produced is 3. Thus:

$$\langle n_{\bar{p}} \rangle = 3 = 100 \cdot p \Rightarrow p = .03 \ll 1 \checkmark$$

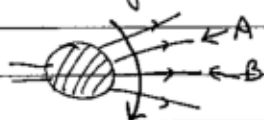
Therefore, since $p \ll 1$, since $N=100$, it is reasonable to assume that:

$$P_{\bar{p}}(n_{\bar{p}}) \approx \frac{\langle n_{\bar{p}} \rangle^{n_{\bar{p}}} e^{-\langle n_{\bar{p}} \rangle}}{n_{\bar{p}}!}$$

Note that "getting a $\bar{p}$ " corresponds to "making a typo".

- ⑪ However, getting a Poisson means decorrelation, no correlation, between "smaller events". What are those "smaller events" in this instance?

Well, order the 100 final particles in some arbitrary way.



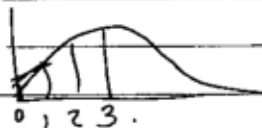
arrows show "sequence of smaller events".

Then no correlation means that "getting a $\bar{p}$ here at A" is represented by a constant probability. Thus, it can have no correlation to "getting a $\bar{p}$ there at B".

Aside: This, a "constant probability" a "no correlation" model would say $F_{AB}^{CD} = F_{AB}^C \cdot F_{AB}^D$, so $f_2 = 0$.

- ⑫ So, assuming that the probability of producing a $\bar{p}$ at each line is a constant, we predict the Poisson.

Eg, $\langle n_{\bar{p}} \rangle = 3 \Rightarrow P(n_{\bar{p}}) = \frac{3^{n_{\bar{p}}} e^{-3}}{n_{\bar{p}}!} =$



- ⑬ Note: consider the shape of a Poisson in a continuous variable:

$$P(x) = \frac{a^x e^{-a}}{x!} \propto \frac{a^x}{\Gamma(x+1)}$$

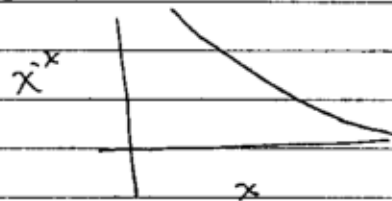
So $P(x) = e^{-a} \cdot a^x \cdot \frac{1}{x!}$ / exact poisson.

What does this look like? For large x we know

$$x! \approx \sqrt{2\pi x} \cdot x^x \cdot e^{-x} \quad \text{So}$$

$$P(x) \approx \frac{e^{-a} \cdot e^{+x \ln a}}{\sqrt{2\pi x} \cdot x^x \cdot e^{-x}} \approx \frac{1}{\sqrt{x}} \cdot x^{-x} \cdot e^{x(1+\ln a)}$$

Well, what does x^{-x} look like?
Does it creep up or down?



$$f(x) = x^{-x} = e^{-x \ln x}$$

$$f'(x) = -e^{-x \ln x} \cdot [x \cdot \frac{1}{x} + \ln x] = -(\ln x + 1) e^{-x \ln x}$$

$$f''(x) = (\ln x + 1)^2 x^{-x} - x^{-x} \frac{1}{x} ?$$

(14) Cleaver to consider shape of $\ln P(x)$ because:

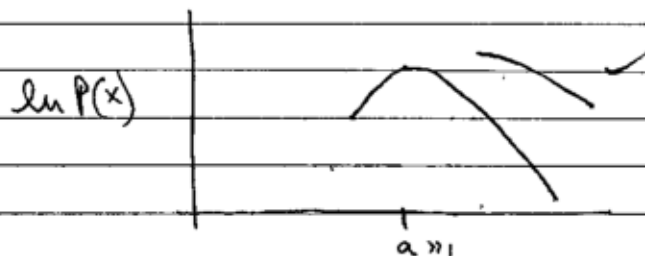
$$\begin{aligned} \ln P(x) &= \ln(e^{-a}) + \ln(a^x) - \ln(x!) \\ &= -a + x \ln a - \{x \ln x - x\} \quad \text{large } x. \end{aligned}$$

$$\ln P(x) = -a + x[\ln a + 1 - \ln x] \approx -x \ln x$$

$$\begin{aligned} \frac{d[\ln P(x)]}{dx} &= [\ln a + 1] - \{x \cdot \frac{1}{x} + \ln x\} = \ln a - \ln x \\ &= -\ln(x/a). \end{aligned}$$

So, if $x \gg 1$, then when $x > \langle x \rangle = a$, slope is negative and increases!

$$\frac{d^2[\ln P(x)]}{dx^2} = -1/x < 0 \text{ but } \rightarrow 0; \text{ always cups down.}$$



So, on the right above $\langle x \rangle$, $P(x)$ gets steeper with no limit. Compare to Horn Zash p 21.

- ⑮ Can we extend idea to where $N \neq \text{constant}$. In any "larger event" in the Poisson limit, the distribution looks the same for any N that is $\gg n$ of interest.

So, in type example, suppose pages varied from 200 letters to 900 letters. But, suppose by count, we determined that $p = 10^{-3}$, i.e. one type per 1000 letters.

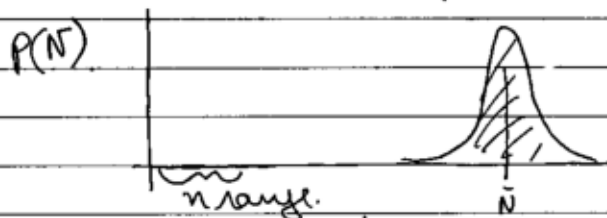
Then ~~$\langle n \rangle$~~ $\langle n \rangle = \text{av. types per page}$ should be proportional to length of page!

Still, the average page might be 600 letters. Then $\langle n \rangle = \text{av. types per average page}$ makes sense. This might still be one.

- ⑯ So, $\langle n \rangle$ means either

- av. # $\bar{p}$ produced per event, when $N = 100$, fixed.
- av. # $\bar{p}$ produced per average event, $N \equiv \bar{N}$.

So, we should have in mind some $\bar{N} \gg n$. There will be some kind of N distribution:

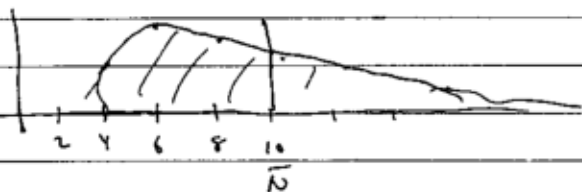


Since we are in effect applying Poisson for each case in this distribution, all N should be $\gg n$.
Then:

$$P_{\bar{N}}(n) = \bar{N} \text{ indep. Poisson, if all } N \text{ in } \mathcal{L} \gg n.$$

Then: $\langle n \rangle = \bar{N} p.$

- ⑰ Is this situation valid in particle stuff? At $p_{\text{lab}} = 500 \text{ GeV}/c$, $\langle n_{\text{ch}} \rangle = \frac{2}{3} \bar{N} \approx 10$, so $\bar{N} = 15$. But, as p. 18 shows, the N distrib is like:



So, still many events have $N \approx 4$.

- ⑮ Dispersion of Poisson? For the binomial, $D = \sqrt{Npq}$.
Taken to the Poisson limit, we will get

$$D = \sqrt{Npq} \approx \sqrt{Np} \approx \sqrt{\langle n \rangle} \quad \text{or} \quad \boxed{D^2 \approx \langle n \rangle}$$

- ⑯ By the way, $D^2 = \langle n^2 \rangle - \langle n \rangle^2$ in general, so
for a Poisson:

$$\langle n^2 \rangle \approx D^2 + \langle n \rangle^2 = \langle n \rangle + \langle n \rangle^2$$

$$\begin{aligned} \text{And } \langle n(n-1) \rangle &= \langle n^2 - n \rangle = \langle n^2 \rangle - \langle n \rangle \\ &= \langle n \rangle + \langle n \rangle^2 - \langle n \rangle \\ &= \langle n \rangle^2 \end{aligned}$$



Thus, $\langle n(n-1) \rangle - \langle n \rangle^2 = 0$ for a Poisson. This is F_{AB}^{cc} related f_2 .

- ⑰ However, for particles, it seems that $D \approx \frac{1}{2} \langle n \rangle$.
This $\Rightarrow$:

$$\langle n^2 \rangle = D^2 + \langle n \rangle^2 = \frac{1}{4} \langle n \rangle^2 + \langle n \rangle^2 = \frac{5}{4} \langle n \rangle^2$$

$$\Rightarrow \langle n(n-1) \rangle = \langle n^2 \rangle - \langle n \rangle = \frac{5}{4} \langle n \rangle^2 - \langle n \rangle$$

$$\Rightarrow \langle n(n-1) \rangle - \langle n \rangle^2 = \underbrace{\frac{1}{4} \langle n \rangle^2}_{\text{yes! See p50 of HZ}} - \langle n \rangle \neq 0!$$

- ⑱ Assuming that $\langle n \rangle \sim \ln s$, then $f_2 \sim (\ln s)^2$ is predicted by multiplicity data alone.

Practice Integration (Stanley)

① Compute $I = \int d\Omega_2 \frac{1}{(a - \hat{p}_2 \cdot \hat{p}_1)(b - \hat{p}_2 \cdot \hat{p}_3)}$ where $\hat{z} \equiv \hat{p}_1 \cdot \hat{p}_3$

② Start with: $\frac{1}{\alpha\beta} = \int_0^1 dx \frac{1}{[\alpha x + \beta(1-x)]^2}$ HARD WAY

Thus:

$$I = \int_0^1 dx \int d\Omega_2 \cdot \frac{1}{D^2}$$

where

$$\begin{aligned} D &= (a - \hat{p}_2 \cdot \hat{p}_1)x + (b - \hat{p}_2 \cdot \hat{p}_3)(1-x) \\ &= [\alpha x + b(1-x)] - \hat{p}_2 \cdot [x\hat{p}_1 + (1-x)\hat{p}_3] \\ &= A - B\hat{p}_2 \cdot \hat{n} \end{aligned}$$

where $A = (a-b)x + b$ $B^2 = x^2 + (1-x)^2 + 2x(1-x)\hat{z}$

③ Choose $\hat{z} = \hat{n}$, so $\int d\phi \rightarrow 2\pi$ and need:

$$\int_{-1}^1 dz' \frac{1}{[A - Bz']^2} = \frac{2}{A^2 - B^2}$$

$$\begin{aligned} A^2 - B^2 &= (a-b)^2 x^2 + b^2 + 2bx(a-b) - x^2 - (1-x)^2 - 2xz(1-x) \\ &= (a-b)^2 x^2 + b^2 + 2b(a-b)x - x^2 - 1 - x^2 + 2x - 2zx + 2zx^2 \\ &= x^2 [(a-b)^2 - 1 - 1 + 2z] + x [2b(a-b) + 2 - 2z] + [b^2 - 1] \\ &= x^2 [(a-b)^2 - 2 + 2z] + x [2(ab - b^2 + 1 - z)] + [b^2 - 1] \\ &= \alpha x^2 + \beta x + \gamma \end{aligned}$$

where:

$$\begin{aligned} \alpha &= (a-b)^2 - 2 + 2z \\ \beta &= 2(ab - b^2 + 1 - z) \\ \gamma &= b^2 - 1 \end{aligned}$$

Thus:

$$\begin{aligned} I &= 2\pi \cdot 2 \cdot \int_0^1 \frac{dx}{\alpha x^2 + \beta x + \gamma} \checkmark \\ &= 4\pi \cdot \frac{1}{R} \ln \left[\frac{2\alpha x + \beta - R}{2\alpha x + \beta + R} \right] \Big|_0^1 \checkmark \\ &= \frac{4\pi}{R} \ln \left[\frac{(2\alpha + \beta - R)(\beta + R)}{(2\alpha + \beta + R)(\beta - R)} \right] \checkmark \end{aligned}$$

$$R \equiv \sqrt{\beta^2 - 4\alpha\gamma}$$

$$\begin{aligned}
 \textcircled{4} \quad 2\alpha + \beta &= 2[(a-b)^2 - 2 + 2z] + 2(ab - b^2 + 1 - z) \\
 &= 2[\cancel{a^2} - \cancel{2ab} + \cancel{b^2} - \cancel{2} + \cancel{2z} + \cancel{ab} - \cancel{b^2} + \cancel{1} - \cancel{z}] \\
 &= 2[a^2 - ab - 1 + z] \equiv -\delta \quad \checkmark \\
 -\beta &= 2[b^2 - ab - 1 + z] \quad \checkmark
 \end{aligned}$$

Thus:

$$I = \frac{4\pi}{R} \ln \left[\frac{(\delta + R)(\beta + R)}{(\delta - R)(\beta - R)} \right] \quad \checkmark$$

$$\delta = 2(1 - z + ab - a^2) \quad \checkmark$$

$$\beta = 2(1 - z + ab - b^2) \quad \checkmark$$

⑤ Compute R :

$$\begin{aligned}
 R^2 &= \beta^2 - 4\alpha\gamma \\
 &= 4(ab - b^2 + 1 - z)^2 - 4[(a-b)^2 - 2 + 2z][b^2 - 1] \\
 &= 4[(ab - b^2)^2 + (1 - z)^2 + 2(1 - z)(ab - b^2) - \{(a-b)^2 - 2 + 2z\}(b^2 - 1)] \\
 &=
 \end{aligned}$$

$$\begin{aligned}
 &= 4[\cancel{a^2 b^2} + \cancel{b^4} - \cancel{2ab^3} + \cancel{1 + z^2} - \cancel{2z} + \cancel{2ab} - \cancel{2zab} - \cancel{2b^2} + \cancel{2zb^2} - \cancel{a^2 b^2} - \cancel{b^4} + \cancel{2ab^3} + \cancel{2b^2} - \cancel{2zb^2} \\
 &\quad + a^2 + b^2 - 2ab - 2 + 2z] = 4[a^2 + b^2 + z^2 - 1 - 2abz] = 4k(a, b, z).
 \end{aligned}$$

Thus: $R = 2\sqrt{k(a, b, z)} \quad \checkmark$

⑥ Compute $\delta\beta + R^2$: Compute $\delta\beta$:

$$\begin{aligned}
 \delta\beta &= 4[(1 - z + ab) - a^2][(1 - z + ab) - b^2] \quad \checkmark \\
 &= 4[(1 - z + ab)^2 - (1 - z + ab)(a^2 + b^2) + a^2 b^2] \quad \checkmark \\
 &= 4\{1 + z^2 + a^2 b^2 - 2z + 2ab - 2abz - a^2 - b^2 + a^2 z + b^2 z - a^3 b - ab^3 + a^2 b^2\} \quad \checkmark \\
 &= 4\{-a^2 - b^2 + 1 + z^2 - 2abz + 2a^2 b^2 - 2z + a^2 z + b^2 z + 2ab - a^3 b - ab^3\}
 \end{aligned}$$

$$\begin{aligned}
 \delta\beta + R^2 &= 4\{2z^2 - 4abz + 2a^2 b^2 - 2z + a^2 z + b^2 z + 2ab - a^3 b - ab^3\} \quad \checkmark \\
 &= 4\{ab(-a^2 - b^2 + 2ab + 2 - 2z) - z(-a^2 - b^2 + 2ab + 2 - 2z)\} \quad \checkmark \\
 &= 4(-a^2 - b^2 + 2ab + 2 - 2z)(ab - z) \quad \checkmark
 \end{aligned}$$

⑦ Compute $\delta + \beta$:

$$\delta + \beta = 2[(1 - z + ab - a^2) + (1 - z + ab - b^2)] = 2[2 - 2z + 2ab - a^2 - b^2] \quad \checkmark$$

So far:

$$\begin{aligned}
 I &= \frac{4\pi}{R} \ln \left[\frac{\delta\beta + R^2 + (\delta + \beta)R}{\delta\beta + R^2 - (\delta + \beta)R} \right] \quad \checkmark \\
 &= \frac{4\pi}{R} \ln \left[\frac{(\delta\beta + R^2)/2(\delta + \beta) + R/2}{\text{"} - R/2} \right]
 \end{aligned}$$

Now $\frac{\delta\beta + R^2}{2(\delta + \beta)} = \frac{4(ab - z)}{2 \cdot 2} = ab - z \quad \frac{R}{2} = \sqrt{k}$

The result is:

$$I = \frac{2\pi}{\sqrt{k}} \ln \left[\frac{ab - z + \sqrt{k}}{ab - z - \sqrt{k}} \right]$$

$$I = -\frac{2\pi}{\sqrt{k}} \ln \left[\frac{z - ab + \sqrt{k}}{z - ab - \sqrt{k}} \right]$$

⑧ Alternative method. Use

EASY WAY

$$\frac{1}{\alpha\beta} = \int_0^\infty \frac{dx}{[\alpha x + \beta]^2}$$

$$\begin{aligned} \text{Then: } D &= (a - \hat{p}_2 \cdot \hat{p}_1)x + (b - \hat{p}_2 \cdot \hat{p}_3) \\ &= ax + b - \hat{p}_2 \cdot [x\hat{p}_1 + \hat{p}_3] \\ &= A - B \hat{p}_2 \cdot \hat{n} \end{aligned}$$

where now: $A = ax + b$

$$B^2 = x^2 + 1 + 2xz \quad \text{if } -1 \leq z \leq 1, \text{ then } B^2 \geq 0, \forall x.$$

Again, $\int d\phi \rightarrow 2\pi$ and

$$\begin{aligned} I &= 2\pi \cdot \int_0^\infty dx \int_{-1}^1 dz' \cdot \frac{1}{[A - Bz']^2} = 2\pi \int_0^\infty dx \cdot \frac{2}{A^2 - B^2} \\ &= 4\pi \int_0^\infty dx \frac{1}{(ax+b)^2 - (x^2+1+2xz)} \quad \checkmark \text{ valid for all } a, b, z. \end{aligned}$$

$$\begin{aligned} D &= a^2x^2 + b^2 + 2abx - x^2 - 1 - 2xz \\ &= x^2(a^2 - 1) + x(2ab - 2z) + (b^2 - 1) \\ &= \alpha x^2 + \beta x + \gamma. \end{aligned}$$

$$\text{Now: } \int_0^\infty dx \cdot \frac{1}{\alpha x^2 + \beta x + \gamma} = -\frac{1}{R} \ln \left(\frac{\beta - R}{\beta + R} \right)$$

$$\begin{aligned} \text{where } R^2 &= \beta^2 - 4\alpha\gamma = (2ab - 2z)^2 - 4(a^2 - 1)(b^2 - 1) \\ &= 4a^2b^2 + 4z^2 - 8abz - 4a^2b^2 + 4a^2 + 4b^2 - 4 \\ &= 4[z^2 + a^2 + b^2 - 1 - 2abz] \\ &= 4 \cdot k \end{aligned}$$

$$\Rightarrow I = \frac{-4\pi}{2\sqrt{k}} \cdot \ln \left[\frac{2ab - 2z - 2\sqrt{k}}{2ab - 2z + 2\sqrt{k}} \right] = -\frac{2\pi}{\sqrt{k}} \ln \left[\frac{z - ab + \sqrt{k}}{z - ab - \sqrt{k}} \right] = I$$

→ ok if $ab > 0$, otherwise in $\alpha\beta$.
→ moral: some ways are better than others!

On the integral $\int \frac{dx}{ax^2+bx+c}$

① Claim is that $I = \frac{1}{R} \ln \left(\frac{2ax+b-R}{2ax+b+R} \right)$.

② Check: $\frac{dI}{dx} = \frac{1}{R} \cdot \frac{(2ax+b+R)}{(2ax+b-R)} \cdot \left[\frac{(2ax+b+R) \cdot 2a - (2ax+b-R) \cdot 2a}{(2ax+b+R)^2} \right]$

$$= \frac{1}{R} \cdot 4a \cdot (2ax+b) \times$$

$$= \frac{1}{R} \cdot \frac{4aR}{(2ax+b-R)(2ax+b+R)} = \frac{4a}{(2ax+b)^2 - R^2}$$

$$= \frac{4a}{4a^2x^2 + 4axb + b^2 - [b^2 - 4ac]} = \frac{4a}{4a^2x^2 + 4axb + 4ac} = \frac{1}{ax^2 + bx + c} \quad \checkmark$$

It must be right

③ Now examine: $\int_{\epsilon}^1 \frac{dx}{ax^2+bx+c} = \frac{1}{R} \left\{ \ln \left(\frac{2a\epsilon+b-R}{2a\epsilon+b+R} \right) - \ln \left(\frac{2a+b-R}{2a+b+R} \right) \right\}$

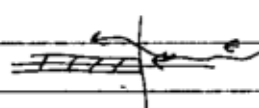
As $\epsilon \rightarrow 0$ and $\epsilon \rightarrow 0$. Now, in particular: $(b-R)$ is > 0 if $a > 0, c > 0$

$$\lim_{\epsilon \rightarrow 0} \ln \left[\frac{2a\epsilon+b-R}{2a\epsilon+b+R} \right] \quad R = \sqrt{b^2 - 4ac}$$

So, if $a > 0, b > 0, c > 0$, then $(b-R) > 0$ $(b+R) > 0$ and we have:

$$\lim_{\epsilon \rightarrow 0} \ln \left[\frac{\epsilon + (b-R)}{\epsilon + (b+R)} \right] = \ln \left[\frac{b-R}{b+R} \right] = \ln(\text{a pos. \#}) \quad \checkmark$$

④ But, suppose $b > 0, a > 0$ but $c < 0$. Then: $(b-R)$ is < 0
and $(b+R)$ is > 0

$$\lim_{\epsilon \rightarrow 0} \ln \left[\frac{\epsilon + (b-R)}{\epsilon + (b+R)} \right]$$


As $\epsilon \rightarrow 0$, we hit branch point!

Choose a branch, so $\ln(-a) = \ln(a e^{+i\pi}) = \ln a + i\pi$.

Then: $\ln \left[\frac{\epsilon + (b-R)}{\epsilon + (b+R)} \right] \rightarrow \ln \left[\frac{R-b}{b+R} \right] + i\pi$

Then integral gives: $\frac{1}{R} \left\{ \ln(1) - \ln \left(\frac{R-b}{b+R} \right) - i\pi \right\} = -\frac{1}{R} \left[\ln \left(\frac{R-b}{b+R} \right) + i\pi \right]$

So all the trouble is in that second term:

$$I = -\frac{1}{R} \ln \left(\frac{a\epsilon+b-R}{a\epsilon+b+R} \right) \quad \text{as } \epsilon \rightarrow 0^+$$

⑤ So $I = -\frac{1}{R} \ln \left[\frac{ae+b-R}{ae+b+R} \right]$

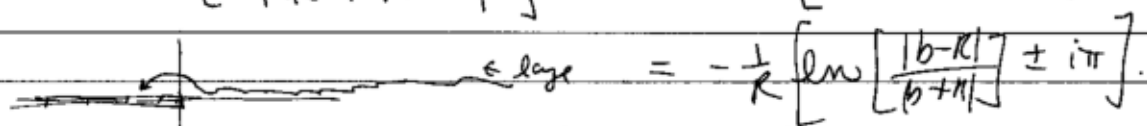
Case I: a, b, c all $> 0 \Rightarrow (b-R) > 0$ so $I = -\frac{1}{R} \ln \left[\frac{b-R}{b+R} \right]$

Case II: $ab > 0$ but $c < 0$ so $(b-R) < 0$. Then $I = ?$

$$I = -\frac{1}{R} \left[\ln \left(\frac{R-b}{b+R} \right) \pm i\pi \right], \text{ not even real!}$$

Case III: $cb > 0, a < 0$. Then: $(b-R) < 0$ again.

$$I = -\frac{1}{R} \ln \left[\frac{-|a|e - |b-R|}{-|a|e + |b+R|} \right] = -\frac{1}{R} \ln \left[\frac{|a|e + |b-R|}{|a|e - |b+R|} \right]$$



Case IV: $ac > 0, b > 0 \dots$

The only time I can see clearly what is happening is if $b-R > 0$.
and:

$$\begin{aligned} (b-R) > 0 &\Rightarrow b > R \Rightarrow b > 0 \text{ and } ac > 0 \\ (b+R) > 0 &\Rightarrow b > -R \end{aligned}$$

or

$$\begin{aligned} (b-R) < 0 &\Rightarrow b < R \Rightarrow b < 0 \text{ and } ac > 0 \\ (b+R) < 0 &\Rightarrow b < -R \end{aligned}$$

$$ac > 0 \Rightarrow R < |b|. \quad b < 0 \Rightarrow R < -b \Rightarrow b < -R.$$

So the real safety condition is $ac > 0$?

But $a < 0$ is also suspicious. For some ϵ , this will be neg, so not "smooth".

$$\left(\frac{-|a|\epsilon + c}{-|a|\epsilon + d} \right)$$

Insist that:

$$\boxed{a > 0, c > 0}$$

Thus:

$$\int_0^{\infty} \frac{dx}{ax^2+bx+c} = -\frac{1}{R} \ln \left[\frac{b-R}{b+R} \right]$$

only if $a > 0$ and $c > 0$

same!

$$\text{ie, } \int_0^{\infty} \frac{dx}{ax^2+bx+c} = - \int_0^{\infty} \frac{dx}{-ax^2-bx-c} = +\frac{1}{R} \ln \left[\frac{-b-R}{-b+R} \right] = \frac{1}{R} \ln \left[\frac{b+R}{b-R} \right]$$

if $a < 0, c < 0$.

Restate:

If a, b, c all > 0 , or if $a, c > 0$ but $b < 0$, then in either case,

$$\frac{|b-R|}{|b+R|} > 0$$

Because: $ac > 0 \Rightarrow R < |b|$. $b > 0 \Rightarrow R < b$ so $(b-R) > 0$
 $b < 0 \Rightarrow R < -b$ so $(b-R) < 0$
 but $(b+R) < 0$ also.

So, as long as $ac > 0$,
 I trust the integral,
 for either sign of b . But I prefer $a > 0$.

Case 1: a, b, c all > 0 . Then:

$$\int_0^{\infty} \frac{dx}{ax^2+bx+c} = \int_0^{\infty} \frac{dx}{|a|x^2+|b|x+|c|} = -\frac{1}{R} \ln \left[\frac{|b|-R}{|b|+R} \right] = -\frac{1}{R} \ln \left(\frac{b-R}{b+R} \right).$$

Case 2: $a < 0, c < 0, b > 0$ $= +\frac{1}{R} \ln \left(\frac{b+R}{b-R} \right) > 0$.

$$\int_0^{\infty} \frac{dx}{ax^2+bx+c} = \int_0^{\infty} \frac{dx}{-|a|x^2+|b|x-|c|} = -\int_0^{\infty} \frac{dx}{|a|x^2-|b|x+|c|}$$

$$= +\frac{1}{R} \ln \left[\right]$$

Case 2 $\int_0^{\infty} \frac{dx}{|a|x^2 \pm |b|x + |c|} = -\frac{1}{R} \ln \left[\frac{\pm |b|-R}{\pm |b|+R} \right]$.

So: $\int_0^{\infty} \frac{dx}{-|a|x^2 \pm |b|x - |c|} = -\int_0^{\infty} \frac{dx}{|a|x^2 \mp |b|x + |c|} = +\frac{1}{R} \ln \left[\frac{\mp |b|-R}{\mp |b|+R} \right]$.

So: $\int_0^{\infty} \frac{dx}{ax^2+bx+c} = \frac{1}{R} \ln \left[\frac{-b-R}{-b+R} \right] = \frac{1}{R} \ln \left[\frac{b+R}{b-R} \right]$.

$a < 0 \quad c < 0$

So, result is same no matter what. All you need is $ac > 0$.

Facts about Coordinates

- ① A coordinate itself is not a "vector" because you cannot show that it has vectorial transformation properties:

$$x^i \neq \frac{dx^i}{dx^j} x^j$$

Example: in spherical coordinates, if x^i were a true vector, then the quantity:

$$g_{ij} x^i x^j$$

would be a scalar. But $x^i = (r, \theta, \phi)$, so

$$g_{ij} x^i x^j = 1 \cdot r^2 + r^2 \cdot \theta^2 + r^2 \cdot \sin^2 \theta \cdot \phi^2$$

Then we would say that, since $g_{ij} x^i x^j = \text{scalar}$, we can evaluate it in a cartesian frame. There, we get r^2 , so clearly a contradiction.

- ② The quantity $\frac{dx^i}{dp}$ is a true vector, assuming p is a scalar.
Proof:

$$\frac{dx^i}{dp} = \frac{dx^i}{dx^j} \frac{dx^j}{dp} \quad \checkmark$$

Thus, we expect $g_{ij} dx^i dx^j$ to be a true scalar.
Here,

$$dx^i = (dr, d\theta, d\phi)$$

$$g_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 = ds^2$$

In a cartesian we get:

$$ds^2 = g_{ij} dx^i dx^j = g'_{ij}(x') dx'^i dx'^j = dx^2 + dy^2 + dz^2$$

- ③ Remember that "scalar" does not always mean a # like π . It includes scalar field. Scalar fields vary from event to event.

example: $ds^2 = ds^2(r, \theta, \phi) = \text{scalar field!}$

5.

- ④ Suppose $\vec{A}$ is a vector in E^3 . We can imagine some true contravariant components for A in a spherical coord system. I do not now know how to express these contravariant components in terms of cartesian components, but I can say:

$$\vec{A} \cdot \vec{A} = g_{ij} A^i A^j = (A^1)^2 + r^2 (A^\theta)^2 + r^2 \sin^2 \theta (A^\phi)^2$$

Since $\vec{r}$ is not a true vector, neither is $\hat{r}$, and $\vec{A} \cdot \hat{r}$ is not a scalar. Thus, we cannot compare:

$$\vec{A} \cdot \hat{r} = (A^1, A^\theta, A^\phi) \cdot (1, \theta, \phi) = (A^1)^2 + r^2 A^\theta \theta + r^2 \sin^2 \theta A^\phi \phi$$

$$\vec{A}' \cdot \hat{r}' = (\vec{A} \cdot \hat{r}) \equiv r A$$

This is just nonsense.

- ⑤ However, Weinberg seems to suggest that we can identify:

$$(\vec{A} \cdot \hat{r}) = r A = A^r$$

$$(\vec{A} \cdot \hat{\theta}) = A = r A^\theta$$

$$(\vec{A} \cdot \hat{\phi}) = A = r \sin \theta A^\phi$$

This identification works as far as ds^2 goes. But is it really true?

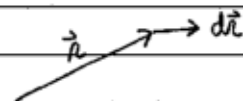
- ⑥ Examine hypothesis ⑤ for special case: $\vec{A} = d\vec{r}$ at $\vec{r}$. Then we know that $A^1 = dr$, $A^\theta = d\theta$, $A^\phi = d\phi$. Can we prove that:

$$d\vec{r} \cdot \hat{r} = (dr)$$

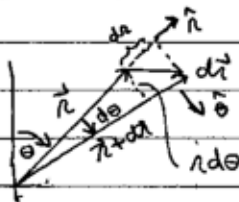
$$d\vec{r} \cdot \hat{\theta} = r(d\theta)$$

$$d\vec{r} \cdot \hat{\phi} = r \sin \theta (d\phi)$$

?



Note: here, $d\vec{r} \neq dr \hat{r}$. $d\vec{r}$ is a general thing, should call it $d\vec{x}$. Geometrically, the first two seem logical:



I think we can check these equations indirectly by going to Euclid:

$$d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$\begin{aligned} \text{Thus, } d\vec{r} \cdot \hat{r} &= dx(\hat{x} \cdot \hat{r}) + dy(\hat{y} \cdot \hat{r}) + dz(\hat{z} \cdot \hat{r}) \\ &= dx \cdot x/r + dy \cdot y/r + dz \cdot z/r \\ &= \frac{d\vec{r} \cdot \vec{r}}{r} = (x dx + y dy + z dz)/r \\ &= ? \end{aligned}$$

Is this dr ? Well,

$$dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy + \frac{\partial r}{\partial z} dz = \frac{x}{r} dx + \frac{y}{r} dy + \frac{z}{r} dz = d\vec{r} \cdot \hat{r} \quad \checkmark$$

The other two are harder to show, but I now believe it.

Finally, circularly we would say $dr^i = (dr, d\theta, d\phi)$

$$\Rightarrow d\vec{r} = (dr)\hat{r} + (r d\theta)\hat{\theta} + (r \sin\theta d\phi)\hat{\phi}$$

⑦ Accepting ansatz ③ [and Weinberg's claim], we find:

$$\begin{aligned} A^r &= \vec{A} \cdot \hat{r} = r A \\ r A^\theta &= \vec{A} \cdot \hat{\theta} = \theta A \\ r \sin\theta A^\phi &= \vec{A} \cdot \hat{\phi} = \phi A \end{aligned}$$

where A are the "normal" components:

$$\vec{A} = (r A)\hat{r} + (\theta A)\hat{\theta} + (\phi A)\hat{\phi}$$

⑧ Application: in Weinberg p403 we have

$$f = \frac{d\vec{r} \cdot d\vec{r} + k(\vec{r} \cdot d\vec{r})^2}{1 - k\vec{r} \cdot \vec{r}}$$

In cartesian this is obvious.

In sphericals, we have:

$$\begin{aligned} d\vec{r} &= (dr)\hat{r} + (r d\theta)\hat{\theta} + (r \sin\theta d\phi)\hat{\phi} \\ \vec{r} \cdot d\vec{r} &= dr \cdot r \\ \vec{r} &= r\hat{r} \\ \vec{r} \cdot \vec{r} &= r^2 \\ d\vec{r} \cdot d\vec{r} &= ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= dr^2 + r^2 d\Omega^2 \end{aligned}$$

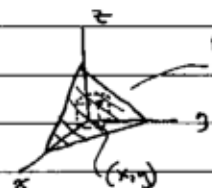
Thus,

$$f = \frac{dr^2 + r^2 d\Omega^2 + k dr \cdot r^2}{1 - k r^2} = \frac{dr^2}{1 - k r^2} + r^2 d\Omega^2$$

Relation between gradient and normal to a surface

① In $\mathbb{R}^3$, we can define a 2-dim. surface by:

$$z = f(x, y) \quad \text{OR} \quad g(x, y, z) = 0.$$

Eg,  The (111) plane is: $x+y+z=1$.

$$\text{So: } z = (-x-y+1) = f(x, y) \quad \checkmark$$

$$(x+y+z-1) = g(x, y, z) = 0.$$

② The two-dimensional gradient of $f(x, y)$ is an arrow in the domain x - y which points in the direction you would go to get the largest increase in $f(x, y)$:

Why? $f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x} (x - x_0) + \frac{\partial f}{\partial y} (y - y_0)$

$$\Leftrightarrow f(\vec{x}) = f(\vec{x}_0) + \vec{\nabla} f \cdot (\vec{x} - \vec{x}_0)$$

$$f(\vec{x} + \Delta \vec{x}) = f(\vec{x}) + \vec{\nabla} f \cdot \Delta \vec{x}$$

Clearly, if $\Delta \vec{x}$ is parallel to $\vec{\nabla} f$, this term takes its maximum value for given $|\Delta \vec{x}|$, so $\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$ is the direction of steepest ascent.

In our example above, $\vec{\nabla} f = -\hat{x} - \hat{y}$, so you head back to the origin from (x, y) to get ascent. ~~(x, y, z) to (0, 0, 1)~~

So $(-\vec{\nabla} f)$ gives you ~~the~~ the azimuth of the fall line."

③ Now what is the 3-gradient of $g(x, y, z)$??

$$g(\vec{x} + \Delta \vec{x}) = g(\vec{x}) + \vec{\nabla} g \cdot \Delta \vec{x} \quad \Delta \vec{x} = (dx, dy, dz) \text{ now}$$

If $\vec{x}$ lies on $g(\vec{x}) = 0$, then:

$$g(\vec{x} + \Delta \vec{x}) = \vec{\nabla} g \cdot \Delta \vec{x}$$

In our example, $\vec{\nabla} g = \hat{x} + \hat{y} + \hat{z}$.

So, we have said: if $\vec{x}$ lies on $g(\vec{x})=0$, then

$$g(\vec{x} + \Delta\vec{x}) = \nabla g \cdot \Delta\vec{x} \quad \text{for any } \Delta\vec{x}.$$

Now, if $\Delta\vec{x}$ also lies on $g(\vec{x})=0$ surface, then

$$g(\vec{x} + \Delta\vec{x}) = 0 \Rightarrow \nabla g \cdot \Delta\vec{x} = 0 \quad \forall \Delta\vec{x} \text{ on } g.$$

Therefore, we conclude that ∇g is a vector in $\mathbb{R}^3$ which is normal to the surface $g=0$.

In our example, it is true.

④ Generalizations. Let $g(x_1, x_2, \dots, x_n) = 0$ describe some surface in $\mathbb{R}^n$. Then

$\nabla_n g$ will be normal to that surface.

(i.e., if $\Delta\vec{x}$ = "vector" on surface, $\nabla g \cdot \Delta\vec{x} = 0$!)

⑤ Let $\vec{P}$ be a point on $g=0$. i.e., $g(\vec{P})=0$. If we desire to leave point $\vec{P}$ in $\mathbb{R}^n$ in a direction that is normal to $g=0$, then we should go to $d\vec{P}$ such that:

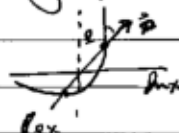
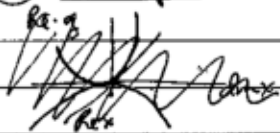
$$\Delta\vec{P} \propto \nabla g. \quad \text{eg, } d\vec{x} = \nabla g \times d\eta$$

where η is a one-dim. differential.

⑥ Let $g(x_1, \dots, x_n) = 0$ be surface in $\mathbb{C}^n$ instead of $\mathbb{R}^n$. Then still, $\nabla_n g$ = "normal" in sense that $\nabla_n g \cdot \Delta\vec{x} = 0$.

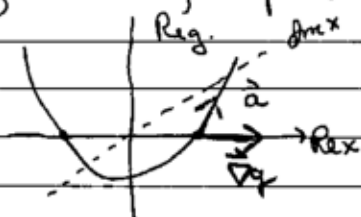
The proof is exactly the same. The Taylor expansion works for real & complex variables. It's just a little hard to picture.

⑦ Example: Let $g(x) = x^2 - 4$ in $\mathbb{C}$. Then $\nabla g = 2x$.



Next sheet please!

⑦ Example: let $g(x) = x^2 - 4$, $x \in \mathbb{C}$. We have claimed that $\nabla g = 2x$ is "normal" to the surface $g=0$. Try drawing a picture:



At $x = 2 + i0$, $\nabla g = 4$, i.e., ∇g points along the real axis.

If we consider a little arrow $\vec{a}$ lying on $g=0$ (wait, how do we draw this?)

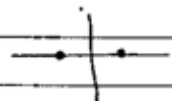
Where is "surface" $g=0$?

$$g(z) = z^2 - 4 = (x + iy)^2 - 4 = x^2 - y^2 - 4 + 2ixy.$$

$$\operatorname{Re} g = 0 \Rightarrow x^2 - y^2 - 4 = 0$$

$$\operatorname{Im} g = 0 \Rightarrow 2xy = 0 \Rightarrow x \text{ or } y = 0.$$

But, if $y=0$, then $x = \pm 2$.



If $x=0$, then $y^2 = -4$, no solution. There is no place where $g=0$, hence no $g=0$ surface. "Surface" is two points in $\mathbb{C}$.

⑧ Example: $g(x,y) = x + y$. Now already too many variables to plot the domain. But, $g=0 \Rightarrow 2$ conditions on 4 variables, so $g=0$ is now a true surface of 2-intrinsic dimensions. (locally a plane).

Normal is: $\sim \nabla g$, so

$$\begin{aligned} dx &= \frac{\partial g}{\partial x} d\eta \\ dy &= \frac{\partial g}{\partial y} d\eta \end{aligned}$$

Altho. $\partial g / \partial x$ and $\partial g / \partial y$ are fixed complex numbers at some (x,y) on $g=0$, $d\eta$ is an arbitrary complex #.

Thus, the departure from $\vec{x}$ along the normal "plane" does indeed have two degrees of freedom. i.e., the normal "plane" is in fact a plane.

⑨ Example: $g(x_1, x_2, \dots, x_n) = 0$. $d\vec{x}_0 = \nabla g d\eta$.

Even here, the normal is still a plane. $g=0$ is a surface of $2n-2 = 2(n-1)$ dimensions, generally not a plane. note:

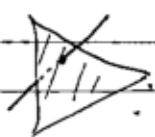
$$\dim \text{ of surface } g + \dim \text{ of normal plane} = 2n.$$

⑩ Verification of "normal surface".

▷ In $\mathbb{E}^3$, draw a plane for $g=0$, eg, $x+y+z-1=0$.

Pick a point on this plane, $\vec{r}$.

The "surface" which is normal to $g=0$ at $\vec{r}$ is a line.



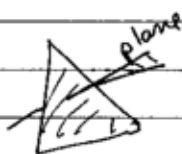
$$\text{Here, } \dim(g=0) + \dim(\vec{v}) = 2 + 1 = 3 \checkmark$$

▷ In $\mathbb{E}^3$ draw a line for $g=0$. This requires $g_1=0$ and $g_2=0$, two constraints on 3 variables. Eg, $x=2, y=2$.

Then the surface normal to $g=0$ is a plane. Again:

$$\dim(g_i=0) + \dim(\vec{v}) = 1 + 2 = 3.$$

⑪ Now back to $g(\vec{z}_1, \vec{z}_2)=0$. This is a plane in $\mathbb{E}^4$. (2 constr. on 4 var.). But, the normal surface is also a plane, somehow: This is what $\mathbb{E}^4$ does for you.



Dispersion Relation Exercise

1. Real analytic functions. We know from the Schwarz reflection principle that if a function $f(z)$ is real on some segment of the real z -axis, then it is true that $f(z^*) = f^*(z)$. That is, f is real analytic.
2. Consider $w = z^{\frac{1}{2}}$. We know that $z = 0$ is the branch point and that we are free to choose the branch cut wherever we like. The branch cut merely serves to define analytic branches of the function. Since there are two branches, there are two Riemann sheets. However, if we choose the cut to lie along the positive real axis, the function thus defined is not real analytic, since on the negative real axis the function is imaginary. If, on the other hand, the cut is chosen to lie along the negative real axis, the function then is real analytic.
3. For a real analytic function cut somewhere along the real axis, the discontinuity of the function across the cut is $2i$ times the imaginary part of the function above the cut.
4. In the dispersion relation technique, a contour around a point z is unwrapped and rewrapped around all cuts and poles of the function. Also, there is always a great circle piece to be dealt with. As long as the magnitude of the function tends to 0 as $|z|$ gets large, the great circle contribution may be dropped.
5. If $f(z)$ is analytic (real analytic) in the entire z plane except for one cut along the real axis from a to b , the dispersion relation is:

$$f(z) = \frac{1}{\pi} \int_a^b \frac{\text{Im} \{f(z' + i\epsilon)\} dz'}{z' - z} \quad z \text{ complex}$$

6. As an example, consider the function $z^{-\frac{1}{2}}$. It is cut along the negative real axis; it is real analytic by 1. above; it vanishes on the great circle. The imaginary part of this function just above the cut is

$$\text{Im} \{(z' + i\epsilon)^{-\frac{1}{2}}\} = \text{Im} \{[|z'| e^{i(\pi - \epsilon)}]^{-\frac{1}{2}}\} = |z'|^{-\frac{1}{2}} e^{-i\pi/2} = -|z'|^{-\frac{1}{2}}$$

and the dispersion relation may be written as.

$$\frac{1}{\sqrt{z}} = z^{-\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \frac{dz'}{\sqrt{z'}(z' + z)} \quad (= \frac{1}{\pi} \int_{-\infty}^0 \frac{[-|z'|^{-\frac{1}{2}}] dz'}{z' - z})$$

7. We can look up this integral in Gradshteyn Ryzhik page 285 number 3. Here, $B(x, y)$ is the beta function of page 950 in terms of gamma functions.

$$\int_0^\infty \frac{x^{-\frac{1}{2}} dx}{(1 + \beta x)} = \beta^{-\frac{1}{2}} B(\frac{1}{2}, \frac{1}{2}) = \frac{\pi}{\sqrt{\beta}}$$

8. For z just above the cut, we may use the rule

$$\frac{1}{z \pm i\epsilon} = P\left(\frac{1}{z}\right) \mp i\pi \delta(z)$$

to write the general dispersion relation as follows:

$$f(z + i\epsilon) = \frac{1}{\pi} \int_a^b \frac{\text{Im} f(z') dz'}{z' - z - i\epsilon} = \frac{1}{\pi} P \int_a^b \frac{\text{Im} f(z') dz'}{z' - z} + i \text{Im} f(z + i\epsilon)$$

Theorem on exponentiated differential operators.

① What is: $e^{\alpha D_x} [x f(x)]$?

$$= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} D_x^n [x f(x)] \quad D_x^n [x f(x)] = x D_x^n f(x) + n D_x^{n-1} f(x)$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} x D_x^n f(x) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} n D_x^{n-1} f(x)$$

Let $m = n-1$ so 2nd term is: $\sum_{m=0}^{\infty} \frac{\alpha^m \cdot \alpha}{m!} D_x^m f = \alpha e^{\alpha D_x} \cdot f$

Thus:

$$= x e^{\alpha D_x} f + \alpha e^{\alpha D_x} f = (x + \alpha) e^{\alpha D_x} f$$

② Theorem then is:

$$e^{\alpha D_x} (x f(x)) = (x + \alpha) e^{\alpha D_x} (f(x))$$

③ Here is another rather obvious theorem:

$$e^{\alpha D_x} f(x) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} D_x^n f(x) = f(x) + \alpha \left. \frac{\partial f}{\partial x} \right|_x + \frac{\alpha^2}{2!} \left. \frac{\partial^2 f}{\partial x^2} \right|_x + \dots$$

$$= f(x + \alpha)$$

This operator translates the function as we well know:

$$e^{\alpha D_x} f(x) = f(x + \alpha)$$

Some algebraic facts.

① $x^2 + bx + c = (x + \alpha)(x + \beta)$ where $\alpha = +\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4c}$
 $\beta = +\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4c}$

Math. Problem

① Consider:
$$f(a) = \oint_0 dz e^{z^2} \cdot \int_{-\infty}^{\infty} dx e^{-x(a-z)}$$

$$= \oint dz e^{z^2} \cdot g(a, z)$$

We have a problem here: the internal integration diverges for all values of a and z . Yet, we wonder if it is not still possible to get some meaning for $f(a)$? The usual suggestion is to break the thing into two pieces, so:

$$g(a, z) = g_+(a, z) + g_-(a, z)$$

$$g_+(a, z) = \int_0^{\infty} dx e^{-x(a-z)} = \frac{1}{(z-a)} \quad \text{when } \operatorname{Re}(z) > a$$

$$g_-(a, z) = \int_0^{\infty} dx e^{+x(a-z)} = \frac{1}{(a-z)} \quad \text{when } \operatorname{Re}(z) < a$$

But $\operatorname{Re}(z) = 0$ in our example, so:

$$g_+(a, z) = \frac{1}{z-a} \quad \operatorname{Re}(a) < 0$$

$$g_-(a, z) = \frac{1}{a-z} \quad \operatorname{Re}(a) > 0.$$

Then let: $f(a) = f_+(a) + f_-(a).$

$$f_+(a) = \oint dz e^{z^2} \int_0^{\infty} dx e^{-x(a-z)}$$

$$= \oint dz e^{z^2} \cdot \frac{1}{(z-a)}$$

$$\operatorname{Re}(a) < 0$$

$$\operatorname{Re}(a) < \operatorname{Re}(z).$$

$$= \oint_{25} dz e^{z^2} \cdot \frac{1}{(z-a)}$$

Here we have taken a to be real and negative, and we are allowed to slide the contour to the right because the contour has "weak ends". Similarly,

$$f_-(a) = \oint dz e^{z^2} \frac{1}{(a-z)}$$

$$\operatorname{Re}(a) > \operatorname{Re} z$$

$$= - \oint_{-25} dz e^{z^2} \frac{1}{(z-a)}.$$

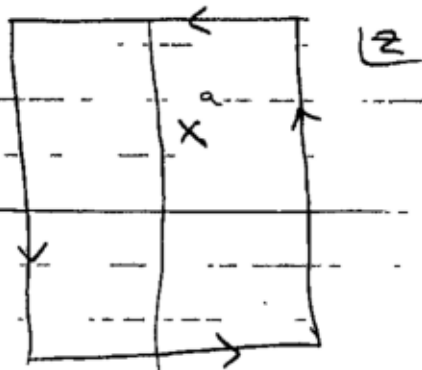
2. Now, we extend the definition of both functions to the complete plane.

This gives us:

$$f_+(a) + f_-(a) = \int_{25}^{\uparrow} dz e^{z^2} \frac{1}{(z-a)} - \int_{-25}^{\downarrow} dz e^{z^2} \frac{1}{(z-a)}$$

$$= \left(\int_{25}^{\uparrow} - \int_{-25}^{\downarrow} \right) dz e^{z^2} \frac{1}{(z-a)}$$

Contour appears as:
Add top + bottom
pieces because
of "weakness"
condition, e^{z^2} has



Then

$$= \oint_a dz e^{z^2} \frac{1}{z-a} = \underline{\underline{2\pi i e^{a^2}}}$$

So:

$$f(a) = 2\pi i e^{a^2}$$

3. Question: what the hell have I really done here???

4. Suppose you try to interchange:

$$f(a) = \int_{-\infty}^{\infty} dx e^{-x^2} \cdot \underbrace{\int_{I_1}^{\uparrow} dz e^{z^2} e^{xz}}_{I_1} \quad z = ip$$

Well

$$I_1 = i \int_{-\infty}^{\infty} dp e^{-p^2} e^{ipx} = i \sqrt{\frac{\pi}{1}} \cdot e^{(-ix)^2/4}$$

$$= i \sqrt{\pi} \cdot e^{-x^2/4}$$

Then

$$f(a) = i \sqrt{\pi} \int_{-\infty}^{\infty} dx e^{-\frac{1}{4}x^2 - ax} \quad I_2 = \sqrt{\frac{\pi}{1/4}} e^{a^2/4 \cdot 1/4}$$

$$= 2\pi i e^{a^2}$$

$$= 2\sqrt{\pi} e^{a^2}$$

Comments Related to the Math Problem.

1. Hassam and I looked closer at the functions:

$$g(a, z) = \int_{-\infty}^{\infty} dx e^{-x(a-z)} \quad z = \text{pure imag}$$

$$= g_+(a, z) + g_-(a, z)$$

$$g_+(a, z) = \int_0^{\infty} dx e^{-x(a-z)} = \frac{1}{z-a} \quad \text{when } \text{Re } a > \text{Re } z$$

$$g_-(a, z) = \int_{-\infty}^0 dx e^{-x(a-z)} = \frac{-1}{z-a} \quad \text{when } \text{Re } a < \text{Re } z$$

Our first conclusion was that there was no region of the a -plane in which both g_+ and g_- are defined. Therefore, there is no place in the a -plane where you can "add" these two functions to get some meaning for $g(a, z)$ itself.

2. But, we overlooked one thing: when a lies on the imaginary axis in the a plane, the complete integral $g(a, z)$ is defined in the distribution sense, ie,

$$\int_{-\infty}^{\infty} dx e^{-x(a-z)} = 2\pi \delta(ia - iz) \quad \text{if } (a-z) = \text{imaginary}$$

$$= g(a, z)$$

However, knowing $g(a, z)$ as a distribution on a line in the a -plane is not enough to analytically continue g to the entire a -plane. We decided that you cannot continue a delta function to complex argument, at least via this definition as an exponential. Ie, the exponential diverges if argument of delta leaves the real line (ie, a leaves the imaginary line).

3. However, in our example we had another stage of integration sitting outside, so that when a lies on the imag axis, the double integral really does converge:

$$f(a) = \int da e^{az} \cdot 2\pi \delta(a'' - z'') = i \int_{-\infty}^{\infty} dz'' e^{-z''} \cdot 2\pi \delta(a'' - z'')$$

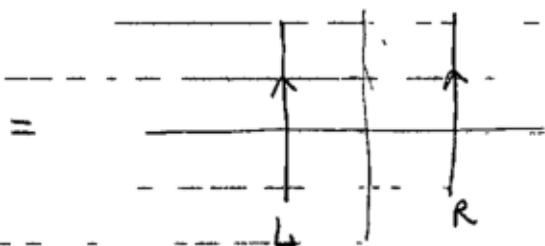
$$= 2\pi i e^{a^2} \quad \text{when } a = ia''$$

Now that we know $f(a)$ on a line in the a -plane in the regular function sense, we may continue $f(a)$ to the entire a -plane! I think this line of reasoning is actually reasonable. It may even allow a definition of complex delta function.

4. Finally, some comments about our original way of looking at this. Recall that we added a temporary converging factor, broke the contour into two pieces, then set factor to zero. It turns out that was correct!

$$f(a) = \oint dz e^{+z^2} [g_+(a, z) + g_-(a, z)]$$

$$= \oint_{\text{left}} dz e^{z^2} g_+(a, z) + \oint_{\text{right}} dz e^{z^2} g_-(a, z)$$



Now, if $g_-(a, z) = \frac{-1}{z-a}$ $g_+(a, z) = \frac{1}{z-a}$
 This gives earlier result.

Green's Theorem and Applications

① The Theorem states:

$$\int_V d^3x [\psi \nabla^2 \phi - \phi \nabla^2 \psi] \\ = \int_S d\vec{a} \cdot [\psi \vec{\nabla} \phi - \phi \vec{\nabla} \psi]$$

② Proof: we do know that

$$\int_V d^3x \vec{\nabla} \cdot \vec{A} = \int_S d\vec{a} \cdot \vec{A}$$

So let $\vec{A} = \psi \vec{\nabla} \phi - \phi \vec{\nabla} \psi \rightarrow \vec{\nabla} \cdot \vec{A} = \psi \nabla^2 \phi - \phi \nabla^2 \psi$, Q.E.D.③ Application to non-rel quantum mechanics: for a particle in some potential we have:

$$\frac{d}{dt} P_r(t) = \frac{d}{dt} \int_V d^3x \psi^* \psi = \int d^3x [\dot{\psi}^* \psi + \psi^* \dot{\psi}] \\ = (-i) \int d^3x [\psi^* H \psi - (H \psi)^* \psi]$$

where $H = -\frac{\nabla^2}{2m} + V$. For a free particle this becomes:

$$\frac{dP_r(t)}{dt} = \frac{+i}{2m} \int_V d^3x [\psi^* \nabla^2 \psi - \nabla^2 \psi^* \psi]$$

Now use Green's Theorem to get $= \frac{+i}{2m} \int_S d\vec{a} \cdot [\psi^* \vec{\nabla} \psi - \vec{\nabla} \psi^* \psi]$ Thus, we may identify the flux as:

$$\frac{dP_r(t)}{dt} = - \int_S d\vec{a} \cdot \vec{S} \Rightarrow \vec{S} = \frac{i}{2m} [\psi^* (\vec{\nabla} \psi) - (\vec{\nabla} \psi)^* \psi]$$

Also, $\vec{\nabla} \cdot \vec{S} + \frac{dS}{dt} = 0 \Leftrightarrow$ this obvious equation.This is the only way to find out what the flux is. Note
that a real ψ can have no flux.

④ Application in electrostatics: Recall how Green's Function is response to a unit charge:

$$\nabla^2 G(\underline{x}, \underline{x}') = -4\pi \delta(\underline{x} - \underline{x}')$$

Green's Theorem is:

$$\int d^3x [\phi \nabla^2 \psi - \psi \nabla^2 \phi] = \int d\vec{a} \cdot [\phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi]$$

Also we know that

$$\nabla^2 \phi = -4\pi \rho$$

Thus, let $\phi = \phi$, and $\psi = G(\underline{x}, \underline{x}')$.

$$\text{LHS} = \int d^3x' [\phi(\underline{x}') \nabla'^2 G(\underline{x}, \underline{x}') - G(\underline{x}, \underline{x}') \nabla'^2 \phi']$$

$$= -4\pi \phi(\underline{x}) + 4\pi \int d^3x' G(\underline{x}, \underline{x}') \rho(\underline{x}')$$

$$\text{RHS} = \int d\vec{a} \cdot [\phi(\underline{x}) \vec{\nabla} G(\underline{x}, \underline{x}') - G(\underline{x}, \underline{x}') \vec{\nabla} \phi]$$

Ergo,

$$\phi(\underline{x}) = \int d^3x' \rho(\underline{x}') G(\underline{x}, \underline{x}')$$

$$+ \frac{\int d\vec{a}'}{4\pi} [G(\underline{x}, \underline{x}') \vec{\nabla}' \phi(\underline{x}') - \phi(\underline{x}') \vec{\nabla}' G(\underline{x}, \underline{x}')]]$$

There is lots of freedom in obtaining a $G(\underline{x}, \underline{x}')$, usually you can get a G such that $G=0$ on some surface. (Dirichlet).

Then:

$$\phi(\underline{x}) = \frac{1}{4\pi} \int d\vec{A}' \cdot \phi(\underline{x}') \vec{\nabla}' G(\underline{x}, \underline{x}')$$

↑ potential everywhere.

↑ potential on your surface.
known green's function

The Generalized Fourier Integral.

① Start off with some $F(x)$. Break the usual Fourier integral into two pieces:

$$f^{(+)}(k) = \int_0^{\infty} dx e^{-ikx} F(x)$$

$$f^{(-)}(k) = \int_{-\infty}^0 dx e^{-ikx} F(x) = \int_{-\infty}^0 dx e^{+ikx} F(-x)$$

Choose $a=0$, though this is not necessary. Then state both at once:

$$f^{(\pm)}(k) = \int_0^{\infty} dx e^{\mp ikx} F(\pm x)$$

The recovery formula is:

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} f^{(+)}(k)$$

② So far, k has been real, and we have had to assume that $F(x) \rightarrow 0$ at $x = \pm\infty$. Suppose $F(x)$ does not have this behavior. We can generalize the above projections like so:

$$f^{(+)}(k' - ib) = \int_0^{\infty} dx e^{-i[k' - ib]x} F(x) = \int_0^{\infty} dx e^{-ik'x} e^{-bx} F(x) \quad b > 0$$

ie, run k down to the lower half plane ($k' = \text{real}$). Go down as far as you need to converge this integral. For $f^{(-)}$, we do the opposite:

$$f^{(-)}(k' + ib) = \int_{-\infty}^0 dx e^{-i[k' + ib]x} F(x) = \int_{-\infty}^0 dx e^{+ik'x} e^{-bx} F(-x)$$

③ We now have projections. It is easy to verify that the recovery formula becomes:

$$F(x) = F^{(+)}(x) + F^{(-)}(x)$$

$$\text{where } F^{(+)}(x) = \frac{1}{2\pi} \int_{-\infty - ib}^{+\infty - ib} dk e^{ikx} f^{(+)}(k) = \frac{1}{2\pi} \cdot e^{-bx} \int_{-\infty}^{\infty} dk' e^{ik'x} f^{(+)}(k' - ib)$$

$$F^{(-)}(x) = \frac{1}{2\pi} \int_{-\infty + ib}^{+\infty + ib} dk e^{ikx} f^{(-)}(k) = \frac{1}{2\pi} \cdot e^{-bx} \int_{-\infty}^{\infty} dk' e^{ik'x} f^{(-)}(k' + ib)$$

In fact, by inserting the projections, you get:

$$F^{(+)}(x) = F(x) \Theta(x)$$

$$F^{(-)}(x) = F(x) \Theta(-x)$$

④ Summary:

$$f^{(\pm)}(k' \mp ib) = \int_0^{\infty} dx e^{\mp i[k' \mp ib]x} F(\pm x)$$

$$\text{or: } f^{(\epsilon)}(k' - \epsilon ib) = \int_0^{\infty} dx e^{-\epsilon i[k' - \epsilon ib]x} F(\epsilon x) \quad \epsilon = \pm, -$$

$$\text{or: } f^{(\epsilon)}(k' - i\epsilon b) = \int_0^{\infty} dx e^{-i\epsilon k'x} \underbrace{e^{-b\epsilon x}}_{\uparrow} F(\epsilon x)$$

This last form shows how b adds converging power. The recoveries are:

$$F(x) = \frac{1}{2\pi} \sum_{\epsilon} e^{\epsilon bx} \int_{-\infty}^{+\infty} dk' e^{ik'x} f^{(\epsilon)}(k' - i\epsilon b)$$

$$\text{or: } F(x) = \frac{1}{2\pi} \sum_{\epsilon} \int_{-\infty - i\epsilon b}^{+\infty - i\epsilon b} dk e^{ikx} f^{(\epsilon)}(k)$$

⑤ Example:

$$F(x) = e^x$$

$$b=2$$

$$f^{(+)}(k' - 2i) = (1 + ik')^{-1}$$

$$f^{(-)}(k' + 2i) = (3 - ik')^{-1}$$

$$\text{Then } F(x) = \frac{1}{2\pi} \left\{ e^{2x} \int_{-\infty}^{\infty} dk' e^{ik'x} (1 + ik')^{-1} + e^{-2x} \int_{-\infty}^{\infty} dk' e^{ik'x} (3 - ik')^{-1} \right\}$$

$$= \frac{1}{2\pi} \left\{ e^{2x} \cdot 2\pi e^{-x} \theta(x) + e^{-2x} \cdot 2\pi e^{+2x} \theta(-x) \right\}, \text{ by closing contours}$$

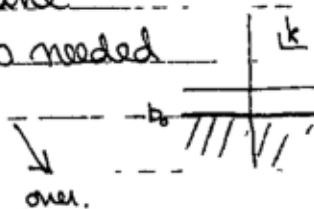
$$= e^x \theta(x) + e^x \theta(-x) = e^x \quad \checkmark$$

⑥ Recycle:

$$\left[\begin{aligned} f^{(\epsilon)}(k) &= \int_0^{\infty} dx e^{-i\epsilon kx} F(\epsilon x), \quad k = k' - i\epsilon b, \quad \epsilon = \pm, - \\ F(x) &= (2\pi)^{-1} \sum_{\epsilon} \int_{C(-i\epsilon b)} dk e^{ikx} f^{(\epsilon)}(k) \end{aligned} \right]$$

where $C(-i\epsilon b)$ means a contour of $k = k' - i\epsilon b$

⑥.5 Remark: $f^{(\pm)}(k)$ is analytic in k in lower half plane below line $\text{Im} k = -b$, where b is the value of b needed to just converge the integral.



② Double - Fourier Theorem:

original:

$$f(k_1, k_2) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{-i[k_1 x_1 + k_2 x_2]} F(x_1, x_2)$$

$$F(x_1, x_2) = (2\pi)^{-2} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 e^{+i[k_1 x_1 + k_2 x_2]} f(k_1, k_2)$$

generalized:

$$\begin{cases} f^{(\epsilon_1, \epsilon_2)}(k_1, k_2) = \int_0^{\infty} dx_1 \int_0^{\infty} dx_2 e^{-i[\epsilon_1 k_1 x_1 + \epsilon_2 k_2 x_2]} F(\epsilon_1 x_1, \epsilon_2 x_2) \\ F(x_1, x_2) = (2\pi)^{-2} \sum_{\epsilon_1, \epsilon_2} \int_{-i\epsilon_1 b_1}^{\infty} dk_1 \int_{-i\epsilon_2 b_2}^{\infty} dk_2 e^{i[k_1 x_1 + k_2 x_2]} f^{(\epsilon_1, \epsilon_2)}(k_1, k_2) \end{cases}$$

where $k_i = k'_i - i\epsilon_i b_i$. Choose $b_1, b_2 > 0$ so integrals converge.

New, "restricted diagonalization theorem"

① Suppose we have: $A(g) = \int dg_1 B(g_1) C(g_1^{-1}g)$

where $\int dg_1$ means an integral only over some piece of G .

I claim that the diag. still works:

② Apply $\int dg D_{mn}^{\sigma}(g)$ to both sides, where $\int dg$ is a full group integral. Then:

$$A_{mn}^{\sigma} = \int dg D_{mn}^{\sigma}(g) \cdot \int dg_1 B(g_1) C(g_1^{-1}g)$$

Now change order to get, with group property,

$$A_{mn}^{\sigma} = \int dg_1 B(g_1) \sum_s D_{ms}^{\sigma}(g_1) \cdot \int dg D_{sn}^{\sigma}(g_1^{-1}g) C(g_1^{-1}g)$$

③ Crucial point: $\int dg = \int d[g_1, g_2]$. g_1 is some fixed $\in G$.
Thus:

$$\int_{g \in G} dg = \int_{g_2 \in G} d[g_1, g_2] = \int_{g_2 \in G} dg_2$$

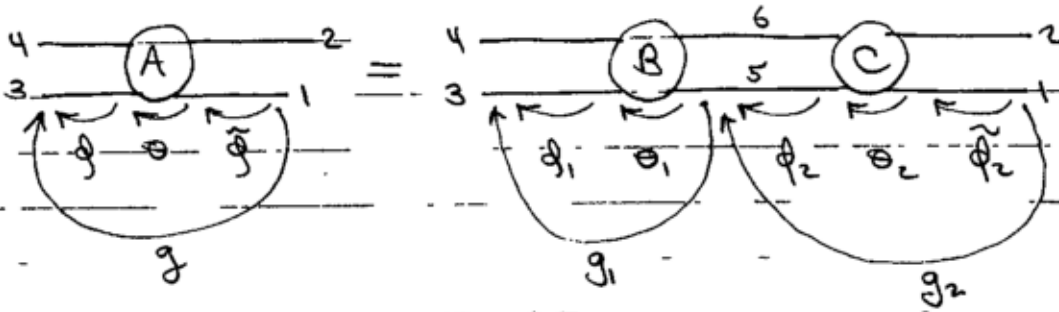
The fact that g_1 does not run over all of G does not matter!
So:

$$\begin{aligned} A_{mn}^{\sigma} &= \sum_s \int dg_1 B(g_1) D_{ms}^{\sigma}(g_1) \cdot \int dg_2 D_{sn}^{\sigma}(g_2) C(g_2) \\ &= \sum_s B_{ms}^{\sigma} \cdot C_{sn}^{\sigma} \end{aligned}$$

The new thing is that now B_{ms} is not a full projection!

Reduces elastic unitarity using new theorem.

① The variable $\tilde{\phi}_1$ will never appear. We have:



$$B(q_1) = \langle 34 | T^\dagger | R_z(\phi_1) R_y(\theta_1) | 56 \rangle = e^{-i\phi_1(h_3-h_4)} T_{34,56}^{(-)}(s, \theta_1)$$

$$C(q_2) = e^{-i\phi_2(h_5-h_6)} e^{-i\tilde{\phi}_2(h_1-h_2)} T_{56,12}(s, \theta_2)$$

② Our equation is:

$$A_{34,12}(g) = \sum_{5,6} \int d\phi_1 d(\cos\theta_1) B_{34,56}(q_1) C_{56,12}(q_1, g)$$

At once, our diag. equation is:

$$A_{mn}^l = \sum_{5,6} \sum_s B_{ms}^l C_{sn}^l$$

③ The projection are:

$$A_{mn}^l = (8\pi^2)^{-1} \int d\phi d\tilde{\phi} d(\cos\theta) \cdot e^{im\phi} e^{-in\tilde{\phi}} \cdot d_{mn}^l(\theta)^* \cdot e^{-i\phi(h_3-h_4)} e^{-i\tilde{\phi}(h_1-h_2)} \cdot \text{disc}_s T_{34,12}(s, \theta) \cdot c$$

$$= \frac{1}{2} c \cdot \delta_{m, h_3-h_4} \delta_{n, h_1-h_2} \cdot \underbrace{\int_{-1}^1 dz d_{mn}^l(\theta)^* \text{disc}_s T_{h_3-h_4, h_1-h_2}}_{T_{h_3-h_4, h_1-h_2}^j}$$

Similarly for C.

$$C_{sn}^l = \frac{1}{2} c \cdot \delta_{s, h_5-h_6} \delta_{n, h_1-h_2} \cdot C_{h_5-h_6, h_1-h_2}^j$$

④ Now look at projection of B :

$$B_{ms}^j = \int dg_1 \, D_{ms}^\sigma(g_1) B(g_1)$$

$$g_1 = (\phi_1, \theta_1, 0) \quad dg_1 = d\phi_1 d(\cos\theta_1)$$

$$D_{ms}^{\sigma*}(g_1) = e^{+i\phi_1 m} d_{ms}^\sigma(\theta_1) \cdot 1$$

$$\begin{aligned} \text{Thus: } B_{ms}^j &= \int d\phi_1 d(\cos\theta_1) \cdot e^{i\phi_1 m} d_{ms}^{\sigma*}(\theta_1) \cdot e^{-i\phi_1(h_1-h_2)} B(\theta) \\ &= \sum_{m, h_1-h_2} B_{ms}^j \end{aligned}$$

There is no constraint here on s !!

⑤ Now go back to

$$A_{34,12,mn}^l = \sum_{s,6} \sum_s B_{34,56,ms}^l C_{56,12,sn}^l$$

$$\begin{aligned} &\sum_{m, h_3-h_4} \sum_{n, h_1-h_2} A_{34,12}^{34,12}(h_3-h_4, l, h_1-h_2) \\ &= \sum_{h_5, h_6} \left(\sum_s \sum_{m, h_3-h_4} B_{34,56}^{34,56}(h_1-h_2, l, s) \right) \\ &\quad \times \left(\sum_s \sum_{n, h_1-h_2} C_{56,12}^{56,12}(h_5-h_6, l, h_1-h_2) \right) \end{aligned}$$

$$\text{Then: } \boxed{A(h_3-h_4, l, h_1-h_2) = \sum_{h_5, h_6} B(h_1-h_2, l, h_5-h_6) C(h_5-h_6, l, h_1-h_2)}$$

⑥ Same answer, but better method!

Partial Chain Diagonalizations.

1. In general, if we have a long "chain" group integral equation, only the rightmost group integral need run over the entire group manifold. The first group integrations can be arbitrarily partial and the thing will still diagonalize. However, the projections will be correspondingly partial:

$$A(g) = \int dg_1 \int dg_2 B(g_1) C(g_2) D(g_2^{-1} g_1^{-1} g)$$

$$\Rightarrow \tilde{A}_{mn} = \sum_{s_1, s_2} \tilde{B}_{ms_1} \tilde{C}_{s_1 s_2} \tilde{D}_{s_2 n}$$

but,

$$\tilde{B}_{ms_1} = \int dg_1 \tilde{D}_{ms_1}(g_1) B(g_1)$$

↑ a "partial" projection...

Unitarity, Hermiticity, etc.

There is still some nonuniformity in my understanding of these ideas as they apply to finite matrices and as they apply to differential equations. For matrices these concepts seem trivial and need not be related even to operators. For DE's on the other hand, there seem to be questions of domain, invariance of certain scalar products, etc etc. Want here to show that these ideas are really the same.

To get a uniform notion, let's be careful to define the words "hermitian" and "unitary" properly. We shall define these ideas only for operators in a Hilbert space (ie, with a scalar product); the matrix notion will be a mere by product. Here are the definitions:

$$\text{hermitian: } \langle \sigma f | i \rangle = \langle f | \sigma i \rangle \quad \text{or} \quad \sigma^\dagger = \sigma$$

$$\text{unitary: } \langle \sigma f | \sigma i \rangle = \langle f | i \rangle \quad \text{or} \quad \sigma^\dagger \sigma = 1$$

First, let's use these definitions to get out the usual matrix statements: Suppose we have a finite dimensional Hilbert space with an orthonormal basis given by states i . Let the dimensionality of the space be n . In this special case we get:

$$\text{herm: } \sigma_{ij} = \langle i | \sigma | j \rangle = \langle \sigma i | j \rangle = \langle j | \sigma i \rangle^* \\ = \sigma_{ji}^*$$

$$\text{uni: } \delta_{ij} = \langle i | j \rangle = \langle i | \sigma^\dagger \sigma | j \rangle = \langle \sigma i | \sigma j \rangle \\ = \langle i | \sigma^\dagger \sigma | j \rangle = \sigma_{ik}^* \sigma_{kj}$$

$$\text{uni: } \sigma_{ij}^\dagger = \langle i | \sigma^\dagger | j \rangle = \langle j | \sigma | i \rangle^* = \sigma_{ji}^*$$

Nothing new or interesting here. Now think of σ as a differential operator and consider hermiticity: We must have some kind of function scalar product. Work in one variable simplest case of $\sigma = d/dx$. Then

$$\langle f | g \rangle = \int dx f^*(x) g(x)$$

$$\langle \sigma f | i \rangle = \langle f | \sigma i \rangle \Rightarrow \int dx [\tilde{\sigma} f(x)]^* g(x) = \int dx f^*(x) [\tilde{\sigma} g(x)]$$

$$? \Rightarrow \int dx [d/dx f(x)]^* g(x) = \int dx f^*(x) [d/dx g(x)]$$

$$\rightarrow = \text{desired, plus } f^*(x) g(x) \Big|_a^b$$

This example shows that the idea of a hermitian differential operator actually depends on the interval of the scalar product and the nature of the functions considered. You must make these parts go away. In this case, you can achieve that by either: (1) letting interval be infinite. Then, if a function has finite norm on that interval, it must vanish at infinity, thus the parts vanish. (2) If you use a finite interval, you can still win by making all your functions periodic on that interval.

Thus, I now see that the idea of hermiticity is really associated with the idea that the "parts" term vanishes. For a more general second order differential operator, the "parts" stuff is called "the bilinear conjunct" of the two functions.

Even if this conjunct is nonvanishing, you can still speak of the "adjoint" of your original differential operator. Apart from the problem of the conjunct not vanishing, this adjoint is the hermitian conjugate. Now, an operator which is self adjoint is only hermitian by my definition of hermitian if the conjunct vanishes, i.e., if the scalar product is chosen right, or if your functions are restricted to be periodic on the interval or whatever.

So, as far as I am concerned, a differential operator is hermitian if it is self-adjoint and the conjunct vanishes. You can arrange for the conjunct to vanish by choosing your scalar product in a certain way, or by imposing a boundary condition on your useable functions. Remember that "conjunct" refers to those left over parts terms.

Now we want to know if an exponentiated hermitian operator is unitary. The answer is going to be yes and here is why: any power of a hermitian operator is also hermitian, as we prove below. If we define the exponentiated operator via its power series, then boom. Watch:

$$H \text{ hermitian} \Rightarrow \langle Hf | g \rangle = \langle f | Hg \rangle \quad \forall f, g$$

$$\Rightarrow \langle H^2 f | g \rangle = \langle Hf | Hg \rangle = \langle f | H^2 g \rangle \quad \forall f, g \Rightarrow H^2 \text{ hermitian, etc.}$$

$$\begin{aligned} \langle Uf | Ug \rangle &= \langle e^{iH} f | e^{iH} g \rangle = \left\langle \sum_{n=0}^{\infty} \frac{i^n}{n!} H^n f \middle| e^{iH} g \right\rangle \\ &= \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \langle f | H^n e^{iH} g \rangle = \langle f | e^{-iH} e^{iH} g \rangle = \langle f | g \rangle \end{aligned}$$

Example: $e^{iH} = e^{-i(x^2/2x)} = e^{2/x}$

Thus, whether or not you have a differential operator, the hermiticity of H always guarantees the unitarity of H exponentiated with an i . If you look inside, you will see that the fact that the conjunct vanishes is used over and over again to make this work. By the way, notice that a function must be infinitely differentiable in order to be acted upon by any exponentiated differential operator. In the case here of d/dx , it turns out that the shift invariance of the scalar product is equivalent to the condition that the conjunct vanish.

Application: in group theory we are used to constructing UIR's by producing a Hilbert space with Hermitian generators as diff ops. If we take any normal scalar product and adjust it in some way, we may break the hermiticity of the generators by causing the conjunct not to vanish; thus, we also break the unitarity of the representation. Nevertheless, your non-unitary representation which results will still have the group property.

Hilbert Spaces and all that. A. Facts leading up to definition of Hilbert Space

1. We start with a linear space = vector space. This object has all the usual properties of vector addition. Also, it is defined over either a real or complex field, and thus has other usual properties like distributiveness.
2. The most primitive structure you can impose on such a linear space is a metric $d(x,y)$. This says something about the "distance" between two vectors, but it says nothing about one vector itself, except that the distance between a vector and itself is zero. - This is a metric space.

The whole point of having a metric = notion of distance between two points is that you can talk about convergence of sequences in the space. The notion of convergence of vectors is defined simply in terms of convergence of the metric $d(x_n, x)$ to zero.

The definition of a "convergent sequence" x_n to x ; and of a "Cauchy sequence" is clear. The usual problem with Cauchy sequences is that $d(x_n, x_m)$ may get very small, but the limit point x may not lie in your space! A convergent sequence is always Cauchy, but Cauchy is only convergent if all limit points are in the space. When this is the case, metric space is "complete in the metric".

3. Suppose you go back to your bare linear space and define the notion of length of a vector, called norm, rather than notion of distance between two vectors. You now have a normed linear space. This norm, or length concept, naturally defines a distance-between-two-vectors concept in this way: $d(x,y) = \|x-y\|$. In other words, the distance between two vectors is taken as the length of the single difference vector. Very natural indeed. Very euclidean. If the norm is chosen right (say usual euclidean length), then the space will be complete in the natural metric. Definition: Banach Space = normed linear space complete in its natural (induced) metric.
- 3.1 Existence of a norm implies the existence of a metric; namely, the natural metric.
4. Now go back once more to the bare linear space. This time, give it a scalar product. Roughly, this is a way of multiplying two vectors to get a scalar. Recall that bare vector space only provides for the addition of vectors. We now have a much higher structure.

First, this scalar product induces a natural norm: $\|x\| = \text{SQRT}(x,x)$. The "scalar product" of a vector with itself defines a natural length, though you could in principle define length of x in some other way.

Second, you take your naturally induced norm $\|x\|$ and form from it a natural metric, just as in 3 and 3.1 above.

If you choose your scalar product nicely (and your vector space of course), then the space will hopefully be complete in the metric that comes out at the end of the tunnel: scalar product, norm, metric. A scalar product space which has the natural norm and natural metric and which is complete in this metric is called a Hilbert Space.

B. properties of a Hilbert Space ; notion of basis

1. Notion of separability. We want ultimately to have a basis in the Hilbert space. In order to have this, you must be able to say that, given a vector f , you can get arbitrarily close by taking a linear combination $\sum a_i s_i$ where the set (s_i) is called a spanning set of vectors. Obviously, for this even to make sense, the set (s_i) must be a countable set, otherwise you could not put them in order $i=1, 2, 3, \dots$.

[A set (s_i) is said to "span" the Hilbert Space if you can get very close to f by doing $\sum a_i s_i$. In mathematical language, the set of vectors you can make by doing $\sum a_i s_i$ must be dense in the Hilbert Space.]

If a Hilbert Space is "separable", then [you can definitely find a countable spanning set] and you can make sense out of $\sum a_i s_i$. We never deal with HS's which are not separable.

If your HS is of finite dimension, then any spanning set must be countable because the spanning set is of finite order (duh). A finite dimensional HS is always separable.

2. Suppose we have a spanning set s_i for some Hilbert Space. We can use this set to get arbitrarily close to any element of this HS, but we might not be able to get exactly to that element for some reason.

This is easily understood in terms of the complete metric mentioned earlier. Let $f = \sum a_i s_i$. Then if this sequence is a Cauchy sequence, the distance between two elements in the tail gets as small as you like, $d(f_n, f_m)$. But we know that this does not necessarily mean that the sequence converges to some f unless the metric is complete.

Probably Cauchy convergence is enough to make $\sum a_i s_i$ dense in HS. Thus, s_i is a spanning set. If the induced metric is complete, then

The nice example is L^2 . Here, t^n forms a spanning set but not a basis because you cannot exactly expand $/t/$ in terms of t^n . After all, any linear combination of t^n is differentiable, but $/t/$ is not. However, $/t/$ is in L^2 . Notice that the basis t^n is not orthonormal.

3. Lets reorganize the last paragraph. The notion of a spanning set has to do with being able to get close to a function by summing $\sum a_i s_i$ up to some large N . But the fact that spanning set s_i is a spanning set does not guarantee that you can exactly arrive at your desired vector by an infinite expansion.

However, [if you can do infinite expansion,] then spanning set is a basis. Theorem: (start with any spanning set (not a basis, say). Then orthonormalize the spanning vectors using Gram-Schmidt. Then this orthonormal spanning set will be a basis!) Thus, you can find an infinite expansion for $/t/$ in terms of the $P_j(t)$. ~~what I said before about differentiability was wrong~~. You can not infinite expand $/t/$ in t^n , however, and not orthonormal.

4. In the above paragraph, we define "basis" as a spanning set which allows infinite representation of any element f of HS. Such a basis is by definition a "complete basis" in the sense of completeness relation $1 = \sum (e_i | e_i /$ where I take to be orthonormal. You prove this completeness relation by applying to arbitrary vector f . The coefficients of the infinite representation of f are then just $(f | e_i) = a_i$.
5. This idea of completeness in "complete basis" is not related to the idea of a complete metric. To have a Hilbert Space in the first place, the metric must be complete.

Pseudofunction distributions.

1. The simple example is this:

$$\langle \frac{1}{x}, q \rangle = \int_{-\infty}^{\infty} \frac{1}{x} q(x) dx \quad ??$$

The function under the integrand fails to be locally integrable, so $1/x$ is not a sensible distribution. I.e., given a test function, it is not clear how this distribution is going to give a definite number. Compare with the singular delta distribution where the rule is clear on what to do for any test function:

$$\langle \delta, q \rangle = \int_{-\infty}^{\infty} \delta(x) q(x) dx = q(0)$$

In fact, neither the delta nor the $1/x$ are locally integrable, thus neither is a classical distribution in this sense.

2. The problem with $1/x$ is of course what to do at the pole,; over, under, or principle part? We can do as we like. Stakgold defines $pf(1/x)$ by taking the principle part. So we take:

$$\langle \frac{1}{x}, q \rangle = \int_{-\infty}^{\infty} \frac{1}{x} q(x) dx$$

Now the rule is complete. For any test function you will get something definite. The pole singularity always self-cancels in a PP integration. Sometimes this kind of distribution is called a pseudofunction distribution because it is "almost" a classical function distribution.

3. Example number 2. Consider:

$$f(x) = \lim_{R \rightarrow \infty} \left[\frac{\cos Rx}{x} \right]$$

$$\begin{aligned} \text{Then: } \langle f(x), q \rangle &= \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} \frac{\cos Rx}{x} q(x) dx \\ &= \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} q(y/R) \frac{\cos y}{y} dy \\ &= q(0) \cdot \int_{-\infty}^{\infty} \frac{\cos y}{y} dy = 0 \end{aligned}$$

Thus:

$$\lim_{R \rightarrow \infty} \left[\frac{\cos Rx}{x} \right] = 0$$

This means: if you apply to a test function and integrate, leaving out one "point", then you get 0 for any test function.

Derivation of $\delta(x-y) = -\frac{1}{2\pi i} \oint_{\infty} d\lambda g(x|y|\lambda)$

① Define a space of functions D_I by $u \in D_I \Rightarrow u = \sum_{i=1}^I a_i u_i(x)$

where $L u_i(x) = \lambda_i u_i(x)$, $L = \text{symmetric}$. Also, define

$$D = \lim_{I \rightarrow \infty} D_I \quad \text{so} \quad u \in D \Rightarrow u = \sum_{i=1}^{\infty} a_i u_i(x).$$

② Define $M_I = \max |\lambda_i| \quad i = 1, \dots, I$.

③ Lemma for $|\lambda| > M_I$, the operator $O_{\lambda} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{L}{\lambda}\right)^n$

exists on the space D_I and may be identified with

$(\lambda - L)^{-1}$. See proof below at the end.

④ Let $u \in D_I$. It's easy to show that

$$(\lambda - L)^{-1} \cdot u = \sum_{i=1}^I \frac{1}{\lambda - \lambda_i} a_i u_i$$

Now you get:

$$\frac{1}{2\pi i} \oint_{|\lambda| > M_I} d\lambda (\lambda - L)^{-1} \cdot u(x) = -u(x).$$

⑤ Now take $\lim$ of left side as $M_I \rightarrow \infty$, keep $u \in D$. Equation is still true:

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint_{|\lambda|=R} d\lambda (\lambda - L)^{-1} u(x) = -u(x).$$

⑥ Now take $\lim$ ④ as $I \rightarrow \infty$. Get

$$\lim_{I \rightarrow \infty} \left[\frac{1}{2\pi i} \oint_{|\lambda| > M_I} d\lambda (\lambda - L)^{-1} u(x) \right] = -u(x) \quad u \in D.$$

Now replace $[\]$ with $\lim_{R \rightarrow \infty} \int_{|x|=R}$. Then lim does nothing and you get result.

$$\frac{-1}{2\pi i} \lim_{R \rightarrow \infty} \oint_{|x|=R} (L-x)^{-1} = 1 \text{ on } D.$$

① Next: easy to show that

Lemma 2: $(L_x - \lambda)^{-1} u(x) = \int_a^b g(x|y; \lambda) u(y) dy.$

② Therefore:

$$= \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \oint_{|x|=R} (L-x)^{-1} u(x)$$

$$= \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \oint_{|x|=R} (L-x)^{-1} \int_a^b g(x|y; \lambda) u(y) dy$$

$$= \int_a^b dy u(y) \left[\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \oint_{|x|=R} (L-x)^{-1} g(x|y; \lambda) dx \right]$$

$$= u(x) \Rightarrow$$

$$\delta(x-y) = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \oint_{|x|=R} (L-x)^{-1} g(x|y; \lambda)$$

$$\text{for } x, y \in (a, b).$$

Comment on $(\lambda - L)^{-1}$.

① Let $u(x) = \sum_{i=1}^I a_i u_i(x)$.

② Define:

$$f_L^\lambda = \text{operator} \equiv \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{+1}{\lambda}\right)^n L^n$$

③ Then,

$$f_L^\lambda u(x) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{+1}{\lambda}\right)^n L^n \left(\sum_{i=1}^I a_i u_i(x) \right)$$

$$= \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{+1}{\lambda}\right)^n \sum_{i=1}^I a_i L^n u_i(x) \quad / \text{unif. conv. on } I$$

$$= \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{+1}{\lambda}\right)^n \sum_{i=1}^I a_i (\lambda_i)^n u_i(x)$$

$$= \frac{1}{\lambda} \sum_{i=1}^I a_i u_i(x) \sum_{n=0}^{\infty} \left(\frac{\lambda_i}{\lambda}\right)^n$$

If $|\lambda| > \max |\lambda_i| \forall i$ up to I , then $\sum$ converges uniformly over i and order interchange is OK.

$$= \frac{1}{\lambda} \sum_{i=1}^I a_i u_i(x) \frac{1}{1 - \frac{\lambda_i}{\lambda}}$$

$$= \sum_{i=1}^I a_i \frac{1}{\lambda - \lambda_i} u_i(x)$$

④ On the space D_I of functions $u = \sum_{i=1}^I a_i u_i(x)$, these two operators coincide, provided $|\lambda| > M_I$.

$$f_L^\lambda = (\lambda - L)^{-1} \text{ on } D_I \text{ if } |\lambda| > M_I.$$

Proof:

$$(\lambda - L) f_L^\lambda u = (\lambda - L) \sum_{i=1}^I a_i \frac{1}{\lambda - \lambda_i} u_i(x) = u.$$

Thus $f_L^\lambda = (\lambda - L)^{-1}$; right, left same.

⑤ However, we do not claim that $f_L^\lambda = (\lambda - L)^{-1}$ on the full space.

$$D = \lim_{I \rightarrow \infty} D_I.$$

This is so because the spectrum $\{\lambda_i\}$ may be unbounded.

⑥ Now reconsider:

$$f_L^\lambda \sum_{i=1}^{\pm} a_i u_i(x) = \sum_{i=1}^{\pm} a_i \frac{1}{\lambda - \lambda_i} u_i(x).$$

Thus:

$$\frac{1}{2\pi i} \oint_{|\lambda| > \max |\lambda_i|} f_L^\lambda u(x) d\lambda = \frac{1}{2\pi i} \oint_{|\lambda| > \max |\lambda_i|} \sum_{i=1}^{\pm} a_i \frac{1}{\lambda - \lambda_i} u_i(x) d\lambda = \sum a_i u_i = u(x).$$

Therefore: $\frac{1}{2\pi i} \oint_{|\lambda| > \text{all } \lambda_i} f_L^\lambda d\lambda = -1.$

I.e., this integral is the identity operator on the space D_I .

⑦ Now take $\lim_{I \rightarrow \infty}$. For each I we can find some

$\delta I \Rightarrow |\lambda_I| > \text{all } \lambda_i, i=1, I$. So, for any I

we have:

$$\frac{1}{2\pi i} \oint_{|\lambda| > |\lambda_I|} f_L^\lambda d\lambda = 1 \quad \forall I$$

⑧ Therefore,

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint_{|\lambda|=R} f_L^\lambda d\lambda = 1 \text{ on all } D.$$

This is the identity operator for any function on the space D .

$$u \in D \Rightarrow u = \sum_{i=1}^{\infty} a_i u_i(x).$$

green function

① Consider $(L_x - \lambda) g(x|y; \lambda) = \delta(x-y)$

Apply $\int dy f(y)$ to get

$$\int dy f(y) (L_x - \lambda) g(x|y; \lambda) = \delta(x)$$

② Assume uniform convergence near x range of interest
of $\int dy f(y) g(x|y; \lambda)$

then interchange to get

$$(L_x - \lambda) \cdot \int dy g(x|y; \lambda) f(y) = f(x)$$

③ Therefore,

$$G_x^\lambda \cdot f = \int dy g(x|y; \lambda) f(y) = (L_x - \lambda)^{-1} f$$

$$\text{I.e., } \boxed{(L_x - \lambda)^{-1} f(x) = \int dy g(x|y; \lambda) f(y)}$$

↑ Lemma 2.

rewrite 1 = rule. We know that $f_L^\lambda = \sum \dots$ exists
as an operator on $D^\pm$ if $\lambda > \lambda_\pm$. Here, f_L^λ can
be identified with $(L - \lambda)^{-1}$.

In the lim $|\lambda| \rightarrow \infty$, $f_L^\lambda = (L - \lambda)^{-1}$ on full D .

Thus:

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint_{|A|=R} d\lambda (L - \lambda)^{-1} = 1$$

Interesting Regge-related integrals

$$\textcircled{1} \int_1^\infty dz Q_j(z) P_\alpha(z) = \frac{1}{(j-\alpha)(j+\alpha+1)}$$

$$\textcircled{2} \int_1^\infty dz Q_j(z) Q_{-\alpha-1}(z) = \frac{\psi(j+1) - \psi(-\alpha)}{(j-\alpha)(j+\alpha+1)} \quad \text{no pole at } j = -\alpha-1!$$

The Legendre Transform Trick

① Consider some $f(x)$. Suppose $\frac{df}{dx} = u(x)$ or $df(x) = u(x)dx$.

Given, then, that $u = \frac{df}{dx}$, how can you find a function $F(u)$

such that $x = \frac{dF}{du}$? If, you are given u as the derivative of some f wrt x , you want to see x as the derivative of some F wrt u . In first case you get $u(x)$, in second $x(u)$.

If you can find such an F , then you have effectively found an inverse to $u(x)$.

② Try: $F \equiv ux - f$. Then $\frac{dF}{du} = u \frac{dx}{du} + x - \frac{df}{du}$

or $\frac{dF}{du} = u \frac{dx}{du} + x - \frac{df}{dx} \frac{dx}{du} = x$. Thus you are done: $x = \frac{dF}{du}$

③ Summary: Given, $u(x) = df/dx$, what is $x(u)$?

Answer: define $g = f - ux$. Then $x(u) = - \frac{dg}{du}$.

④ This trick is used on p 92 of Abus+Lee to invert $\phi = \partial\mathcal{L}/\partial\dot{x}$ to get $\dot{x} = -\partial\mathcal{L}/\partial\phi$.

⑤ Also discussed with extra spectator variable on p 216 Goldstein. Explains idea of $H = p\dot{q} - L$ to replace $\dot{q}$ with p as variable.

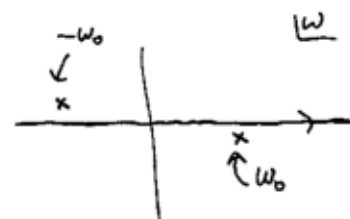
Greens Function Exercise

① Solve: $\left[\frac{d^2}{dt^2} + \omega_0^2\right] f(t) = \delta(t)$. Use Fourier to say:

$$f(t) = \int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t} \quad F(\omega) = \frac{1}{2\pi} \int dt f(t) e^{-i\omega t}$$

Then: $LHS = \int d\omega F(\omega) [-\omega^2 + \omega_0^2] e^{i\omega t} = RHS = \frac{1}{2\pi} \int d\omega e^{i\omega t}$

Claim: $F(\omega) = \frac{1}{2\pi(\omega_0^2 - \omega^2)} = \frac{1}{(\omega_0 + \omega)(\omega_0 - \omega) 2\pi}$



② Choose $\omega_0 =$ slightly complex, as shown. Then:

$$f(t) = \oint_C d\omega \frac{e^{i\omega t}}{2\pi[\omega_0^2 - \omega^2]}$$

For $t > 0$, close on top  where $e^{i\omega t} \sim e^{i[i|\omega|t]} \sim e^{-|\omega|t}$ decays.

So

$$f(t) = \oint_{\omega=-\omega_0} d\omega \frac{e^{i\omega t}}{[2\omega_0](\omega_0 + \omega)2\pi} = \frac{2\pi i}{2\pi} \frac{e^{-i\omega_0 t}}{[2\omega_0]}$$

For $t < 0$, close other way

$$f(t) = \oint_{\omega=\omega_0}^{\odot} d\omega \frac{e^{i\omega t}}{(2\omega_0)(\omega_0 - \omega)2\pi} = \frac{2\pi i}{2\pi} \frac{e^{+i\omega_0 t}}{(2\omega_0)}$$

③ Solution is therefore: $f(t) = \frac{2\pi i}{2\pi 2\omega_0} \left[\theta(t) e^{-i\omega_0 t} + \theta(-t) e^{+i\omega_0 t} \right]$

Conclusion: if $\omega_0 = \omega_0 - i\epsilon$, then

$$\left[\frac{d^2}{dt^2} + \omega_0^2\right] f(t) = \delta(t) \Rightarrow f(t) = \frac{+i}{2\omega_0} \left[\theta(t) e^{-i\omega_0 t} + \theta(-t) e^{+i\omega_0 t} \right]$$

Verify: $f(t) = \frac{i}{2\omega_0} \{ \phi(t) e^{-i\omega_0 t} + \phi(-t) e^{+i\omega_0 t} \} = \frac{i}{2\omega_0} \{ \phi(t) e^{-i\omega_0 t} + (\phi \leftrightarrow -\phi) \}.$

$$f'(t) = \frac{i}{2\omega_0} \cdot [\phi(t) (-i\omega_0) e^{-i\omega_0 t} + \phi(-t) e^{+i\omega_0 t}] = \frac{i}{2\omega_0} e^{-i\omega_0 t} [\delta(t) - i\omega_0 \phi(t)].$$

$$f''(t) = \frac{i}{2\omega_0} e^{-i\omega_0 t} \{ [\delta'(t) - i\omega_0 \delta(t)] + [\delta(-t) - i\omega_0 \phi(-t)] (-i\omega_0) \} + (\dots)$$

$$= \frac{[-2i\omega_0 \delta(t)] i e^{-i\omega_0 t}}{2\omega_0} \times$$

$$= \frac{i}{2\omega_0} e^{-i\omega_0 t} \{ \delta'(t) - 2i\omega_0 \delta(t) + (i\omega_0)^2 \phi(t) \}$$

$$+ \frac{i}{2\omega_0} e^{+i\omega_0 t} \{ \delta'(-t) - 2i\omega_0 \delta(-t) + (i\omega_0)^2 \phi(-t) \}$$

Look at $\delta(t)$ term: $\frac{i}{2\omega_0} [-2i\omega_0] \delta(t) [e^{-i\omega_0 t} + e^{+i\omega_0 t}]$ / does not look right.

$$\int_{-\infty}^{\infty} \phi(t) \delta(t) f(t) dt$$

$$\hookrightarrow = + \underline{\underline{\delta(t) \cdot 2}}$$

Next,
$$e^{-i\omega_0 t} \delta'(t) + e^{+i\omega_0 t} \delta'(-t) = -(-i\omega_0) e^{-i\omega_0 t} \delta(t) + (i\omega_0) e^{+i\omega_0 t} \delta(t) = 2i\omega_0 \delta(t)$$

So these two terms yield: $\frac{i}{2\omega_0} [2i\omega_0 \delta(t)] = \underline{\underline{-\delta(t)}}$

Lastly: $\frac{i}{2\omega_0} [-\omega_0^2] \{ e^{-i\omega_0 t} \phi(t) + e^{+i\omega_0 t} \phi(-t) \} = -\omega_0^2 f(t).$

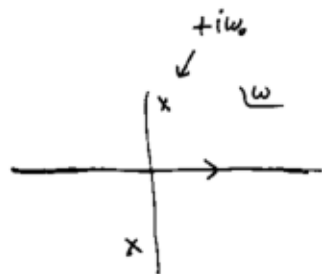
Thus, $f''(t) = +\delta(t) - \omega_0^2 f(t) \Rightarrow \left(\frac{d^2}{dt^2} + \omega_0^2 \right) f(t) = \delta(t) \checkmark$

Another Green's Fctn Exercise

$$\left[\frac{d^2}{dt^2} - \omega_0^2 \right] f(t) = \delta(t)$$

Same as before, but now $F(\omega) = \frac{-1}{2\pi(\omega_0^2 + \omega^2)}$

$$= \frac{-1}{2\pi(\omega + i\omega_0)(\omega - i\omega_0)}$$



Poles no longer close to axis. So $f(t) = \int_C d\omega \frac{e^{i\omega t} (-1)}{2\pi [\omega + i\omega_0][\omega - i\omega_0]}$

Again, for $t > 0$ close top to get $\frac{i e^{i\omega_0 t} (-1)}{(2i\omega_0)} = -\frac{e^{-\omega_0 t}}{2\omega_0}$

For $t < 0$, close bottom to get $\frac{-i e^{-i\omega_0 t} (-1)}{2\pi (-2i\omega_0)} = -\frac{1}{2\omega_0} e^{+\omega_0 t}$

So result is: $f(t) = -\frac{1}{2\omega_0} [\theta(t) e^{-\omega_0 t} + \theta(-t) e^{+\omega_0 t}] = -\frac{1}{2\omega_0} e^{-\omega_0 |t|}$

Conclusion: for real ω_0

$$\left[\frac{d^2}{dt^2} - \omega_0^2 \right] f(t) = \delta(t) \Rightarrow f(t) = -\frac{1}{2\omega_0} e^{-\omega_0 |t|}$$

Green's Exercise #3.

① If you want to solve $D_x f(x) = h(x)$, you need to solve $D_x g(x-y) = \delta(x-y)$.
Then $f(x) = \int dy g(x-y) h(y)$.

② Suppose you want to solve: $D_x^{\mu\nu} f(x) = h^{\mu\nu}(x)$. Then you need to solve $D_x^{\mu\nu} g_{\alpha\beta}(x-y) = \delta_\alpha^\mu \delta_\beta^\nu \delta(x-y)$. Then $f(x) = \int dy g_{\mu\nu}(x-y) h^{\mu\nu}(y)$.
I.e., when your operator has Lorentz indices, more work needed.

③ Example: $[\Box^2 g^{\mu\nu} - \partial^\mu \partial^\nu] g_{\alpha\beta}(x) = \delta_\alpha^\mu \delta_\beta^\nu \delta(x)$. Solve this for green's function. Use

$$g_{\alpha\beta}(x) = (2\pi)^{-4} \int d^4 k e^{ikx} G_{\alpha\beta}(k)$$

$$\delta^\mu(x) = (2\pi)^{-4} \int d^4 k e^{ikx}$$

Then $[-k^2 g^{\mu\nu} + k^\mu k^\nu] G_{\alpha\beta}(k) = \delta_\alpha^\mu \delta_\beta^\nu$

④ Looks suspicious. This must be true for all μ, ν , given fixed α, β .
If we choose $\mu = \alpha, \nu = \beta$ we get

$$[-k^2 g^{\alpha\beta} + k^\alpha k^\beta] G_{\alpha\beta}(k) = 1 \quad / \text{no sum} \Rightarrow G_{\alpha\beta} \neq 0.$$

But, if we choose $\mu = \alpha, \nu \neq \beta$ we get

$$[-k^2 g^{\alpha\nu} + k^\alpha k^\nu] G_{\alpha\beta} = 0 \Rightarrow G_{\alpha\beta} = 0.$$

Seems a contradiction. Only OK if $[-k^2 g^{\alpha\nu} + k^\alpha k^\nu] = 0$.

⑤ Start over. You are trying to solve this

$$D_x^{\mu\nu} G_{\mu\nu}(x) = \delta_{\mu'}^{\mu} \delta_{\nu'}^{\nu} \delta^4(x-y).$$

In momentum space you want to solve

$$D^{\mu\nu} G_{\alpha\beta} = \delta_{\alpha}^{\mu} \delta_{\beta}^{\nu}$$

Let $(\mu\nu) = i$ $(\alpha\beta) = j$. Then $D^i G_j = \delta^i_j$. For each fixed j you are trying to solve this for G_j , but solution must work for all i . I.e., in matrix notation,

$$\begin{pmatrix} D \end{pmatrix} \begin{pmatrix} G \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}. \quad DG^T = 1$$

Given a D , is there always a G which solves this? Solution seems to be:

$$D^i G_j = \delta^i_j \rightarrow G_j = \frac{D_j}{D^2} \quad \text{where } D^2 = \sum_i D_i D_i$$

Try this solution: $D^i \left[\frac{D_j}{D^2} \right] = \delta^i_j$

Only consistent if $D^i D_j = \delta^i_j D^2$. Why should this be true? This is a very special D ! Are there any such D ? Look at two dim

$$D = \begin{pmatrix} a \\ b \end{pmatrix}. \quad \text{Let } a^2 + b^2 = 1.$$

$$\left. \begin{array}{l} aa = 1 \\ ab = 0 \\ ba = 0 \\ bb = 1 \end{array} \right\} \text{no solution!}$$

Conclusion: a rank-2 tensor greens function does not exist

Comment on Greens Exercise #3.

The way I have done it, the two Lorentz indices taken together act as a single index. In effect, here was my supposed Green's Equation:

$$D^{uv} G_{ab} = d^{uv}_{ab} d(x-y)$$

Knowledge of this Green function was supposed to allow solution of equations like

$$D^{uv} h = f^{uv}(x)$$

I found that such a greens function simply does not exist. In simpler vector notation you are trying to solve this:

$$D G^+ = 1 \quad \text{in the same sense as } uu^+ = \text{whatever in Dirac.}$$

I showed that there were no possible D's with solutions in two dimensions.

Green's Exercise #4.

It turns out that the above "exercise" was the wrong one, ie, it is not relevant to propagators and so on. The interesting "tensor" greens equation is this:

$$D^{uv} G_{va} = d^u_a d(x-y)$$

Ie, you have usual matrix multiplication. The kind of equation you can solve with knowledge of this Green function is:

$$D^{uv} f_{ua} = h^u_a \quad \text{or} \quad D^u_v f^v_a = h^u_a$$

Again, ordinary matrix multiplication. In general, here is how you solve this Greens equation. You go over to k-space where the equation is simply a matrix equation:

$$D^u_v(k) G^v_a(k) = d^u_a$$

The thing has a solution as long as $\det D \neq 0$. Recall that if the matrix D is a "projection operator" then that means it has some non-trivial "kernel" or "nullspace", ie, a large part of the domain is mapped into 0. Obviously such an operator cannot be 1:1 and therefore cannot be invertible. Therefore you must find that $\det P_{ij} = 0$.

Example: Consider the operator $g_{uv} d_a d^a = d_a d_u$. In k-space this is $g_{uv} k^2 - k_u k_v$.

There are several ways to tell that this operator is non-invertible. First, you can show using trace thing or whatever that the $\det T_{uv} = \text{scalar}$. So compute this det in frame where $k = k(1,0,0,0)$ and you at once find that $\det=0$. The other way to know this at once is to realize that $k_v M^{vu} = 0$. This fact tells you that M_{uv} is a projection operator. Reason: any vector is $V_u = (V \cdot k) \frac{k_u}{k^2} + \text{rest}_u$.

The operator always maps the part of V_u proportional to k_u into nada, so M_{uv} is a projection operator. It follows that:

- a) $\det M = 0$
- b) M^{-1} does not exist (M is not one-to-one operator)

Example: However, if you shift coefficient a little so M is no longer "conserved" then you can invert:

$$D_{uv}(k) = \left[g_{uv} k^2 - k_u k_v \left(1 - \frac{1}{\alpha}\right) \right] \Rightarrow G_{uv} = \frac{1}{k^2} \left[g_{uv} - (1 - \alpha) \frac{k_u k_v}{k^2} \right]$$

Verify by showing that $D^u_v G^v_a = \delta^u_a$.

A study of the arcsin function

① Define $\sin^{-1}(x)$ like so:

$$8.92 \quad \sin^{-1}(x) = i \operatorname{sh}^{-1}\left(\frac{x}{i}\right) = i \operatorname{sh}^{-1}(-ix) = -i \operatorname{sh}^{-1}(ix).$$

But define $\operatorname{sh}^{-1}(ix)$ like so:

$$8.55 \quad \operatorname{sh}^{-1}(ix) = \ln[ix + \sqrt{1-x^2}]$$

Then:

$$\sin^{-1}(x) = -i \ln[ix + \sqrt{1-x^2}]$$

If $x = \sin \theta$, then $\ln[i \sin \theta + \cos \theta] = i\theta$ + you get $\sin^{-1}(x) = \theta$ ✓.

② Now use $\sqrt{1-x^2} = \mp i \sqrt{x^2-1}$, $\operatorname{Im} x \geq 0$. Then

$$\sin^{-1}(z) = -i \ln[iz \mp i \sqrt{z^2-1}]$$

$$= -i \left\{ \underbrace{\ln(i)}_{i\frac{\pi}{2}} + \ln[z \mp \sqrt{z^2-1}] \right\} = \frac{\pi}{2} - i \ln[z \mp \sqrt{z^2-1}]$$

Result

$$\boxed{\sin^{-1}(z) = \frac{\pi}{2} - i \ln[z \mp \sqrt{z^2-1}], \operatorname{Im} z \geq 0}$$

③ Corollaries: cut structure
(recall L6L5527)

z

← branch here.

* main

A) for $z > 1$, $\operatorname{Re}[\sin^{-1}(z)] = \frac{\pi}{2}$ for all $z > 1$ (of course true at $z=1$)

$$\operatorname{Im}[\sin^{-1}(z)] = -\ln[z \mp \sqrt{z^2-1}], \operatorname{Im} z \geq 0, i\mathbb{R}, \pm i\mathbb{R}.$$

Theory of Partial Fractions

- ① Consider a proper rational function of the form $g(x)/f(x)$. Expand the denom into factored product:

$$f(x) = (x-a_1)(x-a_2) \dots (x-a_n) = \text{degree } n.$$

Since fraction is proper, degree of $g(x)$ is $n-1$ or less. Therefore, $g(x)$ is fully determined by its value at n points, which we can take as the $g(a_i)$, assuming all a_i are different. Thus:

$$g(x) = \frac{g(a_1)(x-a_2)(x-a_3) \dots (x-a_n)}{(a_1-a_2)(a_1-a_3) \dots (a_1-a_n)} + \text{cyclic}$$

But identify $(a_1-a_2)(a_1-a_3) \dots (a_1-a_n) = f'(a_1)$. Thus,

$$\boxed{\frac{g(x)}{f(x)} = \frac{g(a_i)/f'(a_i)}{(x-a_i)} + \text{cyclic} \quad \text{all } a_i \text{ different.}}$$

Example $\frac{2x-3}{(x-a)(x-b)} = \frac{(2a-3)/(a-b)}{x-a} + \frac{(2b-3)/(b-a)}{x-b}$

- ② Next, suppose all the a_i are the same. Then in order to specify $g(x)$ of degree $n-1$ you still need n values of $g(x)$; go like so

$$g(x) = g(a) + g'(a)(x-a) + \frac{g^{(2)}(a)}{2!}(x-a)^2 + \frac{g^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{g^{(n-1)}(a)}{(n-1)!}(x-a)^{(n-1)}$$

Now you see that $g(a) = g(a)$, $g'(a) = g'(a)$ and so on. We use the value of g at a , and the values of $n-1$ derivatives at the same point, — n numbers to determine $g(x)$. In this case

$$\boxed{\frac{g(x)}{(x-a)^n} = \frac{g(a)}{(x-a)^n} + \frac{g'(a)}{(x-a)^{n-1}} + \frac{g^{(2)}(a)/2!}{(x-a)^{n-2}} + \dots + \frac{g^{(n-1)}(a)/(n-1)!}{(x-a)}}$$

Example: $\frac{2x-3}{(x-a)^3} = \frac{(2a-3)}{(x-a)^3} + \frac{2}{(x-a)^2}$

A fascinating formula.

$$\frac{x}{x-a} = \frac{x-a+a}{x-a} = 1 + \frac{a}{x-a}$$

so

$$\left[\frac{x}{x-a} = 1 + \frac{a}{x-a} \right]$$

Next: $\frac{x^2}{x-a} = x \left(1 + \frac{a}{x-a} \right) = x + a \left(1 + \frac{a}{x-a} \right) = x + a + \frac{a^2}{x-a}$

$$\frac{x^3}{x-a} = x^2 + ax + a^2 \left(1 + \frac{a}{x-a} \right) = x^2 + ax + a^2 + \frac{a^3}{x-a}$$

$$\begin{aligned} \frac{x^4}{x-a} &= x^3 + x^2a + xa^2 + a^3 \left(1 + \frac{a}{x-a} \right) \\ &= x^3 + ax^2 + a^2x + a^3 + \frac{a^4}{x-a} \end{aligned}$$

Looks like

$$\left(\frac{x^{n+1}}{x-a} \right) = \left[x^n + ax^{n-1} + a^2x^{n-2} + \dots + ax + a^n \right] + \frac{a^{n+1}}{x-a}$$

Generalized Binomial Theorem.

$$(a_1 + a_2 + a_3 + \dots + a_{N+1})^n = \left[\sum_{i=1}^{N+1} a_i \right]^n$$
$$= \sum_{n_1, n_2, \dots, n_N = 0}^{\infty} \binom{n}{n_1} \binom{n_1}{n_2} \binom{n_2}{n_3} \dots \binom{n_{N-1}}{n_N} \cdot (a_1)^{n-n_1} (a_2)^{n_1-n_2} (a_3)^{n_2-n_3} \dots (a_N)^{n_{N-1}-n_N} (a_{N+1})^{n_N}$$

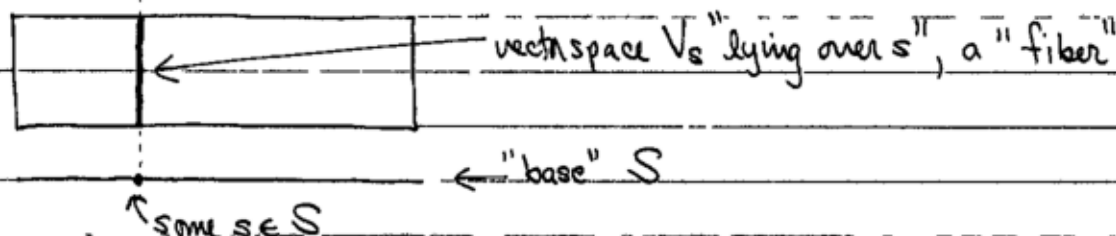
Fiber Bundles

The concept of a vector bundle, and its application in induced representation theory.

1. First, review the idea of a vector field. For each point in some domain S , you define a vector in some vector space V . Thus, a vector field is a mapping from S to V , so $f: S \rightarrow V$. Notice that the vector space which you map into is "fixed" in the sense that it is the same vector space for each element s of the domain.

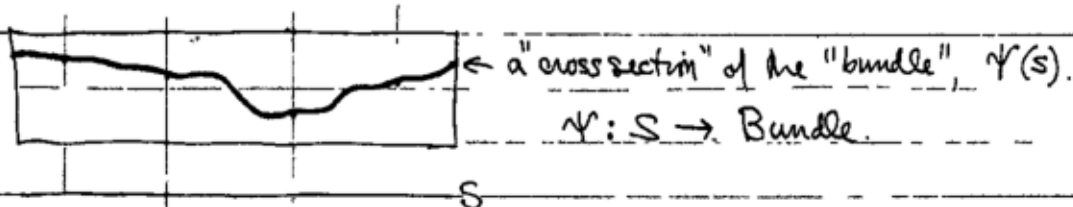
2. However, it is conceivable that the vector space into which you map might be a different vector space for each $s \in S$. Mackey gives the picturesque example of letting S be the surface of a sphere, and V_s is a two-dim vector space defined as the plane tangent to the sphere at s . Then you can have a mapping $f: s \rightarrow V_s$ such that each point on the sphere is mapped into some vector in the particular plane which is tangent at s .

3. The idea of having the mapped-into vector space "move" with s is what vector bundles are all about. To each $s \in S$ we can assign a whole vector space V_s . One usually speaks of S as the "base" and over each s in the base S there "lies" the vector space V_s referred to as a "fiber". This is because in the simple picture about to be drawn, V_s does indeed look like a fiber:



As you let s wander over S , you get a whole "bundle" of these fibers. The system consisting of S along with all the vector spaces V_s is referred to as a "vector bundle".

4. Suppose we have a map which takes each $s \in S$ into a particular vector in the fiber over s , i.e., in V_s . Of course in our picture above, a "vector in the fiber over s " is represented by just a point. As you let s wander through S , you get a set of such points, one point in each fiber. This set of points is called a "cross section" of the "fiber bundle". I.e., this cross section is a map from S into the bundle = set of all V_s 's, to which I have not assigned a name.



If all the vector spaces V_s were really the same vector space V , then this "cross section" would be called a vector field. So, cross section is a generalization of the concept of a vector field.

5. Focus down for a moment on one of the fibers V_s . Let us equip this fiber with a (perhaps s -dependent) scalar product so that it becomes a Hilbert Space. Perhaps the elements of the Hilbert Space V_s are functions of some variable x belonging to some other space X . Then presumably the scalar product of two elements of V_s looks like this: (say)

$$(f_1^{(s)}, f_2^{(s)})_{V_s} = \int_X d\mu(x) f_1^{(s)}(x) f_2^{(s)}(x)$$

where $d\mu(x)$ is some measure on X . We are never going to draw the space X in any picture. The point is that somehow, the scalar product of two vectors in the fiber V_s is defined somehow which makes V_s a Hilbert Space. Then we might call our vector bundle a "Hilbert bundle", but I have not seen that name anywhere. ✓

6. Given such a "hilbert bundle", we can ask: is there some sensible way we can define a spalar product of two cross sections through the bundle? Ie, we would like to form a new space whose elements are cross sections, and we want this space to become a vector space and then a Hilbert space. Let us call this new space $\mathcal{L}$. To make $\mathcal{L}$ a linear space, we need a reasonable definition of the addition of two cross sections, and multiplication of a cross section by a scalar. This is trivial, just do it in the obvious way, as you would for a vector field. For a scalar product, let's use:

$$(\psi_1, \psi_2)_{\mathcal{L}} = \int_S d\mu(s) (\psi_1^{(s)}, \psi_2^{(s)})_{V_s}$$

where $d\mu(s)$ is some reasonable measure on S . This means that the scalar ~~of~~ product of two cross sections is defined in terms of the scalar product in all the various fibers V_s . The norm of a cross section is of course defined in obvious way. Reasonable cross sections should of course have a finite norm. ✓

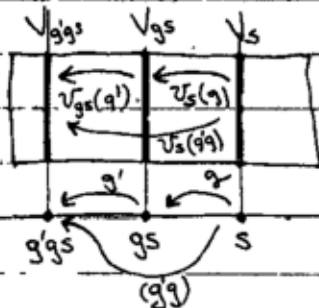
7. Notice that so far the word "group" has not appeared. It now appears. Just suppose that S is a homogeneous space under some group G (here, "homogeneous space under G " means that any two elements of S are connected by some group element g ; other terminology: " G is transitive on S "). (The usual example of a homogeneous space is the surface of a sphere under $O(3)$ rotations about the sphere's origin).

8. Now, suppose that, when some $s \in S$ is mapped into $gs \in S$ by group element g , the fiber over s is somehow mapped into the fiber over gs . Of course in such a fiber-to-fiber mapping, all the points on the fiber can be shuffled around, so there are infinitely many such mappings given any pair of fibers (assuming these fibers are function spaces, say). Thus we have arrived at the idea of a group element acting on a fiber giving another fiber. Here is a possible notation for such a map:

$$U_s(g): V_s \rightarrow V_{gs}$$

9. Instead of considering all such mappings $U_s(g)$, let's restrict to those which are "sensible" in the sense of transitivity (called "consistent"). By this we mean that if g takes fiber 1 into fiber 2, and if g' takes fiber 2 into fiber 3, then we want $(g'g)$ to take fiber 1 into fiber 3 in the same way.

To repeat the same statement graphically and algebraically:



$$\begin{aligned} U_s(g): V_s &\rightarrow V_{gs} \\ U_{gs}(g'): V_{gs} &\rightarrow V_{g'gs} \\ U_s(g'g): V_s &\rightarrow V_{g'gs} \end{aligned}$$

$$U_{gs}(g') U_s(g) = U_s(g'g)$$

consistency condition

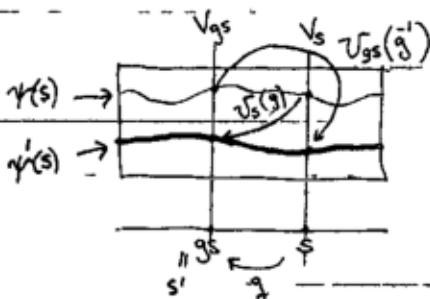
In other words, we want a point in

the initial fiber V_s to end up at the same point in $V_{g'gs}$ no matter which way we transport it, two steps or one step. ✓

10. Remark on Mackey: this equation appears in Mackey's 1968 book on page 29, but I really think little of his choice of notation (ie, x for elemtn of G , reverse notation for group operations, etc.) ✓

ie, assume we have a map $U_s(g)$, defined $\forall g \in G, s \in S$, which is consistent

11. Now comes the important question: how shall we define the action of a group element on a cross section? This can be done in a very natural way via the mapping $U_s(g)$ defined above. We want to start with an old cross section and generate a new one $\psi'(s)$. Call the old one $\psi(s)$. Idea is this:



$$\psi'(s) = \psi'(gs) = U_s(g) \psi(s) \checkmark$$

$$\text{OR: } \psi'(s) = U_{gs}(g') \psi(gs) \checkmark$$

15. Notice that $U_{s_0}(g)$ is defined for all g , ~~not~~ just for $g \in H$. However, for general g , this map takes you to some other fiber. However, consider instead the map $U_s(g^{-1})$. You start with some g . ^{and reference s_0 .} This determines an $s = gs_0$. We see this thing as a map from the fiber over s , to the reference fiber over s_0 . However, you can think of this $U_{gs_0}(g^{-1})$ as being a map from G to the reference fiber. Hermann refers to this map as $\tilde{\Psi}(g)$. I.e., $\tilde{\Psi}: G \rightarrow V_{s_0}$.

16. But so what? Now I am getting lost, so will stop. Well, I think the next move is to associate with this map $U_s(g^{-1})$ a particular special cross section. Pick some $s \in S$. This determines a g such that $gx s = gs_0$ because S is homogeneous. Now, $U_s(g^{-1})$ takes a point in the fiber over s and transports it into the reference fiber.

17. Aha! There is a difference between my $U_s(g^{-1})$ and Hermann's $\tilde{\Psi}(g)$. His $\tilde{\Psi}(g)$ is a specific point on the reference fiber, whereas my $U_s(g^{-1})$ does not give you a specific point on this fiber unless you let the thing act on some particular point in the fiber over s . So to relate these two objects, we have to get more specific. Let $\Psi(s)$ be a particular cross section. Then at s , $\Psi(s)$ is a particular point on the fiber over s , so we can now say:

$$U_s(g^{-1}) \Psi(s) = \tilde{\Psi}(g)$$

$$\text{OR } U_{gs_0}(g^{-1}) \Psi(gs_0) = \tilde{\Psi}(g)$$

This is Hermann's equation 9-1. Notice that he just uses g^{-1} for $U_s(g^{-1})$.

17.5 Upon translating his equation 9.2 I find: well, no need. We already know that $U_{s_0}(h)$ defines a representation of H , mapping V_{s_0} to itself. We can picture $\tilde{\Psi}(g)$ as a point on the reference fiber, even though g not nec in H . Thus, under the rep of H , $\tilde{\Psi}(g)$ will get mapped into some other point still on V_{s_0} . I have shown that:

$$U_{s_0}(h) \tilde{\Psi}(g) = \tilde{\Psi}(gh) \quad , \quad \text{hence} \quad U_{s_0}(h^{-1}) \tilde{\Psi}(g) = \tilde{\Psi}(g) \quad (9-2)$$

18. What does this mean? It means that we have "realized" this rep of H in such a way that V_{s_0} is the Rep space, and G is the M-space.

19. I think we are getting near the punch line of Hermann's analysis. First, we know that there is a 1-to-1 correspondence between the space of functions $\tilde{\Psi}(g)$ and the space of cross sections $\mathcal{L}$. this is because we can form one from the other uniquely:

$$U_s(\tilde{g}) \Psi(s) = \tilde{\Psi}(g)$$

$$\Psi(s) = U_{s_0}(g) \tilde{\Psi}(g)$$

Now the claim is that if we restrict the space $\mathcal{L}$ to just those $\tilde{\Psi}$ which satisfy ^(9.2) ~~2.2~~, ie, to those $\tilde{\Psi}$ that form the rep space for the rep of H, then in fact the space of cross sections $\mathcal{L}$, with the same restriction, forms a Rep space for a rep of G itself. But how does this work!?

20. Clearly this is just what Vilenkin said. The equation (9-2) of Hermann must correspond to that property-2 of Vilenkin. Looking at Vilenkin's form for the scalar product in $\mathcal{L}$, it looks like his elements of $\mathcal{L}$, called f , were actually cross sections over G as a base. ie, Vilenkin's scalar product is then exactly the scalar product of two cross sections through this bundle.

Ideas on Connecting the Fiber Bundle Technology to the Induced Rep Theory.

1. Assume that G has subgroup H . Let S be the ~~M -space~~ base of a homogeneous vector bundle which we shall associate with group G . Let s_0 be the point in S whose invariance group is H . Let V_{s_0} be the reference fiber (it lies over s_0). Since V_{s_0} is a vector space, really we should refer to an element in this reference fiber as a vector, for example, $\underline{f}$. Assume that the reference fiber is a Hilbert Space, so there exists a scalar product. Moreover, assume that the reference fiber is the carrier space (ie, rep space) of some representation of group H . Denote the rep operators of this rep by $U_{s_0}(h)$. Clearly, these operators map the reference fiber into itself, so $U_{s_0}(h) \underline{f} = \underline{f'}$.
2. Earlier we wrote $\underline{f}$ as $\psi(s_0)$, the value of some cross section over s_0 . Thus we should keep in mind that the cross section is vector valued, although we neglected to put a bar under ψ .
3. So far we only know $U_{s_0}(g)$ when g is in H . However, I suspect that the requirement that the vector bundle be homogeneous (ie, consistent) will in fact give us $U_{s_0}(g)$ for all g in G , not just in H . For example, the consistency condition requires that $U_{s_0}(g) U_{s_0}(h) = U_{s_0}(gh)$. If this does not fully determine $U_{s_0}(g)$, it certainly restricts it a lot. Perhaps arbitrary $U_s(g)$ is determined just from $U_{s_0}(h)$. I have not yet determined whether or not this is true. Hold.
4. Let us assume for now that $U_s(g)$ is known for all s in S and for all g in G . Then a subset of this large set of operators forms a rep of H whose carrier space is the reference bundle. ~~Note that because $U_s(g)$ is known,~~ If we are given just one vector on just one fiber, ^{does} then $U_s(g)$ determined uniquely an entire cross section through the bundle? In particular, given one point on the reference fiber, ~~it~~ ^{can} be associated with one cross section? The fact that space S is homogeneous allows us to define this notation: $g = s/s_0$. Is the g so determined unique? I don't think so. For example, $s_0/s_0 = h$ can be any element of H ! Thus, s/s_0 really only specifies a coset of G relative to H and any g in that coset will be equal to s/s_0 .
5. We interrupt the line of flow here to elaborate on this s/s_0 problem. Question: consider these two left cosets of H ; g_1H and g_2H . If these cosets are equal, does that mean $g_1 = g_2$? ~~ASSUMING YES~~ No, it just means that g_1 and g_2 are both in the same coset. (Obviously g_1 is in the coset g_1H). In general there are fewer cosets than group elements. We know that these cosets partition the

group into say N pieces, and each piece contains M elements where M is the order of H . Thus, the order of G must be MN . Thus, $\text{order}(H) = \text{order}(G)/N$. This is the old result that the order of a subgroup must be an integral divisor of the order of the group. The number of cosets, then, is equal to $\text{order}(G)/\text{order}(H)$. This includes the "trivial" coset which is H itself. The point of this discussion is

that if you use as your set S the set of cosets G/H , then the division s_0/s_0 has a unique solution, namely, H . In other words, $s_0 = H$ in this case. Similarly, s/s_0 is also unique. It is really (C_1/H) which is just equal to C_1 where C_1 is some coset. (Why? Because $C_1 = g_1H = g_1HH$, so $C_1/H = g_1H = C_1$). But wait? s/s_0 is supposed to be a group elemtn...?

6. We were working on this question: ~~how exactly~~ does a point on a fiber determine a cross section over S ? I was going to answer: $\mathcal{N}(s) = U_{s_0}(s/s_0) f$ in the case that f was on the reference fiber, but then I noticed that s/s_0 was not unique. Given some s , you really don't know which g to put inside U . Maybe if you take different g 's, both equal to s/s_0 , you might get two different cross section points over s !

7. I think I have to retract this notion that a point on a fiber uniquely determines a cross section, via $U_s(g)$. Suppose the order of subgroup H were 25. Then, a point on ~~the ref.~~ fiber actually determines 25 points on every other fiber because s/s_0 has 25 solutions for each s .

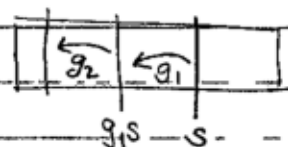
8. Instead, let's work with a different idea. Suppose you have a ~~see~~ cross section over S . Take its value over some s . Find the 25 solutions to s/s_0 . For each of those 25 g_i in G , define $\underline{f}(g_i) = U_s(g_i^{-1})\mathcal{N}(s)$. Notice that all these $\underline{f}$'s so obtained lie on the reference fiber. Thus, for each point s in S , we obtain 25 points on the reference fiber. Now, repeat this process for all s in S . Then, each point on the cross section maps into 25 points on the reference fiber. Considered as a map from S to V_{s_0} , we have a 25-to-1 mapping. However, if we consider this cross section as just some kind of fixed, auxiliary object, then ~~can be~~ considered as a map from G to V_{s_0} we have a 1-to-1 mapping. ! It is this map that Hermann calls $\mathcal{N}$. Of course the map is vector valued. Note: Vilenkin calls this same map $\underline{f}(g)$.

9. So, with each cross section over the bundle, we may associate a unique map from G to V_{s_0} , which map is a one-to-one map. Namely, $\underline{f}(g) = U_s(g^{-1})\mathcal{N}(s)$ where $s = gs_0$. Perhaps ought to call the cross section $\underline{f}(s)$.

10. However, if we are given a 1-to-1 map from G to V_{S_0} , that map by itself does not define a unique cross section! For each g in G , sure, we have a unique point on V_{S_0} . But, maybe 25 different g 's all give the same s , so we would end up trying to pin $\star(s)$ at 25 different values. This is the same problem we had back ~~in~~ in (7.). A point on the reference fiber always maps into 25 points on all other fibers, so unique cross section is impossible.

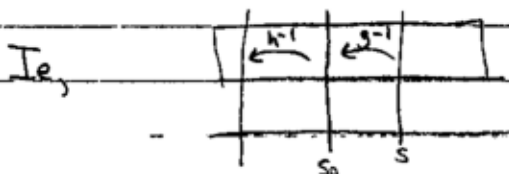
11. Question: is there some extra condition we could impose on this map $\star(g)$ such that a unique cross section is determined? Lets go back and look at 9. We found that given a cross section, we could get a unique map. That map so determined has a special property which I will now show. Remember that consistency required that :

$$U_{g_2 s}(g_2) U_s(g_1) = U_s(g_2 g_1)$$



Let $g_1 s = s_0$, so $s = g_1^{-1} s_0$. Let $g_2 = h^{-1}$. Let $g_1 = \bar{g}$. Then as special case: ($s = \bar{g}^{-1} s_0 = g s_0$)

$$U_{s_0}(h^{-1}) U_s(\bar{g}) = U_s(h^{-1} \bar{g})$$



Using this consistency condition in this form, we see that:

$$\begin{aligned} \tilde{\Psi}(g) &\equiv U_s(\bar{g}) \Psi(s) \Rightarrow U_{s_0}(h^{-1}) \tilde{\Psi}(g) = U_{s_0}(h^{-1}) U_s(\bar{g}) \Psi(s) \\ &= U_s(h^{-1} \bar{g}) \Psi(s) \\ &= U_s([g h]^{-1}) \Psi(s) \\ &= \tilde{\Psi}(gh) \end{aligned}$$

Thus, that 1-to-1 map $\tilde{\Psi}$ obtained from the cross section $\star$ actually satisfies this condition, which Hermann calls 9.2.

Interpretation: the space whose elements are these 1-to-1 maps actually forms the carrier space of a realization of the representation U of group H in which the rep operator is a simple right shift! The M -space of this rep is simply the group G . Keep in mind, however, that the map $\tilde{\Psi}$ is vector valued, so $U_{s_0}(h^{-1})$ can shuffle the vector components as well as do the shift.

12. We could convert this rep to a scalar rep à-la Vilenkin page 28. Recall there the idea that any irreducible rep of a group H can be realized as a shift rep acting on scalar functions. By staying with vector functions, we have realized the rep U , whether or not it is irreducible.

13. Now let's go back to the question of trying to determine a unique cross section from the map $\tilde{\Psi}$. Perhaps if the map $\tilde{\Psi}$ obeys the condition 9.2 (a restriction in the map $\tilde{\Psi}$), maybe then we can find a unique Ψ . I'll bet ahead of time that the restriction 9.2 is exactly what you need to make those 25 values of the cross section all the same value. OK, consider:

$$\begin{aligned}\Psi(g_1 s_0) &\equiv U_{s_0}(g_1) \tilde{\Psi}(g_1) \\ \Psi(g_2 s_0) &\equiv U_{s_0}(g_2) \tilde{\Psi}(g_2) \quad \text{where } g_1 s_0 = g_2 s_0 = s\end{aligned}$$

Our worry was that we might be pinning this cross section at 25 different values at the one point s . Here are two of those values; I want to show that they are in fact the same so there is no problem. Notice that the fact that both g_1 and g_2 map s_0 into s means that these g 's are in the same left coset of H :

$$\begin{aligned}g_1 s_0 = g_2 s_0 &\Rightarrow (g_2^{-1} g_1) s_0 = s_0 \Rightarrow g_2^{-1} g_1 = \text{some } h_i \\ &\Rightarrow g_1 = g_2 h_i \Rightarrow g_2 = g_1 h_i^{-1}\end{aligned}$$

Now let's compare the two things we want to show are equal:

$$U_{s_0}(g_1) \tilde{\Psi}(g_1) \stackrel{?}{=} U_{s_0}(g_2) \tilde{\Psi}(g_2)$$

$$\begin{aligned}\text{Well: } U_{s_0}(g_1) \tilde{\Psi}(g_1) &= U_{s_0}(g_1) \tilde{\Psi}(g_2 h_i) = U_{s_0}(g_1) [U_{s_0}(h_i^{-1}) \tilde{\Psi}(g_2)] \quad \text{by 9.2} \\ &= U_{s_0}(g_1) U_{s_0}(h_i^{-1}) \cdot \tilde{\Psi}(g_2) \\ &= U_{s_0}(g_1 h_i^{-1}) \tilde{\Psi}(g_2) = U_{s_0}(g_2) \tilde{\Psi}(g_2) \quad \checkmark\end{aligned}$$

14. We conclude, then, that provided the map $\tilde{\Psi}$ satisfies 9.2, we can get a unique cross section from $\tilde{\Psi}$ by the rule: $\Psi(s) = U_{s_0}(g) \tilde{\Psi}(g)$. Of course the cross section so obtained is the same one you started with to get the map $\tilde{\Psi}$:

$$\begin{aligned}\tilde{\Psi}[\Psi] &= \tilde{\Psi}(g) = U_s(g^{-1}) \Psi(s) = U_s(g^{-1}) U_{s_0}(g) \tilde{\Psi}(g) \\ &= U_{s_0}(1) \tilde{\Psi}(g) = \tilde{\Psi}(g) \quad \text{because } U_{s_0}(e) = 1, \text{ a rep. of } H!\end{aligned}$$

$$\begin{aligned}\Psi[\Psi] &= U_{s_0}(g) \tilde{\Psi}(g) = U_{s_0}(g) U_s(g^{-1}) \Psi(s) \\ &= U_s(g \cdot g^{-1}) \Psi(s) \\ &= U_s(1) \Psi(s) = \Psi(s) \quad \text{I.e., conclude } U_s(1) = 1 \quad \forall s!\end{aligned}$$

15. Therefore, we finally arrive at the result: the set of maps $\tilde{\Psi}: G \rightarrow V_{s_0}$ which satisfy condition 9.2 is isomorphic to the set of cross sections on the bundle.

The induced representation.

1. So far, we have succeeded in 'realizing' our representation of the subgroup H as shift operators on a carrier space whose elements are the functions $\tilde{\psi}(g)$ with G as the M -space. Namely,

$$U_{S_0}(h^{-1}) \tilde{\psi}(g) = \tilde{\psi}(gh) \equiv \tilde{\phi}(g) \quad \text{say.}$$

Define:

$$L(h) \equiv U_{S_0}(h^{-1}).$$

Then: $L(h) \tilde{\psi}(g) = \tilde{\psi}(gh)$

How do we know that $L(h)$ is a rep? The mult rule is OK because it is a shift. The closure requirement is OK because this equation is condition 9.2, thus obviously the functions ~~ψ satisfy~~ satisfy 9.2. I.e., $L(h)$ is a rep because we assume it is a rep. Just for fun, to show closure: assume that $\tilde{\psi}$ satisfies 9.2, show that $\tilde{\phi}$ does also:

$$L(h) \tilde{\psi}(g) = \tilde{\psi}(gh) \text{ is 9.2} \equiv \tilde{\phi}(g)$$

~~$$L(h) \tilde{\phi}(g) = L(h) \tilde{\psi}(gh) = \tilde{\psi}(ghh) = \tilde{\psi}(gh)$$~~
~~$$= \tilde{\phi}(gh)$$~~

$$L(h) \tilde{\phi}(g) = L(h) \tilde{\psi}(gh) = \tilde{\psi}(g \cdot h \cdot h) = \tilde{\psi}([gh]h) = \tilde{\phi}(gh)$$

Thus, $\tilde{\phi}(g)$ also satisfies 9.2.

2. Now define a new shift operator $T(g_1)$ acting on the same space of functions satisfying 9.2. I claim that this new shift operator defines a representation of the entire group G , with the same carrier space as the rep L of group H :

$$T(g_1) \tilde{\psi}(g) = \tilde{\psi}(g_1^{-1}g) \equiv \tilde{\phi}(g)$$

Closure:
$$\begin{aligned} L(h) \tilde{\phi}(g) &= L(h) \tilde{\psi}(g_1^{-1}g) = L(h) \tilde{\psi}([g_1^{-1}g]) = \tilde{\psi}([g_1^{-1}g]h) \\ &= \tilde{\psi}(g_1^{-1}[gh]) = \tilde{\phi}(gh) \quad \checkmark \end{aligned}$$

Again, the mult rule is trivial because it is a shift. Closure is proved above.

Notice that the rep $L(h)$ of H is a right shift whereas this new rep T of G is a left shift. The new rep T is the representation of G induced by the rep

L of H . Its carrier space is the set of 9.2-satisfying 1-to-1 mappings, which set is isomorphic to the set of cross sections on a certain vector bundle.

3. Now, exactly how do we characterize this "vector bundle" whose cross sections form the carrier space for the induced representation? First, the base S of the bundle must be chosen as a homogeneous space such that s_0 has invariance group H . In other words, to say that S is homogeneous implies that action of G on S is somehow defined. We want a "consistent" homogeneous vector bundle, and this means that the action of G on a fiber is somehow defined. In fact, I think I can show that the entire "fiber operator" $U_s(g)$ is completely defined by the smaller set of operators $U_{s_0}(h)$ which transform the reference fiber into itself. I.e., by the representation of H you start with.

4. Now we know that it is always possible to choose S specifically to be the space of left cosets G/H . Therefore, given G and H and the rep L of H , the entire vector bundle is completely ~~xxx~~ determined. Thus, the space of cross sections is determined. The space of 9.2-type functions is also determined. Finally, the induced representation is determined.

5. Notice, by the way, that the induced rep operator $T(g)$ is not equal to $U_{s_0}(g)$ because $U_{s_0}(g)$ takes an element in the reference fiber and maps it into the fiber over s ; thus, $U_{s_0}(g)$ could not give $\tilde{\Psi}(g^{-1}g_1)$ when acting on $\tilde{\Psi}(g_1)$. On the other hand, the rep $L(h)$ was equal to $U_{s_0}(h^{-1})$.

6. Can we express the operator $T(g)$ in terms of the fiber bundle operators $U_s(g)$? If so, then we can see exactly how T takes one cross section and gives another. After playing you can show that:

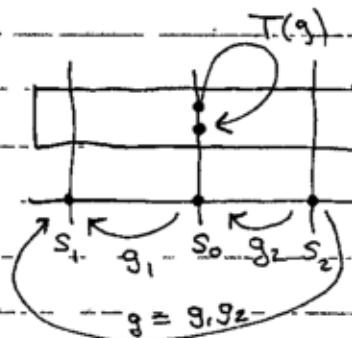
$$[U_{s_0}(g_1) T(g) U_{s_2}(g_2)] \tilde{\Psi}(s_2) = \tilde{\Psi}(s_1)$$

This comes from:

$$T(g) \tilde{\Psi}(g_1) = \tilde{\Psi}(\tilde{g}^{-1} g_1) \equiv \tilde{\Psi}(\tilde{g}_2^{-1})$$

and:

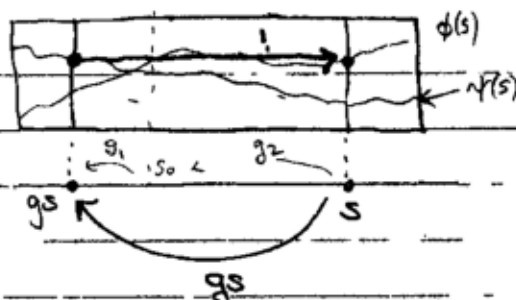
$$\begin{aligned} \tilde{\Psi}(g_1) &= U_{s_1}(\tilde{g}_1^{-1}) \Psi(s_1) \\ \tilde{\Psi}(\tilde{g}_2^{-1}) &= U_{s_2}(g_2) \Psi(s_2) \end{aligned}$$



7. Lets reformulate this idea again:

$$[U_{s_0}(g_1)T(g)U_{s_0}(g_2)]\psi(s) = \psi(gs) \\ \equiv \phi(s)$$

or: $[T(\tilde{g})]\psi(s) \equiv \phi(s) = \psi(gs)$



In other words, start with an old cross section ψ . The value of the new cross section above s is equal to the old cross section above point gs . It's as simple as that. The reason it is so simple is that the "structure" is all built into the fiber bundle on which cross sections are defined.

8. Lets recap:—the operators $\tilde{T}(g)$ act on the space of cross sections in the bundle. In fact, the operator $\tilde{T}(g)$ is merely a shift operator on this space. These operators $\tilde{T}(g)$ form a representation of the group G . The Carrier Space of this simple shift rep is the space of cross sections on the bundle. The M -space is the space S . However, this bundle is not any old bundle and S is not any old space. S is the space of left cosets G/H of G relative to some subgroup H . The bundle over S is the particular bundle such that $U_{s_0}(h)$ forms a particular representation of H . Then the rep $\tilde{T}$ is the rep "induced" by the rep L of H .

9. Finally, a comment about scalar products. Compare:

$$\|\psi(s)\| = \int_{G/H} d\mu(s) (\psi(s), \psi(s))_s$$

$$\|\tilde{\psi}(g)\| = \int_G d\mu(g) (\tilde{\psi}(g), \tilde{\psi}(g))_{s_0} \leftarrow \text{ie, scalar product in } V_{s_0}$$

$$(\tilde{\psi}(g), \tilde{\psi}(g))_{s_0} = (U_s(\tilde{g})\psi(s), U_s(\tilde{g})\psi(s))_{s_0} \\ = (\psi(s), \psi(s))_s$$

I bet these scalar products are equal. The homogeneity of the fiber bundle causes the scalar product at s to be defined by the scalar product at s_0 .

10. Conclusion: the bundle defined by a particular shuffling of the reference fiber provides a space whose cross sections "carry" the induced rep.

Mackey on Induced Representations.

1. Let's start reading on page 19. Start right off with a group G and an M -space he calls S , measure μ . Pick some $s \in S$. Then we presume there is some defined action of G on S so that gs traces out an orbit in S as g moves through G . Perhaps there is some point s_0 and a subgroup H such that $hs_0 = s_0$ for all h in H . Then you might say that in the space S , the point s_0 acts as an origin with respect to H , and then gs would map out a sphere through s , centered at s_0 , again as g runs through G . Maybe the origin should stay put for all of G ? For example, the physical spheres in E^3 under $G = SO(3)$ are spheres or orbits in this math sense. The property of a homogeneous space is that the whole space is just one fat orbit. It can all be gotten to be group transformations.

2. Assume we have a homogeneous space S with an origin s_0 . For each g in G , we can examine $gs_0 = s$, some other point in S . There is a map which takes each g and tells you where that g is going to take s_0 , call this map $\mathcal{N}(g)$. In general, there can be more than one element g which takes s_0 to some s , so $\mathcal{N}$ need not be 1-to-1. Define K_{s_0} to be the subgroup of G such that $ks_0 = s_0$. Now let's ask: when do g_1 and g_2 both take s_0 into the same s ? You can easily show that the answer is: when g_1 and g_2 both lie in the same right coset of subgroup K_{s_0} . So, if you want to redo this mapping $\mathcal{N}$ to make it 1-to-1, you can use the set of right cosets as the domain, instead of the group itself. Then each right coset gets mapped into an $s = gs_0$ where g is any element of that right coset. Clearly, then, the point is that this function $\mathcal{N}$ was constant on ~~the~~ the right cosets of G . The set of such cosets is usually called: G/K_{s_0} .

Ergo, since the set G/K is 1-to-1 with S , we might as well use G/K instead of S . This is the now-familiar notion that any homogeneous space S under G can be thought of as the space of right cosets G/K_{s_0} where K_{s_0} is the "symmetry subgroup" of point s_0 . For example, the point s_0 in S becomes the "point" K in G/K , i.e., the trivial coset. The action of G on S in the realization where S is G/K is given very easily. If Kg_1 is some right coset, then stop. Use left cosets to compensate Mackey's notation. If g_1K is some left coset, then $g(g_1K) = (gg_1)K$, which clearly is another left coset.

By the way, in going from S to G/K , you have to somehow convert your measure μ into a measure defined on the left coset space. It seems to me that if you are later going to deal only with functions which are constant over cosets, you can just use the group Haar measure itself...?

3. We are now in the middle of page 22. Let $\mathcal{F}$ be the set of complex valued functions f which are constant on the ~~left cosets of~~ K_{S_0} and which have a finite square norm (defined by integrating over $S=G/K$ with measure du .) Ie, we have a scalar product. (By the way, if the measure is invariant in the sense of left shift, you can show that... oops, too early) OK, given the space $\mathcal{F}$ equipped with inner product, then you can use that space as the rep space for this rep: $(U(g)f)(g_1) = f(gg_1)$. Keep in mind that the M-space is really G/K although it looks like G here. Now, if your measure is left-invariant, easy to show that this representation is "unitary" in the sense that the ~~scalar~~ norm of a function in the space $\mathcal{F}$ is not changed under the application of the rep operator U . Of course if you define a hermitian conjugate operator U^+ then unitary translates into $U^+U = 1$, as usual.

4. So why go through all this bother? We take a group and a subgroup K ; we define a homogeneous M-space G/K . We define a Rep space of functions on this M-space of finite norm. Then this allows us to define a unitary representation of G as a shift operator acting on function in the rep space. So what!!

5. The reason, according to Mackey, is that a slight modification of the above construction yields the famous inducing method. Before we had a condition 1 which said: functions in $\mathcal{F}$ are constant on left cosets, ie, $f(kg) = f(g)$ for all k in K_{S_0} . Now we are going to assume that we already have some representation of the subgroup K_{S_0} which we call $L(k)$. The Rep space of this rep is called $\mathcal{H}(L)$. Now define a new version of the space $\mathcal{F}$ to contain functions which, though still defined on the space of left cosets G/K , now take their values in the Hilbert Space $\mathcal{H}$. Since $\mathcal{H}$ is a vector-space, these new functions f are vector-valued functions. Now, we replace condition 1 with a condition 1'. Instead of having f be constant on the coset, we allow it to take different values as its argument ranges over a given coset. What values shall we give $f(kg)$ as k ranges over K ? Answer: $L(k) f(g)$. Ie, we let our given representation describe those values. Next, we wish to replace the condition 2 to say that f has a finite norm. We define a new norm as on page 24, a norm constructed on top of the $\mathcal{H}$ -norm.

Now, having defined our new space $\mathcal{F}$ by these conditions, we define a new rep for the whole group G in this way: $U(g;L)f(g_1) = f(gg_1)$. This rep depends parametrically on L because L plays a key role in defining the rep space $\mathcal{F}$. Moreover, the measure invariance guarantees that U also is a unitary rep!

6. Comment on M-spaces. Going back to our original conditions 1 and 2, we restricted to functions f constant over cosets. This meant essentially that the M-space was really G/K , not G itself. With our new conditions and the induced rep, the vector-valued elements f are definitely not constant on the cosets G/K , so I would say that the M-space of the induced rep was in fact the group G .

7. Mackey goes on to describe how you can induce a rep U on G from a rep L on subgroup H without having to deal with any ^{invariant} measure. This is a pleasant little digression into measure theory, which I of course know nothing about. First, we assume that our space S is a "Borel Space". This is a statement about the measureability of the space S and I think it amounts to saying Hausdorff separable plus locally compact. This means that there will exist a positive number associated with every subset of S called the "measure" and denoted $\mu(E)$. Fine. Now, given some measure μ_1 , you can concoct another measure μ_2 by integrating μ_1 over your set E with some positive weight function g .

The Radon-Nikodym Theorem: if two arbitrary measures of space S equal ~~xxx~~ zero on the same subsets of S , then these two measures are connected by some weight function g and therefore in some sense the measures are equivalent i.e., in the same class. Notice that "set of measure zero" suddenly springs to life and means something!!

8. For example, suppose we have a set E in S and a measure $\mu(E)$. We can let some g in G act on E . Hence, define a new measure $\mu_g(E) = \mu(gE)$. If these measures are equivalent for all g in G , then this measure is called "quasi-invariant". OK, the point here is that G/H always has such a quasi-invariant measure, even if it has no "invariant" measure. Then you can execute the inducing program based on this broad notion of quasi invariant measure. Moving from one measure to another within the class merely generates equivalent representations U under inducement.

9. Observation: if you let H = identity subgroup, and L = trivial rep of same, then condition 1 says nothing, condition 2' boils down to condition 2, and the ~~xxx~~ induced rep U defined on the M-space G itself is just the regular rep.

10. Now let's flash-back to page 4,5 in section 1.1. Although the measure μ is defined on sets in the M-space S, invariance of this measure means that $\mu(gE) = \mu(E)$, for all g in some group G . The set E is not in G , so I don't think you would call this a Haar measure. Nevertheless, this is the kind of measure that you need to define scalar product of two functions defined on the M-space S. Now, suppose you have group G , space S , and an invariant measure μ . Define a Hilbert Space of functions on S by the obvious L^2 norm. Elements of this Hilbert space are $f(s)$. Then, you can define an operator U which acts on the Hilbert space $\mathcal{H}$ and which is completely defined in this way: left shift. Because this has the group-multiplicity property, this operator defined on G and $\mathcal{H}$ is, by definition, a representation of the group G . The invariance of the measure on S implies that these operators U are unitary, so this is a unitary representation.

11. We are now at the first indentation, page 27.

Induced Representations

① By starting with a rep of a subgroup of a group, you generate a rep for the entire group, the "induced" rep

② Construction: let $H = \text{s.g. of } G$. Let $\mathcal{K}$ be the rep space for $Q(h)$, a rep of H . Let the elements of $\mathcal{K}$ be scalar functions f , with some scalar product (f_1, f_2) . Of course $f \in \mathcal{K}$ requires $\|f\| = (f, f) \leq \infty$. Notice that the elements of $\mathcal{K}$ do not carry label h , i.e. not $f(h)$. Let us denote by a an arbitrary element of $\mathcal{K}$. Thus, eg, $Q(h)a = \text{some new } a = a'$, under action of Q .

③ Now we shall construct a new rep $T(g)$ and rep space $\mathcal{L}$ as follows. First $\mathcal{L}$. The elements of $\mathcal{L}$ will be labelled by $g \in G$, unlike the elements of $\mathcal{K}$. In order for a vector f to lie in $\mathcal{L}$, two things must be true:

- 1) $\int d[g] (f(g), a) < \infty \quad \forall a \in \mathcal{K}$, for some Borel (Haud) $d[g]$.
- 2) $Q(h)f(g) = f(gh)$.

Assume that given $(G, H, Q(h), \mathcal{K})$, we have been able to construct $\mathcal{L}$ + identify its elements f . Then claim: This defines a rep on all G .

$$T(g)f(g_0) = f(\bar{g}'g_0)$$

④ ~~Proof that this really is a rep. First, the group property:~~

~~$$\begin{aligned} T(g_1 g_2) f(g_0) &= f([g_1 g_2] g_0) = f(\bar{g}_2' [\bar{g}_1' g_0]) \\ &= T(g_2) f(\bar{g}_1' g_0) = T(g_2) T(g_1) f(g_0) \end{aligned}$$~~

~~Well, (3) is error. Should say $T(g)f(g_0) = f(gg_0)$. Then:~~

~~$$\begin{aligned} T(g_1 g_2) f(g_0) &= f(g_1 g_2 g_0) = f(g_1 [g_2 g_0]) = T(g_1) f(g_2 g_0) \\ &= T(g_1) T(g_2) f(g_0) \end{aligned}$$~~

~~$$\begin{aligned} \text{What about } T(g)f(g_0) &= f(g_0 \bar{g}') \text{ Then: } T(g_1 g_2) f(g_0) = f(g_0 \bar{g}_2' \bar{g}_1') \\ &= f([g_0 \bar{g}_2'] \bar{g}_1') \\ &= T(g_1) f(g_0 \bar{g}_2') = T(g_1) T(g_2) f(g_0) \checkmark \end{aligned}$$~~

④ Show that $T(g) f(g_0) = f(g_0^{-1}g)$ is a rep.

Notice that G itself plays the role of X . So

$$\begin{aligned} T(g_1) T(g_2) f(g_0) &= \langle g_0 | T(g_1) T(g_2) f \rangle \\ &= \langle g_1^{-1} g_0 | T(g_2) f \rangle = \langle g_2^{-1} g_1^{-1} g_0 | f \rangle \\ &= \langle (g_1 g_2)^{-1} g_0 | f \rangle = \langle g_0 | T(g_1 g_2) f \rangle = T(g_1 g_2) f(g_0) \end{aligned}$$

⑤ Also, you must show that $f(g_0) \in \mathcal{L} \Rightarrow \tilde{f}(g_0) \equiv f(g_0^{-1}g_0) \in \mathcal{L}$.
If $\mathcal{L}$ must be closed under T . So, we need to show that $\tilde{f}(g)$ satisfies properties 1. and 2. if $f(g)$ does.

$$\begin{aligned} 1. \quad \int dg \, (\tilde{f}(g), a) &= \int dg \, (f(g_0^{-1}g), a) \\ &= \int d[g_0 g'] \, (f(g'), a) \\ &= \int dg' \, (f(g'), a) \quad \text{because measure is left invariant} \\ &< \infty \quad \checkmark \end{aligned}$$

$$2. \quad \text{Show} \quad Q(h) \tilde{f}(g) = \tilde{f}(gh).$$

$$\begin{aligned} \text{We know that } Q(h) f(g) &= f(gh), \\ \text{i.e., } \langle g | Q(h) f \rangle &= \langle gh | f \rangle. \end{aligned}$$

$$\text{So, we also know: } \tilde{f}(g) \equiv f(g_0^{-1}g) \Rightarrow \langle g | \tilde{f} \rangle = \langle g_0^{-1}g | f \rangle.$$

$$\text{So: } \tilde{f}(gh) = \langle gh | \tilde{f} \rangle = \langle g_0^{-1}(gh) | f \rangle = \langle (g_0^{-1}g)h | f \rangle$$

$$= \langle (g_0^{-1}g) | Q(h) f \rangle = \langle g | Q(h) \tilde{f} \rangle$$

$$= [Q(h) \tilde{f}](g) = Q(h) \tilde{f}(g), \text{ sloppily.}$$

Thus, $\mathcal{L}$ is closed (invariant). Then, $\{T(g), \mathcal{L}\} = \text{a rep.}$

Comments on Induced Representations.

1. Recall that the Hilbert space $\mathcal{L}$, which is the "representation space" of the induced representation, is defined by two properties (to be redescribed in a few sentences below). Usually, when you have a representation space for a rep, the elements of that Rep Space are functions. The range of the variable(s) of the functions is a different space, usually called M by Vilenkin. Vilenkin never gives this space M a name. Mackey refers to M as "a G -space". This is the space you usually want to be homogeneous, which is to say, you want G to be transitive on M . Maybe M is the "carrier space" of the representation?? For now, let's call it the M -space.

2. Now, when you talk about induced reps, you start off with some known rep of a subgroup called H . The rep space of this known rep is called $\mathcal{K}$. In turn, this rep space $\mathcal{K}$ is defined on some M -space, say the real line. You need to know about this M -space in order to actually calculate the scalar product of two functions in $\mathcal{K}$, but formally we can ignore the M -space and write the scalar product as (f, g) .

3. Now, the "induced" rep is usually called T . What is the rep space of T , and what is the M -space of T ? The Rep space of T is called $\mathcal{L}$. The M -space of T happens to be G itself!!! We want $\mathcal{L}$ to be a Hilbert space, so it must have a scalar product. This scalar product is defined in a special way, so that in effect the scalar product of $\mathcal{L}$ is "built on top of" the scalar product of $\mathcal{K}$.

$$(f, g)_{\mathcal{K}} \equiv \int dx f^*(x) g(x), \text{ say}$$

$$\begin{aligned} \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right)_{\mathcal{L}} &= \int d[g] \sum_i \left(\begin{pmatrix} f_i(g) \\ f_i(g) \end{pmatrix}, \begin{pmatrix} f_i(g) \\ f_i(g) \end{pmatrix} \right)_{\mathcal{K}} \\ &= \int d[g] \sum_i \int dx \left[f_i(g, x) \right]^* f_i(g, x) \end{aligned}$$

Thus we see that in fact the elements of the Rep space $\mathcal{L}$ are columns of functions defined both on the M -space of $\mathcal{K}$ and on the M -space of $\mathcal{L}$, which we just noted was G itself.

So, the first property which serves to define $\mathcal{L}$ is that the elements of $\mathcal{L}$ must have finite norm, where the norm is of course defined by the scalar product shown above.

4. So the first property defining $\mathcal{L}$ is that the elements of $\mathcal{L}$ must have finite $\mathcal{L}$ -norm, certainly a reasonable requirement. The second property required of $\mathcal{L}$ is that $\mathcal{L}$ must be a Representation Space for the rep Q of subgroup H defined by right-shift. Note that this is not the original rep space $\mathcal{H}$. We have here a rep of H with M -space equal to G , not X . Ie,

$$[Q(h)f](g) = f(gh) \quad \text{"property-2"}$$

In other words, we are assuming that the space $\mathcal{L}$ is closed under the action of this rep Q of the subgroup H . Remember, the first step in getting an induced rep is to start with a known rep of H . Well, here we have that known rep, except here it is defined on $\mathcal{L}$ instead of $\mathcal{H}$.

Notice that in general the requirement that $\mathcal{L}$ be a rep space of Q is a severe restriction on the possible elements $f(g)$ of $\mathcal{L}$. Any old column of functions defined on G (and on X) will not do. That is, if you pull out of the air a column of functions which has finite $\mathcal{L}$ -norm, probably that column of functions will not be an element of $\mathcal{L}$ because the above property-2 will in general not be satisfied, even though property-1 is satisfied.

5. Theorem: not only is $\mathcal{L}$ as defined above a Rep space for rep Q of H , it is also a Rep space for a rep T of all of G defined by left shift!!!

$$[T(g')f](g) = f(g'^{-1}g)$$

In order to prove this theorem, you merely have to show that $\mathcal{L}$ is closed under the action of all $T(g')$ for $g' \in G$, and you have to show the usual "group multiplication property". Of course this latter is always true for a shift. The closure of $\mathcal{L}$ under all $T(g')$ can be shown trivially from the closure of $\mathcal{L}$ under $Q(h)$. Ie, you want to show that $f'(g) = f(g'^{-1}g)$ is an element of $\mathcal{L}$. Well, f' satisfies property-1 due to the shift invariance of the Haar measure. It satisfies property-2 because you just show it (see other sheets).

6. First Significance: the first significance of the "inducing construction" is that given a rep of H , you can construct a rep of G .

7. Example: let H be the identity subgroup of G , and let Q be the trivial rep of H . This means that property-2 is an identity and does not restrict $\mathcal{L}$ in any way. Thus, the Rep space $\mathcal{L}$ of the induced rep T is any finite-norm column of functions on G . This rep is of course called: the "regular rep" of G . Ie, the rep where the M -space is chosen as G itself.

Mackey: "Induced Reps" 1968

Chapter 1: "Group Reps and the Inducing Construction."

1.1 Preliminaries: $S =$ the rep space. Reverse notation: $sx, x \in G$. S must have a "volume element" — i.e. must be a "Borel space" with measure μ . This "Borel" property is same as separable and local compactness. Measure serves to define scalar product, $\int |f(s)|^2 d\mu(s) < \infty$. Called $L^2(S, \mu)$. With this as rep space, $[U_x f](s) = f(sx)$ will be a rep.

1.2 Definitions: "Invariant subspace" M of $\mathcal{H}$. L denotes a rep, so $\mathcal{H}(L) =$ rep space. "Peter-Weyl Theorem" — compact groups are decomposable + have finite dim V.I.R.'s. $\mathcal{U} =$ "interwinning operator" to deal with equivalence. If $\mu(E) = \mu(Ex), x \in G$, this is "right invariant Haar measure". If you let $S = G$, get "regular rep."

1.3 Expansion: mentions how you reduce regular rep of a ^{compact} commutative group to get Fourier-like sum. I.e., $g = \sum \hat{g}_\chi X$ where $X =$ one dim rep. I.e., use "expand" on $g \in L^2(G, \mu)$ is a "direct sum" of V.I.R.'s. Think of $\mathcal{O}(2)$. I.e., g means $g(x)$.

1.4 Direct Integrals: is, a direct continuous sum of V.I.R.'s.

1.5 The inducing Construction. Start with $\{G, S, \mu\}$. sx maps out an "orbit" as x runs through g . Orbits are sort of the disjoint pieces of the space S . If only one orbit, then "transitive" action of G on S . Call s_0 the "origin". Let $\Psi: G \rightarrow S$ i.e., $x \mapsto sx \in S$. Define K_{s_0} as all G s.t. $s_0x = s_0$, i.e. all of G that does nothing to "origin" s_0 . Thus, K_{s_0} is like a "sphere" in G . If $x, y \in K_{s_0}$, then $s_0 = s_0x = s_0y \Rightarrow s_0xy^{-1} = s_0$. Now K_{s_0} is a subgroup, invariant. Thus it has cosets $K_{s_0} \subset G$. Let G/K_{s_0} denote the set of right cosets. When is $\Psi(x) = \Psi(y)$? I.e., when is $sx = sy$. (Not nec $= s_0$!) I.e., when $s(xy^{-1}) = s(1)$. Thus $xy^{-1} = 1 \Rightarrow x = 1 \cdot y$. But $1 \in K_{s_0}$, so x, y must be in same coset. Thus, entire coset maps into same element of S .



The original map $\psi: G \rightarrow S$ is many-to-one, but the map $\psi: G/K_0 \rightarrow S$ is one-to-one. Thus, you can always use G/K_0 as your space S . An element of S is now: (K_0x) , where K_0 is a set $\subset G$. Now instead of saying $S \times X$, we say $(K_0x) \times X$.

So, use G/K_0 as the rep space S . Define some function $f(x)$ to be constant on a given coset, i.e., $f(K_0x) = f(x)$. Such a function can be regarded, therefore, as a function on G/K_0 . (see "induced rep" notes elsewhere).

Mackey's book discusses many other things:

theory of induced reps (nothing on fiber bundles)
imprimitivity theorems

particle spin

why QM HS is complex

induced reps of Lie groups; $SL(n, \mathbb{C})$

ergodic theory, relation to QM and statistics.

Reasonably written, but takes work to understand.

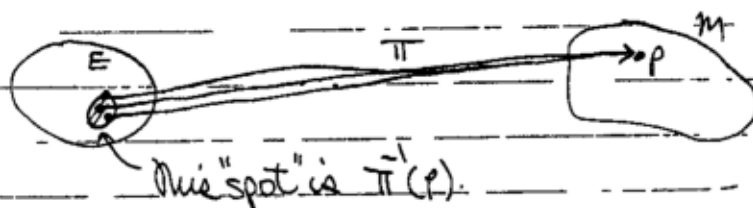
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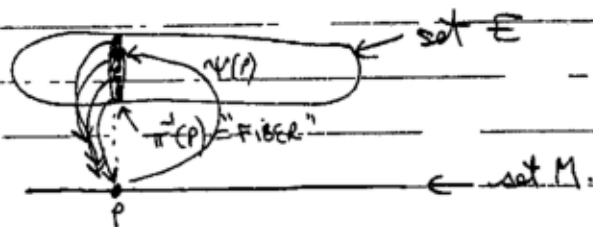
Hermann: Lie Groups for Physicists; poorly written(I find this book very hard to read.)

Chapter 9: vector bundles. Let $\psi_i(x) = \text{vector-valued field}$. Let G act on space $X \ni x$ (like rotation). For $g \in G$ and $x \in X$, define a matrix A_{ij} which can act on ψ_i . I.e., we have a map $\tau: G \times X \rightarrow \text{space of matrices } \ni A$. Consider another map, $m: G \times X \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Or $m: \mathbb{R}^n \rightarrow \mathbb{R}^n$, i.e., a vector ψ_i goes into another vector, $\psi_i \rightarrow \psi'_i = A_{ij} \psi_j$, but ψ_j not at x but at $g \cdot x$. Whew!

p53: Consider $\pi: E \rightarrow M$, an "onto" map.

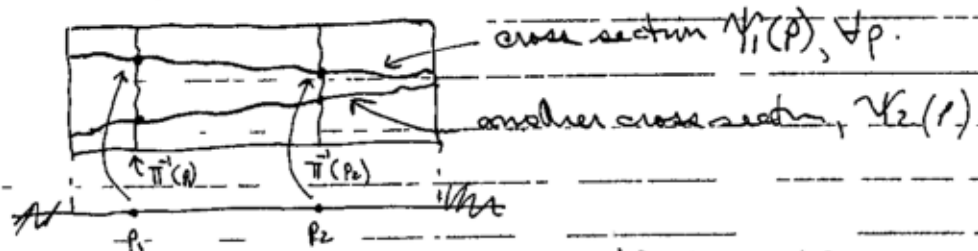


Better picture:



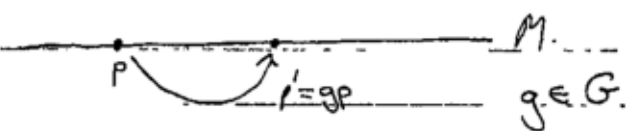
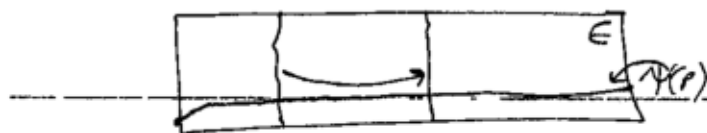
If the set $\pi^{-1}(p)$ which "lies over p " is a real vector space (i.e., closed under addition, etc), then $\pi^{-1}(p)$ is called a "FIBER". If this is true $\forall p \in M$, then the system $\{E, M, \pi\}$ is a "vector bundle".

Now define: $\psi: M \rightarrow E$ such that each p gets mapped into a particular element of the fiber $\pi^{-1}(p)$. Thus, $\psi(p)$ is a "horizontal" slice of E .



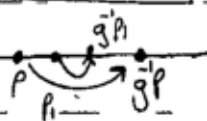
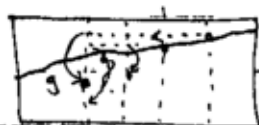
You can add two cross sections: $\psi_1 + \psi_2 = \psi_3$ means $\psi_3(p) = \psi_1(p) + \psi_2(p)$, $\forall p$.

Thus, set of cross sections is a linear space, $\Gamma(E)$.

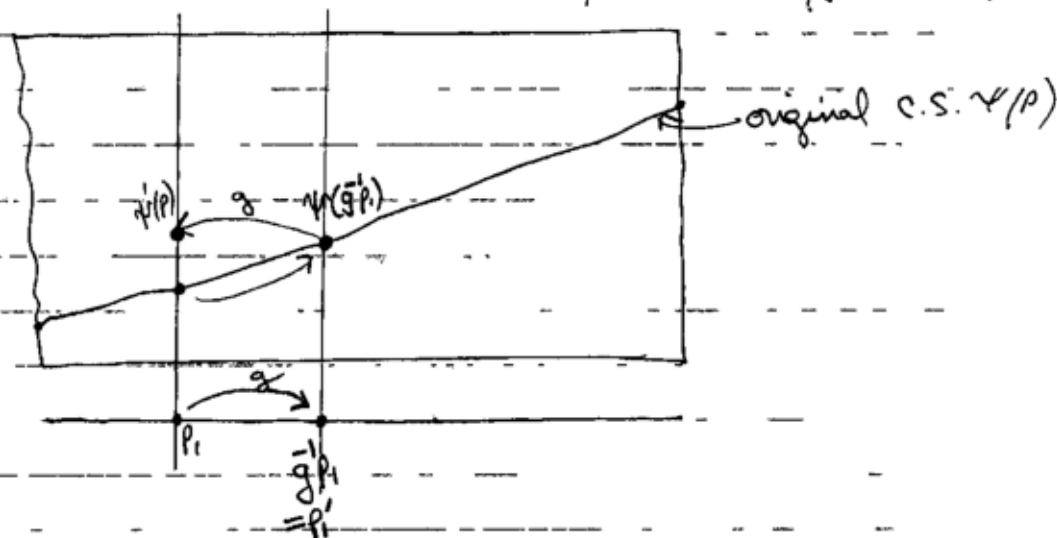


Start with old cross section $\psi(p)$ + create a new one.

$$\psi'(p) = g \psi(gp)$$



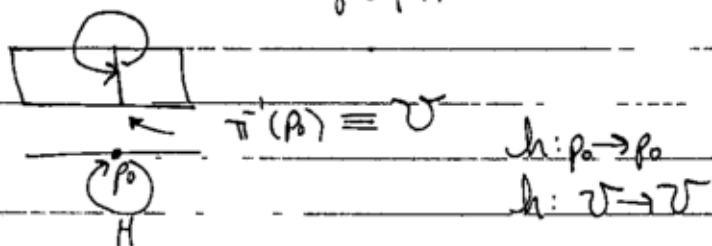
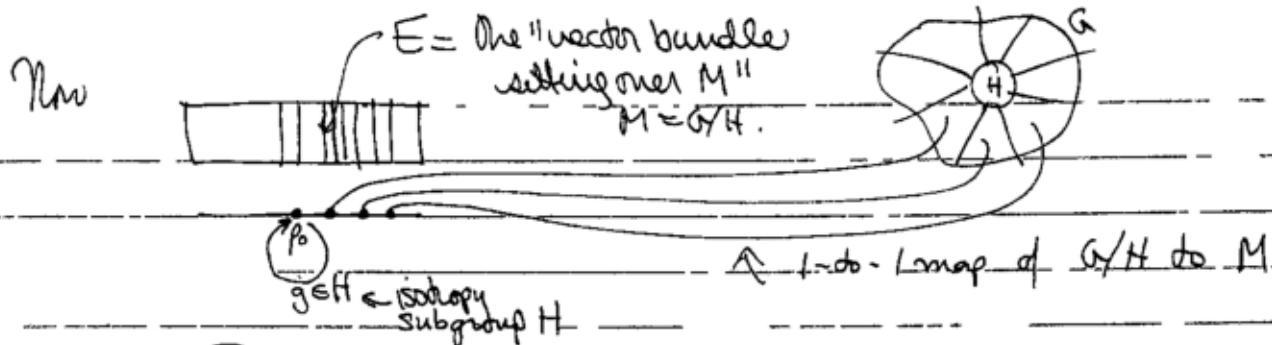
Let's work out two points on $\psi'(p)$. Notice that ~~the set~~ $g \in G$ must act on elements of both spaces, E and M .



Now, let the "points" $\psi^1(p) \dots \psi^n(p)$ be a basis for the vector space (fiber) $\pi^{-1}(p)$, i.e., a particular p . Then, as you let p exhaust M , you have n "basis" cross sections. Any cross section is then

$$\psi(p) = \sum_i f_i(p) \psi^i(p)$$

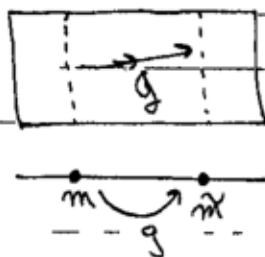
↑
arb. fn on M .



The fiber over p_0 maps into itself under $h \in H$, perhaps rearranging itself. Thus, V is the rep. space for a rep of H called " σ ".

How does G act on M ? (When $M \cong G/H$). Well, $m \in G/H$ means $m = Hg$, i.e., m is some coset. Then G acting on m is very natural:

$$G: M \rightarrow M, \quad g \cdot m \equiv (Hg)g = H(g \cdot g) = H\tilde{g} = \tilde{m} \\ = \text{some other coset.}$$



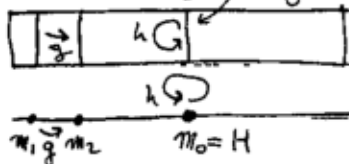
We want g to map the ~~bundle~~ fiber over m into the fiber over $\tilde{m}$.

If G does this to E , then E is a "homogeneous vector bundle" w.r.t. G and H s.g. of G .

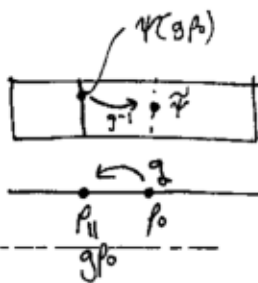
By the way: p_0 is H itself, i.e., $h \cdot p_0 = (H)h = H \cdot 1 = p_0$.
 I.e., H is the "origin" of $M = G/H$.

σ defines a rep σ .

Now, so far we have:



So, what $h \in H$ does to $V = \pi^{-1}(p_0)$ is the linear rep σ . Claim: if you know σ , then the entire homo. vectn bundle is determined, so use σ to label the H.V.B.



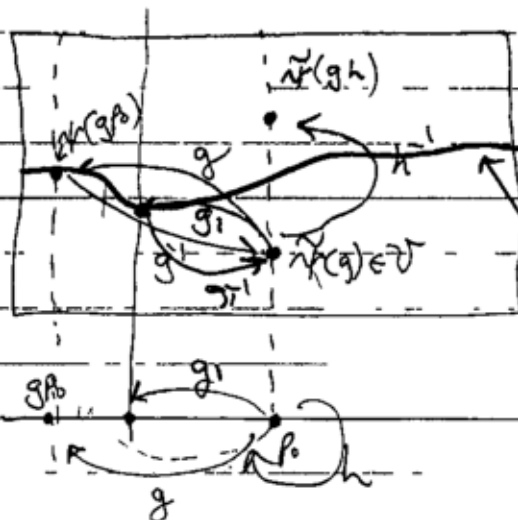
$\tilde{\psi}$ takes a $g \in G$, & gives you an element of U :
 $\tilde{\psi}(g) \in U$

$$\tilde{\psi}(gh) = h^{-1} \tilde{\psi}(g.h.p_0) = h^{-1} \tilde{\psi}(g.p_0) = g$$

But $\psi(g.p_0) = g \cdot \tilde{\psi}(g)$ so

$$= h^{-1} g^{-1} g \tilde{\psi}(g).$$

Wait:



Given ψ , a cs, (9-1) defines $\tilde{\psi}(g)$, a point in U which depends on g .

$\psi(p)$ defined in terms of $\tilde{\psi}(g)$, see (9.3):

What means $\sigma(h)$? $\sigma: U \rightarrow U$, I thought, parametr'd by h .

$$\sigma(h): U \rightarrow U, \text{ e.g., } u_1$$

OK, when g acts on element of U , call it $\sigma(g)$ instead of g , if $g \in H$.

$$\text{So, } \tilde{\psi}(h) = h^{-1} \psi(h.p_0) \text{ means } \sigma(h^{-1}) \psi(h.p_0) = \sigma(h^{-1}) \tilde{\psi}(h.p_0).$$

Not yet clear. Suppose someone hands you a set U and a rep $\sigma(h)$ such that

$$\tilde{\psi}(gh) = \sigma(h^{-1}) \tilde{\psi}(g) \quad \tilde{\psi}(\dots) \in U.$$

I.e., someone tells you how U reshuffles under σ . Then, $\tilde{\psi}: G \rightarrow U$ is known. Each $p \in M$ can be written as $g.p_0$, so $\psi(p) = \psi(g.p_0) = g \tilde{\psi}(g)$.

$$(gh) \tilde{\Psi}(gh) = \tilde{g} \tilde{\Psi}(\tilde{g}) = g \cdot h \cdot h^{-1} \tilde{\Psi}(g) = g \tilde{\Psi}(g) \quad \text{by 9.2.}$$

$$\text{Thus, } \Psi(gp_0) = \Psi(ghp_0) = \Psi(gp_0).$$

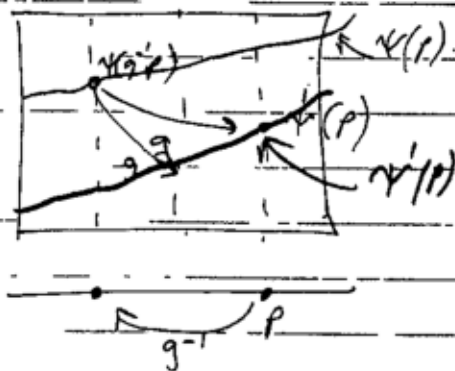
For each map $\Psi: G \rightarrow V$ obeying 9.2 (i.e., for each shuffling of V defined over all of G), there is a special cross section Ψ as in 9.3.



Now go back to def of Ψ :

$$\Psi'(p) = g \cdot \Psi(p) \equiv g \Psi(\tilde{g}^{-1}p).$$

$$\Psi(p) = g(g) \Psi(p) \equiv g \Psi(\tilde{g}^{-1}p).$$



$$\text{Thus, write for } gp_0: \quad g(g) \Psi(gp_0) = g \cdot \Psi(\tilde{g}^{-1}gp_0) \quad \checkmark$$

$$\begin{aligned} \text{But: } g(g)(\tilde{g}^{-1}gp_0) &= g \cdot [(\tilde{g}^{-1}g) \Psi(\tilde{g}^{-1}gp_0)] \\ &= g \cdot [(\tilde{g}^{-1}g)^{-1} \Psi(\tilde{g}^{-1}gp_0)] \\ &= g \cdot [\tilde{g}^{-1} \Psi(\tilde{g}p)] = g \cdot \tilde{\Psi}(\tilde{g}) = g \Psi(\tilde{g}^{-1}g). \end{aligned}$$

$$\text{Thus: } g(g) \Psi(gp_0) = g \tilde{\Psi}(\tilde{g}^{-1}g)$$

$$\text{Meanwhile: } \Psi'(p) = g \Psi(\tilde{g}^{-1}p) = g \Psi(p).$$

So $\Psi'(p) = \tilde{g}^{-1} \Psi'(gp_0)$. Assume we eventually get:

$$[g(g) \Psi](g) = \tilde{\Psi}(\tilde{g}^{-1}g).$$

Fredholm Theory

Solving a (Fredholm) Matrix Equation with Symmetric Kernel via Eigenvectors.

1. This idea is mentioned by Schid and Sorensen. I must have known this once but forgot about it.

2. Suppose you have this equation to solve:

$$A = K + KPA \Rightarrow A(1 - KP) = K \Rightarrow A = K(1 - KP)^{-1}$$

To illustrate the idea, assume P is just a number.

3. The main idea is this: if H is a matrix and $|i\rangle$ is a basis of eigenvectors of that symmetric or hermitian matrix, then it is easy to compute any $f(H)$ acting on a basis vector, ie:

$$H|i\rangle = \lambda_i|i\rangle \Rightarrow f(H)|i\rangle = f(\lambda_i)|i\rangle.$$

So let's now consider this equation:

$$A = V + \frac{1}{\sigma - \beta} V A \Rightarrow A \left[1 - \frac{V}{\sigma - \beta} \right] = V$$

$$\Rightarrow A [\sigma - \beta - V] = V(\sigma - \beta)$$

$$\Rightarrow \underline{A} = (\sigma - \beta) \underline{V} [\sigma - \beta - \underline{V}]^{-1}$$

$$\text{Let } V|i\rangle = \lambda_i|i\rangle. \text{ Then}$$

$$\begin{aligned} \langle i | \underline{A} | k \rangle &= A_{ik} = \langle i | \lambda_j \rangle \langle \lambda_j | A | \lambda_\ell \rangle \langle \lambda_\ell | k \rangle \\ &= (\sigma - \beta) \langle i | \lambda_j \rangle \langle \lambda_j | \frac{\lambda_\ell}{\sigma - \beta - \lambda_\ell} | \lambda_\ell \rangle \langle \lambda_\ell | k \rangle \\ &= (\sigma - \beta) \langle i | \lambda_j \rangle \frac{\lambda_j}{\sigma - \beta - \lambda_j} \langle \lambda_j | k \rangle \end{aligned}$$

So, in this case the eigenvalues of the "potential matrix" tell you the trajectory shift

$$\Rightarrow \boxed{A_{ik} = (\sigma - \beta) \frac{\langle i | \lambda_j \rangle \lambda_j \langle \lambda_j | k \rangle}{[\sigma - \beta - \lambda_j]}}$$

Here BETA is some fixed input trajectory.

Comments on Fredholm Theory.

1. The basic Fredholm Equation is this:

$$\phi(x) = f(x) + \int dx' \phi(x') K(x', x)$$

Here I have reversed the order of the kernel from its usual sense. Suppose we now introduce a dummy variable x'' to get:

$$\phi(x'', x) = f(x'', x) + \int dx' \phi(x'', x') K(x', x)$$

This is the usual way I encounter these equations. Sometimes f and K are closely related. I have also been using this notation for the same equation:

$$\phi(1,3) = f(1,3) + \sum_2 \phi(1,2) K(2,3)$$

2. Now, in certain cases the kernel is separable or factorizeable. I have considered this possibility elsewhere. Here, I will not assume anything of the kind. The solution to this equation is:

$$\phi = f + \phi K \quad \text{in matrix sense.}$$

$$\phi = f \cdot (1-K)^{-1} = \frac{f \cdot \tilde{N}}{1-K} \quad \tilde{N} = |1-K|^{-1} + N$$

So, you can solve this thing. In the continuum sense, you will need to compute the Fredholm Determinant $|1-K|$ and the Fredholm First Minor N , which I give elsewhere. If K is small, you might want to use the leading terms (trace approximation). But if K is factorizeable, then you would do better to use the other techniques for handling this equation in that case, technique which is much simpler.

3. The special cases I have been running into of this equation have:

$$\phi(1,3) \rightarrow A(1,3) \quad f(1,3) \rightarrow K(1,3) \quad K(2,3) \rightarrow P(2) K(2,3)$$

$$\text{so } A(1,3) = K(1,3) + \sum_2 A(1,2) P(2) K(2,3)$$

$$\Rightarrow A = K(1-PK)^{-1}$$

I.e., here is still the general Fredholm solution.

4. One way to approximate the solution to such an equation is to somehow reduce the problem from a continuum problem to a lattice problem. This is what the computer would do. Then you can solve things in the finite matrix sense.

I would like to make up a name for this: then a theorem

Theorem: even when your kernel is non-separable, you can formally solve the integral equation in The Fredholm Sense. You can approximate the solution in The Fredholm Lattice Sense.

Theorem 2: if kernel is (finitely) separable, you can exactly solve the integral equation by (finite) matrix techniques (see elsewhere).

The Theory of Solving Integral Equations with Factorizeable Kernels.

1. Actually, we shall assume separable kernels, then the limit of a simple factorizeable kernel will be taken. The implication here is that the inhomogeneous term is the thing that is separable, and this same term appears as part, though not all, of the true kernel, ie, the thing under the integral other than the function you are solving for. So here is our prototype equation:

$$A(1,3) = K(1,3) + \sum_2 A(1,2) P(2) K(2,3) \quad (1)$$

Actually, the thing I call K is not the true kernel, PK is. I choose the above form because it conforms to my needs of the moment. The 2-sum is written as an integral since a) that's what it is in part, could also include discrete sums; b) I want to keep that "integral equation sum" separate from the "separable kernel sum" which I am about to introduce.

2. Let us now assume that the kernel is separable in this sense:

$$K(1,2) = \sum_{i=1}^m L_i(1) M_i R_i(2) \quad (2)$$

The letters are supposed to mean: left, middle, and right. The object M_i does not depend on 1 or 2, and could have been included into one of the sides. I find it convenient to explicitly allow for such a "middle" object, since in most applications there is such a middle object.

3. Now, what happens when this kernel is substituted in both places in the equation in P 1 above:

$$A(1,3) = \sum_{i=1}^m \left[L_i(1) + \sum_2 A(1,2) P(2) L_i(2) \right] M_i R_i(3) \quad (3)$$

We find that we have completely exposed the dependence of $A(1,3)$ on variables 3. So let's define a new "left" type object script L. Then we have shown that:

$$A(1,3) = \sum_{i=1}^m \mathcal{L}_i(1) M_i R_i(3) \quad (4)$$

$$\text{where } \mathcal{L}_i(1) = L_i(1) + \sum_2 A(1,2) P(2) L_i(2) \quad (5)$$

Now insert (4) into (5) to find that:

$$\mathcal{L}_i(1) = L_i(1) + \sum_{j=1}^m \mathcal{L}_j(1) M_j \left\{ \sum_2 R_j(2) P(2) L_i(2) \right\} \quad (6)$$

4. Now, notice that inside the curly bracket, R is on the left, and L on the right. This happened because this is the way things were pulled apart. Define the bracket to be a matrix X_{ji} element. Also, upgrade the two L's and M to diagonal matrices for ease of manipulation. Then (6) is this matrix equation:

$$X_{ji} \equiv \sum_2 R_j(2) P(2) L_i(2) \quad (7)$$

$$\mathcal{L} = L + \mathcal{L} M X \quad (8)$$

5. Correct a mistatement: we want to convert equation (6) to a matrix equation. At the moment, the L's form a vector. We should interpret this vector as, say, the first row of a matrix, so that $L_i = L_{1i}$ where now L is a matrix. This is the proper way to convert a vector to a matrix. However, the object M is a linking thing, and we want to imagine it as a diagonal matrix as mentioned before. With this in mind, we write (8) as:

$$L_{ii} = L_{1i} + \sum_{jk} L_{ij} M_{jk} X_{ki}$$

$$L = L + L M X$$

6. The solution of this matrix equation is not too hard to find:

$$L = L \cdot (I - MX)^{-1} = L (M^{-1} - X)^{-1} M^{-1}$$

$$L_{ii} = \sum_{jk} L_{ij} (M^{-1} - X)^{-1}_{jk} M^{-1}_{ki} = \sum_j L_j (M^{-1} - X)^{-1}_{ji} M^{-1}_{ii}$$

7. Now, put this solution back into (4) to get the solution to this integral equation. Rewrite (4) as:

$$\begin{aligned} A(1,3) &= \sum_{i,j=1}^n L_{1i}(1) M_{ij} R_{j1}(3) = \sum_{i=1}^n L_i(1) M_i R_i(3) \\ &= \sum_{i=1}^n \sum_j L_j (M^{-1} - X)^{-1}_{ji} M_{ii} R_i(3) \\ &= \sum_j L_j(1) (M^{-1} - X)^{-1}_{ji} R_i(3) \end{aligned}$$

So, my only use of matrix notation really was to solve for script L. Could have used a reverse vector notation. However, matrixization of M is very useful.

8. The solution to our integral equation may now be written in this way:

$$A(1,3) = \sum_{i,j=1}^n \frac{L_i(1) C_{ij} R_j(3)}{|M^{-1} - X|}$$

where / / indicates the determinant of the nxn matrix, and where

$$\begin{aligned} C_{ij} &= [\text{cof}(M^{-1} - X)]_{ji} & M_{ij} &= \delta_{ij} M_{ii} = \delta_{ij} M_i \\ X_{ij} &= \int_2 R_i(2) P(2) L_j(2) \end{aligned}$$

For completeness, I copy here the original equation and the separable kernel form:

$$\begin{aligned} A(1,3) &= K(1,3) + \int_2 A(1,2) P(2) K(2,3) \\ K(1,2) &= \sum_{i=1}^n L_i(1) M_i R_i(2) \end{aligned}$$

the solution

the problem

9. Now we take the limit where there is only one term in the kernel, ie, a factorizeable kernel: Then, the objects R, L, M, X are just numbers. We find:

factorizeable case:

$$A(1,3) = K(1,3) + \sum_2 A(1,2) P(2) K(2,3)$$

$$K(1,2) = L(1) \cdot M \cdot R(2)$$

$$\Rightarrow A(1,3) = \frac{K(1,3)}{\left[1 - M \left\{ \sum_2 R(2) P(2) L(2) \right\}\right]}$$

In the fact. case, the solution is equal to some number times the kernel.

10. To make sure this result is right (since I am using it to check the general result) lets do this directly. We find:

$$A(1,3) = L(1) M R(3) + \sum_2 A(1,2) P(2) L(2) M R(3)$$

$$= \left[L(1) + \sum_2 A(1,2) P(2) L(2) \right] M R(3)$$

$$= \mathcal{L}(1) M R(3)$$

$$\text{where } \mathcal{L}(1) = L(1) + \sum_2 \mathcal{L}(1) M R(2) P(2) L(2)$$

$$= L(1) + \mathcal{L}(1) M X \Rightarrow \mathcal{L}(1) = L(1) / (1 - MX)$$

$$\Rightarrow A(1,3) = L(1) M R(3) / (1 - MX) = K(1,3) / (1 - MX) \checkmark$$

11. Comment: In my applications, the object X_{ij} as defined above turns out to be a symmetric matrix. Since M and M^{-1} are also symmetric matrices, the cofactor matrix C_{ij} would (in this case) also be symmetric. As noted elsewhere, this has the implication that (if X is symmetric) the matrix C_{ij} factorizes at a place where the determinant has a simple ~~max~~ zero.

Let us imagine that X and M both depend on some parameter p, and that the determinant has a simple zero at $p = p_0$. Assume also that X is symmetric. Then of course $A(1,3)$ is also a function of p. $A(1,3)$ is going to have a simple pole at $p = p_0$. The residue of that pole is going to factorize! We have:

$$|M^{-1} - X| \approx C(p_0) \cdot (p - p_0) \text{ near } p = p_0$$

$$\Rightarrow A(1,3) |_{p \approx p_0} = \frac{\left[\sum_{i=1}^n L_i(1) \sqrt{\frac{C_{ii}}{C(p_0)}} \right] \cdot \left[\sum_{j=1}^n R_j(3) \sqrt{\frac{C_{jj}}{C(p_0)}} \right]}{(p - p_0)}$$

12. Restate last result. If the kernel is separable, X is symmetric, and the det has a simple zero at $p = p_0$, then near the pole $A(1,3)$ has the form:

$$A(1,3) \Big|_{p \times p_0} = \frac{\beta^{p_0}(1) \cdot \beta^{p_0}(3)}{(p - p_0)}$$

and M symmetric.

where

$$\beta^{p_0}(1) = \sum_{i=1}^n L_i(1) \left[\frac{C_{ii}}{C(p_0)} \right]^{\frac{1}{2}}$$

$$L_i(1) = R_i(1)$$

and

$$|M^{-1} - X| \approx C(p_0) (p - p_0)$$

$$C_{ii} = \text{cof} (M^{-1} - X)_{ii}$$

Here, the betas are just numbers. Even though we are very close to $p = p_0$, we still cannot trivially compute the residue factors because we still have to perform the inversion of the matrix $(M^{-1} - X)$, i.e., we have to know the diagonal elements of the cofactor matrix. Perhaps there is some trick to computing the diagonal elements of the inverse of a symmetric matrix, but I don't know of any trick to do this, just brute force computer.

13. If we take the factorizable limit of the above result, we get $C_{ii} = 1$ and:

$$\beta^{p_0}(1) = L(1) / \sqrt{C(p_0)} \quad L(1) = R(1)$$

14. Even in the simplest case, however, in order to locate any poles you have to be able to do the integral which is called X .

Extension of Theory to Include a Double-Separable Kernel.

1. Let us reconsider the same integral equation, but in the case where the matrix called M is not diagonal. I.e., the kernel has this form: - - - -

$$K(1,2) = \sum_{i,j=1}^n L_i(1) M_{ij} R_j(2) \quad (2')$$

Finally, let's get this into useful matrix form. We want $\vec{L}$ to visualize L and script- L as horizontal vectors $(x, x, x, \dots)$, whereas R is a vertical vector. Then the object above is just a number and there is no need to show sums. Now here are some of the previous equations:

$$K(1,2) = \vec{L}(1) \cdot M \cdot \vec{R}(2)$$

$$A(1,3) = [\vec{L}(1) + \sum_2 A(1,2) P(2) \vec{L}(2)] \cdot M \cdot \vec{R}(3)$$

$$= \vec{\mathcal{L}}(1) \cdot M \cdot \vec{R}(3)$$

$$\vec{\mathcal{L}} = \vec{L} + \sum A(1,2) P(2) \vec{L}(2)$$

$$X = \sum_2 \vec{R}(2) P(2) \vec{L}(2)$$

$$\vec{\mathcal{L}} = \vec{L} + \vec{\mathcal{L}} M X$$

$$\vec{\mathcal{L}} (1 - M X) = \vec{L}$$

$$\vec{\mathcal{L}} = \vec{L} (M^{-1} - X)^{-1} M^{-1}$$

$$A(1,3) = \vec{L}(1) (M^{-1} - X)^{-1} \vec{R}(3)$$

In other words, everything I did before is still valid. It's just that M is no longer a diagonal matrix. Note that the problem is not at all complicated by this fact. The problem of computing the determinant is just as hard as before. The factorizable limit of course is the same. The pole analysis is the same. Note that it appears that M should be a non-singular matrix! I.e., you expect that $|M| \neq 0$.

2. State results here:

$$A(1,3) = K(1,3) + \sum_2 A(1,2) P(2) K(2,3)$$

$$K(1,2) = \sum_{i,j=1}^n L_i(1) \cdot M_{ij} \cdot R_j(2)$$

$$\Rightarrow A(1,3) = \sum_{i,j=1}^n L_i(1) \cdot C_{ij} \cdot R_j(3) / |M^{-1} - X|$$

where:

$$C_{ij} = [\text{adj}(M^{-1} - X)]_{ij}$$

$$X_{ij} = \sum_2 R_i(2) P(2) L_j(2)$$

but M must also
be symmetric

Alternative Form of the General Solution.

1. I wrote the solution in the way I did because I wanted the matrix in question to be symmetric if X and M were both symmetric. Since the product of two symmetric matrices is not symmetric, the present form I am about to give is not as nice because you don't get explicitly factorizable poles. Of course you know that the poles will still factorize, but not so clear in this notation.

2. On the other hand, this notation will be useful in the case that M or X or both are somehow "small", because then you can use the expansion theorems which we are used to in Fredholm theory. So:

$$A(1,3) = \sum_{ijk} L_i(1) [\log(1-MX)]_{ij} M_{jk} R_k(3) / |1-MX|$$

3. Usually, the combination MX is called K in Fredholm theory. For the numerator, well I will make a new sheet for this right now:

$$A(1,3) = \sum_{ijk} L_i(1) (1-MX)^{-1}_{ij} M_{jk} R_k(3)$$

4. Suppose the matrix MX is factorizable. I.e., suppose that:

$$(MX)_{ij} = A_i B_j$$

Well, this could only really happen if M and X were each factorizable. But if M were factorizable, then your original kernel would be factorizable. I.e., suppose that:

$$M_{ij} = f_i h_j \cdot k$$

Then: $K(1,2) = \left(\sum_i L_i(1) f_i \right) \cdot k \left(\sum_j R_j(2) h_j \right) = L'(1) \cdot k \cdot R'(2).$

But when your kernel is so trivially factorizable, you already know that the solution is very simple, namely:

$$A(1,3) = \frac{L'(1) \cdot k \cdot R'(3)}{1 - k \left[\int R'(z) P(z) L'(z) \right]}$$

Relation between Fredholm Theory and Greens Function of ODE Theory.

1. You know that if you have a differential equation like this:

$$(L-\lambda)\phi = f$$

$$(L-\lambda)g = \delta$$

then there is a related integral equation. The kernel of this integral equation is the greens function of the differential equation. Thus:

$$\phi = (L-\lambda)^{-1} f \Rightarrow \phi(x) = \int g(x,y) f(y) dy$$

This is of course the whole point of using greens functions. You first solve for your greens function, then using it, you can write down the solution to the general ~~hom~~ inhomogeneous equation. (modulô BC's). Note that the associated integral equation is the "inverse equation", not something you get by differentiating both sides (variable does not appear as an endpoint).

2. We know that in general a greens function can be written as a sum involving the eigenvalues of the eigenvalue equation:

$$g(x,y;\lambda) = \sum_n \frac{\phi_n(x) \phi_n(y)}{\lambda - \lambda_n}$$

Then when you contour integrate the greens function, you generate the delta completeness relation:

$$\delta(x-y) = \sum_n \phi_n(x) \phi_n(y)$$

$$\langle x|y \rangle = \sum_n \langle x|n \rangle \langle n|y \rangle.$$

3. Now consider an integral equation whose kernel is the greens function. An obvious theorem is this:

Theorem: if kernel is non-separable, then eigenvalues must cluster at ∞ .

Proof: non-separable means the above sum is infinite (sum for greens function). Since any finite region of the λ -plane only contains finite number of eigenvalues (a property of Fredholm kernel), and since there are an infinite number of eigenvalues, they must be clustering at infinity. Ie, finite cluster points are not allowed.

4. Comment: I have vaguely mixed defining properties of "self-adjoint ODE" and "Fredholm kernel". Probably anything you say for differential equations goes for the related integral equation, though not vice versa. Ie, I think a self-adjoint ODE problem translates into a Hilbert-Schmidt = Fredholm integral equation problem.

An application of GFredholm Theory.

1. I am reading paper of Jones and Teplitz.

2. Consider an integral equation like (2.3) of above ref. As usual, the "parameter" λ is set to one, but there is a hidden or buried parameter j . Call it j . Now, from our simple matrix Fredholm theory, we know that the solution to the integral equation as well as the resolvent will have poles in j at points where $\det(K-1)=0$. We also know that in such a case, the eigenvalue equation $K\phi = 1\phi$ (here $\lambda=1$) has a solution, because after all the pole locations are the eigenvalues of the homogeneous problem.

3. So, let K be the residue of K at the pole it has at $j=j_0$. This value j_0 makes the Fredholm determinant vanish, this it is a pole of the solution to the inhomogeneous problem. Thus, looking at the eigenvalue problem, we see:

$$K\phi = (j-j_0)\phi \quad \Rightarrow \quad \frac{1}{(j-j_0)} \int \bar{K}(x,x') \phi(x') dx' = \phi(x).$$

Thus we have explained the equation which appears in the appendix of above paper. There, $j_0 = -1$.

4. Apparently, in any Fredholm eigenvalue problem,

FREDHOLM THEORY REVISITED: The matrix viewpoint

- ① A typical integral equation has the form:

$$\phi(x) = f(x) + \int_a^b K(x, x') \phi(x') dx'$$

the "inhomogeneous term"

the "kernel"

- ② At once, we may consider this as a vector + matrix statement. In fact, one approx. method actually uses finite interpolating matrices to solve the equation. Of course our matrices here are of infinite dimension.

$$\phi = f + K\phi$$

- ③ Now having written this in matrix notation, we can solve:

$$[I - K]\phi = f \Rightarrow \phi = [I - K]^{-1} f$$

However, $A^{-1} = \hat{A}/|A|$ where $\hat{A}$ = classical adjoint $\equiv \text{adj}(A)$.
Thus:

$$\phi = \frac{(\hat{I} - \hat{K}) f}{|I - K|}$$

- ④ Mikhlin discusses the iterative solution method + construction of a resolvent: If the resolvent exists, then:

$$\phi(x) = f(x) + \int_a^b \Gamma(x, s) f(s) ds$$

where

$$\Gamma(x, s) = \sum_m K_m(x, s)$$

where

$$K_m(x, s) = \int ds_1 ds_2 \dots ds_{m-1} \underbrace{K(x, s_1) K(s_1, s_2) \dots K(s_{m-1}, s)}_{m \text{ - kernels}}$$

Notice:

$$K_1(x, s) = K(x, s)$$

$$K_2(x, s) = \int ds_1 K(x, s_1) K(s_1, s)$$

⑤ We may convert all these statements to matrix language:

$$\phi = f + \Gamma \cdot f$$

$$\Gamma = \sum_{m=1}^{\infty} K_m$$

$$K_m = K^{m.}$$

Thus, the resolvent is: $\Gamma = \sum_{m=1}^{\infty} K^m$

⑥ We may relate the resolvent to our other solution:

$$\phi = (1-K)^{-1} \cdot f = (1+\Gamma) \cdot f$$

$$\Rightarrow (1-K)^{-1} = (1+\Gamma)$$

$$\alpha: 1 = (1-K)(1+\Gamma) = (1+\Gamma)(1-K)$$

$$\Rightarrow \Gamma = K + K\Gamma = K + \Gamma K$$

⑦ Note that six and five are consistent because:

$$(1-K)^{-1} = 1 + \sum_{m=1}^{\infty} K^m = 1 + \Gamma \Rightarrow \Gamma = \sum_{m=1}^{\infty} K^m$$

⑧ Converting back to continuum notation, we can relate the resolvent + kernel:

$$\Gamma(x, z) = K(x, z) + \int K(x, y) \Gamma(y, z) dy \quad + \text{ reverse.}$$

⑨ The homogeneous equation is:

$$\mu \phi(x) = \int_a^b dx' K(x, x') \phi(x')$$

or simply:

$$\mu \phi = K \phi$$

This is just the standard eigenvalue problem. (Here we finally have to show $\mu = \lambda^{-1}$ instead of $\lambda = 1$ as above.)

- ⑩ Many theorems of linear algebra apply to integral equations via this matrix connection. Eg, consider

$$(K-1)\phi = f \quad / \text{this is our integral equation.}$$

Fredholm's Fourth Theorem: let $A \equiv K-1$ so $A\phi = f$.

Imagine that the homo. equation $A^*\phi = 0$ has n nontrivial solutions, ϕ_i $i=1, 2, \dots, n$. This means that the "nullity" or "size of the nullspace" of A^* is n . [By the way, this means that $\text{rank } A^* = N-n$.]

The theorem: $A\phi = f$ has solution(s) iff

$$(f, \phi_i) = 0 \quad \forall \quad i=1, \dots, n.$$

In other words, the space of solutions f [i.e., the range of A] is orthogonal to the nullspace of A^* :

$$\text{Or: } R_A \perp N_{A^*} \quad \text{Or: } R_A = (N_{A^*})^\perp$$

Corollary: if $A\phi = 0$ has only the trivial solution $\phi = 0$, then $R_A =$ the full space, and $A\phi = f$ always has a solution. (†)

The Fredholm Alternative: either (a) or (b) is true:

- (a) $A\phi = 0$ has nontrivial solutions
 or: (b) $A\phi = f$ has solution $\forall f$.

/for more, see p154 Stakgold.

- ⑪ Eigenvalue: to find the eigenvalues of K , i.e.

$$K|\psi_\mu\rangle = \mu|\psi_\mu\rangle$$

One need only solve: $|K - \mu \cdot 1| = 0$. If $\mu = \lambda$, this is the Fredholm determinant:

$$|K - 1| = 0 \rightarrow \text{eigenvalues } \{\lambda_i\}.$$

- ⑫ How is a kernel related to its eigenfunctions? (See p237 Stakgold).

Theorem: at least for symmetric kernels, the eigenfunctions of the kernel span the range of the kernel.

I.e., if $Kf = g$, then $g = \sum_i \langle g, \phi_i \rangle \phi_i$ (p218)

Application:

$$K = K|\phi_i\rangle\langle\phi_i| = \sum_i \mu_i |\phi_i\rangle\langle\phi_i|$$

or: $k(x, y) = \sum_i \mu_i \phi_i^*(x) \phi_i(y)$

or: $K_{kl} = \sum_i \mu_i [V_i V_i^\dagger]_{kl}$

Here, $[V_i V_i^\dagger]$ is a matrix; $KV_i = \mu_i V_i$

These expressions for the kernel are called Mercer's Theorem. (p240). In fact, you can only do such an expansion if spectrum is "quasidefinite" [+ this is the case in m.p. models].

- ⑬ Do not confuse kernels with greensfunctions!

Derivation of Fredholm's First Minor

① We know that:

$$\Gamma = -1 + \frac{\text{cof}(1-K)}{|1-K|} \stackrel{?}{=} \frac{N}{|1-K|}$$

equality is true with N given as follows:

theorem:

② Nikolai claims that N_{ij} is given by: (numerator of resolvent).

$$N_{ij} = K_{ij} \sum_k \begin{vmatrix} K_{ij} & K_{ik} \\ K_{kj} & K_{kk} \end{vmatrix} + \frac{1}{2!} \sum_{k,l} \begin{vmatrix} K_{ij} & K_{ik} & K_{il} \\ K_{kj} & K_{kk} & K_{kl} \\ K_{lj} & K_{lk} & K_{ll} \end{vmatrix} + \dots \text{ etc.}$$

Proof:

③ Expand each determinant by going down first column.

$$N_{ij} = \begin{vmatrix} K_{ij} & K_{ik} & K_{il} & \dots \\ K_{kj} & K_{kk} & K_{kl} & \dots \\ K_{lj} & K_{lk} & K_{ll} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

"n+1 = 3"

$$= \left\{ +K_{ij} K_{kk} - K_{kj} K_{ik} \right\} + \left\{ +\frac{1}{2!} K_{ij} \begin{vmatrix} K_{kk} & K_{kl} \\ K_{lk} & K_{ll} \end{vmatrix} - K_{kj} \cdot \frac{1}{2!} \begin{vmatrix} K_{ik} & K_{il} \\ K_{lk} & K_{ll} \end{vmatrix} + K_{lj} \cdot \frac{1}{2!} \begin{vmatrix} K_{ik} & K_{il} \\ K_{kk} & K_{kl} \end{vmatrix} \right\} + \dots$$

$$= \left\{ +\frac{1}{3!} K_{ij} \begin{vmatrix} 3 \times 3 \end{vmatrix} - K_{kj} \cdot \frac{1}{3!} \begin{vmatrix} 3 \times 3 \end{vmatrix} + K_{lj} \cdot \frac{1}{3!} \begin{vmatrix} 3 \times 3 \end{vmatrix} - K_{mj} \cdot \frac{1}{3!} \begin{vmatrix} 3 \times 3 \end{vmatrix} \right\} + \dots$$

Note that all indices except i and j are implicitly summed.

④ Now it is easy to show that in a $\{ \}$ bracket of $n+1$ terms, the rightmost n terms are all identical. Eg, look at $n+1 = 3$. One need only switch k and l to see that term 3 = term 2. Therefore,

$$N_{ij} = (K_{ij} |1-K|)$$

$$+ [K_{kj} K_{ik} - K_{kj} \begin{vmatrix} K_{ik} & K_{il} \\ K_{lk} & K_{ll} \end{vmatrix} + K_{kj} \frac{1}{2!} \begin{vmatrix} K_{ik} & K_{il} \\ K_{lk} & K_{ll} \end{vmatrix} + \dots]$$

Here we use the known expansion for $|1-K|$. But now, pulling K_{kj} from second group we get

$$N_{ij} = K_{ij} |1-K| + K_{kj} [N_{ik}]$$

⑤ Therefore, the expansion N_{ij} satisfies this integral equation:

$$N = K(1-K) + NK$$

$$\text{or } \left(\frac{N}{1-K} \right) = K + \left(\frac{N}{1-K} \right) K$$

⑥ But we know from $\Gamma + 1 = (1-K)^{-1}$ that the resolvent solves:

$$\Gamma = K + \Gamma K.$$

$$\text{Thus, } \frac{N}{1-K} \stackrel{!}{=} \Gamma.$$

⑦ Since $\Gamma = K + \Gamma K$, we have $\Gamma(1-K) = K$

$$\text{So } \Gamma = K(1-K)^{-1} = \frac{K \widetilde{\text{cof}}(1-K)}{1-K}$$

Thus, as an aside, we have learned how to expand:

$$K \widetilde{\text{cof}}(1-K)$$

$$\text{Namely, } [K \widetilde{\text{cof}}(1-K)]_{ij} = N_{ij}$$

$$\text{⑧ Also, } \Gamma = \frac{-|1-K| \cdot 1 + \widetilde{\text{cof}}(1-K)}{1-K}$$

$$\text{Thus: } \widetilde{\text{cof}}(1-K) = |1-K| \cdot 1 + N$$

So we also have an expansion for $\widetilde{\text{cof}}(1-K)$.

⑨ Finally, $\widetilde{\text{cof}}(1-K) = (1-K)^{-1} |1-K|$, so:

$$(1-K)^{-1} = 1 + \frac{N}{1-K}$$

This gives us an expansion for $(1-K)$ inverse.

⑩ Also, of course, $\Gamma = 1 + K + K^2 + \dots$

Comment on Fredholm.

In Joel's thesis, in his paper on baryon multiplicities, and in my discussion with Geoff back in April 1974 I ran into what I would now call the matrix formalism for multiple channels. I have recently outlined this technique in Joel Thesis Notes. It turns out to be quite trivial. You have there an integral (multiperipheral) equation which diagonalizes completely when transformed to the J -variable. The t -integrations never appear in Joel's simple model. The integral equation then becomes a simple matrix equation of the form:

$$\underline{F}(J) = \underline{G}(J) + [\underline{G}(J)\underline{S}(J)] \underline{F}(J) \\ = \underline{G}(J) + \underline{K}(J)\underline{F}(J)$$

These things are matrices in "channel space" and you might call the matrix K the "kernel" if you like. (It was the kernel in the integral equation before the transformation). The matrix S is always diagonal and contains the appropriate Regge "cut propagators" which attach to G . The kernel G itself has strengths, widths, and possible off diagonal terms.

$$\underline{G} \cdot \underline{S} = \begin{bmatrix} \gamma_1 e^{-\alpha_1 J} & \gamma_3 e^{-\alpha_3 J} \\ \gamma_3 e^{-\alpha_3 J} & \gamma_2 e^{-\alpha_2 J} \end{bmatrix} \begin{bmatrix} \frac{1}{J-\beta_1} & 0 \\ 0 & \frac{1}{J-\beta_2} \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_1 \frac{1}{J-\beta_1} e^{-\alpha_1 J} & \gamma_3 \frac{1}{J-\beta_2} e^{-\alpha_3 J} \\ \gamma_3 \frac{1}{J-\beta_1} e^{-\alpha_3 J} & \gamma_2 \frac{1}{J-\beta_1} e^{-\alpha_2 J} \end{bmatrix} = \underline{K}$$

$$\left\{ \begin{array}{c} \text{---} \textcircled{\gamma_1} \text{---} \\ \text{---} \textcircled{\gamma_2} \text{---} \\ \text{---} \textcircled{\gamma_3} \text{---} \end{array} \right\} \leftarrow G^{\text{'s}}$$

$$\left\{ \text{---} = \frac{1}{J-\beta_1} = - \frac{1}{J-\beta_2} \right\} \leftarrow S^{\text{'s}}$$

$$\left\{ \begin{array}{l} \text{---} \textcircled{\gamma_1} \text{---} = \gamma_1 \frac{1}{J-\beta_1} e^{-\alpha_1 J} \\ \text{---} \textcircled{\gamma_2} \text{---} = \gamma_2 \frac{1}{J-\beta_2} e^{-\alpha_2 J} \\ \text{---} \textcircled{\gamma_3} \text{---} = \gamma_3 \frac{1}{J-\beta_1} e^{-\alpha_3 J} \end{array} \right\} \leftarrow K^{\text{'s}}$$

Whether you think in terms of K 's (Chew) or S 's and G 's (Koplik) the resulting equation is the same. This equation is just a matrix equation, there is no integral in sight. The direct solution is:

$$\underline{F} = \frac{\underline{G} \underline{S} (1 - \underline{K})^{-1} \underline{K} \underline{S}^{-1}}{\det(1 - \underline{K})}$$

$$\underline{K} = \underline{G} \underline{S}$$

To find the poles in $F(J)$ you must look at this determinant. I do not feel that it is correct to refer to this as a Fredholm determinant, nor do I feel there is any meaning whatsoever here for "trace approximation". This is just the solution to a matrix problem.

$$\det(1 - \underline{K}) = 1 - \text{tr } \underline{K} + \dots \quad \text{OK, this is all settled}$$

Genuine Fredholm Determinant.

I think the real application of Fredholm theory is that described back on the first yellow sheet. You have a genuine integral equation (one variable of integration). There is a kernel, and a (different) inhomogeneous term. The so-called "parameter" λ is equal to 1. Although the texts deal with the subject of analyticity in the "parameter", this is not the exact analyticity we are interested in.

So, we have this "second kind" integral equation. If it really is a Fredholm equation (square integrable kernel), then we can write a formal solution in terms of the "resolvent". And, we can formally express the resolvent as Fredholm's first minor divided by the Fredholm Determinant.

Now we add on the variable J just as a parameter that sits quietly in all the components of the original integral equation. This parameter allows us to vary the equation in some sense.

Obviously this parameter will show up in the resolvent and in the Fredholm determinant. If this determinant vanishes for some value of parameter J , then you have found a pole in the solution of the integral equation as a function of J .

The Fredholm determinant is not very obvious. It is the determinant of an infinite dimensional continuum matrix, as one can see by looking at its development and limit via mechanical quadrature. You could certainly construct an approximate finite dimensional matrix from knowledge of the kernel and inhomogeneous term. In the continuum case, I do not know how you calculate such a determinant.

However, from page 17 I see that if the Fredholm "bound" B is smaller than unity, you can write an explicit closed form for the determinant.

$$D(s) = \exp \left[- \sum_{n=1}^{\infty} \frac{A_n(s)}{n} \right]$$

where $A_n = h(K_n) = \int dx_1 \dots dx_n K(x_1, x_2) \dots K(x_{n-1}, x_n) K(x_n, x_1)$

More generally you can write a series for the determinant. Each term in the series is a finite multiple integral over the kernel (p 17). Hopefully, it will converge. The first term beyond the 1 is the "trace". Note that if the kernel factorizes, the trace approximation is exact!!! ✓

Finally, one might think of the original "parameter" λ in terms of the overall scale coupling strength of the kernel(s).

For an example of use of the trace approx, see Chew Rogers and Snider. Their kernel is shown on page 22, along with their integral equation. Then on page 25 the trace approximation is used for Fred's det. Notice that they use λ , but this should not be confused with the real Fredholm "parameter" which is set to unity throughout. ✓

See following page for derivation of "first minor".

See math/matrix notes for the "fred. det" expansion.

Back to Fredholm Theory

1.

Here are some of the conclusions of Nikishin's book 1964.

You start with an equation of the form:

$$\phi(x) - \lambda \int_a^b K(x,s) \phi(s) ds = f(x)$$

This is second kind [ϕ appears twice].

A typical multiperipheral equation is:

$$B(J,t) = B_0(J,t) + \sum_J \int_{-\infty}^0 dt' B'(J,t') G^{JJ}(J,t',t) \quad \text{Chew-Frazer.}$$

Neglecting the λ sums, we have: (J so far just side there as a parameter.)

$$\begin{array}{ll} B \leftrightarrow \phi & B_0 \leftrightarrow f \\ t \leftrightarrow x & G(x',t) \leftrightarrow K(x,s) \\ t' \leftrightarrow s & \lambda \leftrightarrow 1 \end{array}$$

Now, suppose we want to know if there is a place where $\phi(x)$ or $B^J(J,t)$ blows up. Look at the resolvent:

$$\Gamma(s,x;\lambda) = \sum_m \lambda^m K_m(x,s)$$

solution: $\phi(x) = f(x) + \int_a^b \Gamma(s,x;\lambda) f(s) ds.$

We know that: when Γ blows, so does ϕ .

Later we learn that the resolvent solves an equation too:

$$\Gamma(s,x;\lambda) = K(x,s) + \lambda \int_a^b K(x,t) \Gamma(t,s;\lambda) dt$$

After much manipulation, it is shown that:

$$\Gamma(x,s;\lambda) = \frac{D(x,s;\lambda)}{D(\lambda)} \quad \text{where } D(\lambda) = 1 - c_1 \lambda + \frac{1}{2} c_2 \lambda^2 + \dots$$

Fred's Det $\text{where } c_1 = \int dx K(x,x) = \text{trace}(K)$

And, if $K(x,s)$ factorizes into $g_1(x)g_2(s)$, then the full Fred. det is given by:

$$D(\lambda) = 1 - \lambda \text{Trace}(K)$$

There are two unanswered questions

- 1) why is there a connection $\lambda \leftrightarrow$ resolved λ 2) where is $(\det[1 - \lambda K]) = 0$. λ^{15}

1) Why do we connect $\lambda \leftrightarrow J$?

2.

No I think this is a mistake. One way out is this: in our applications, $\lambda=1$, and J = just another parameter. Thus:

$$I(s, x; 1, J) = \sum_m K_m(x, s; J) \quad (\lambda^m = 1^m = 1)$$

$$I(s, x; 1, J) = \frac{D(s, x; 1, J)}{D(1, J)}$$

$$D(1; J) = D(J) = 1 - c_1 + \frac{1}{2} c_2 \dots$$

$$\text{factorizable } K \Rightarrow D(\lambda=1, J) = D(J) = 1 - \text{Trace } K(J).$$

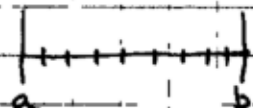
So as parameter J moves, D can hit 0 + you get a pole in solution.

$$\phi(x; J) \text{ blows at } J \Rightarrow D(\lambda=1; J) = 0. \quad \checkmark$$

2) Where is $\det [1 - \lambda K] = 0$?

Consider the "mechanical quadrature" approach to solving the equation.

$$\phi(x) - \lambda \int_a^b K(x, s) \phi(s) ds = f(x)$$



$$\phi(x_i) - \lambda \sum_{j=1}^n K(x_i, s_j) \phi(s_j) = f(x_i)$$

$$\sum_j [\delta_{ij} - \lambda K(x_i, s_j)] \phi(s_j) = f(x_i)$$

$$(\underline{1} - \lambda \underline{K}) \vec{\phi} = \vec{f}$$

$$\underline{K} = K(x_i, s_j) = K_{ij}$$

Now we do have a matrix. The sense is:

As the order of this matrix $\rightarrow \infty$, you approach the exact answer. The solution ϕ blows when:

$$\det(1 - \lambda K) = 0$$

Can we write an expression for this $\det$ in continuum language? I don't even know how to write it in infinite order.

But, this may not be the same $\det$ used by Chew. it is

OK now.

Alternative form Now in reading in Smilgies:

3.

kernel of finite rank:

$$K(s, t) = \sum_{\nu=1}^n a_{\nu}(s) \overline{b_{\nu}(t)} \equiv \sum_{\nu=1}^n a_{\nu} \otimes \overline{b_{\nu}}$$

Then define:

$$K_{\mu\nu} = (a_{\nu}, b_{\mu}) \quad / \text{as in Nikluis}$$

Then we come up with another sense of the matrix condition:

$$\det(\underline{I} - \lambda \underline{K}) = d(\lambda) \quad \underline{K} = K_{\mu\nu}$$

Note: a kernel of this form does not equate to the coupled-channels situation, because only the kernel itself is in the sum.

Note: if $n=1$, then you do have a factorizable kernel and the matrix condition reduces to:

$$\det(\underline{I} - \lambda \underline{K}) = |1 - \lambda(a, b)| = 1 - \lambda \int a(t) b^*(t) dt$$

But this is the bad approximation once again.

Which sense of $\det(1-K)$ is Chew using?

1) The separable kernel sense

2) The mechanical quadrature sense

} read some CS-II definition!

Integral Equations. Need some of this to understand what is meant by:

1) Fredholm determinant

2) trace approximation.

Since I know nothing to speak of about integral equations, now might be a good time. Took out a book from LBL:

IS of Mm P & AM Vol 4 "Integral Equations" S.G. Mikhihin, 2nd Ed, Perg, Mc '64

Intro (preface): This book attempts to consolidate journal work over 20 years (1st ed ~ 1949). The first 133 pages covers the theory, the second 200 pages is applications. This book is translated from the Russian.

Outline:

I. 1. Fredholm Equations (-66)

2. Symmetric Equations (theory of HILBERT-SCHMIDT)

3. Singular Equations (113-134)

(67-112)

II. 1. Dirichlet

2. Biharmonic

3. Schwarz

4. Potentials

5. Symmetric Appl.

6. Singular Appl.

Comment: he notes that the theory of i.e.'s works best in 1 and 2 dim. problems, fails frequently to solve full 3 dim. problems.

1.1 Classification of Integral Equations

Your basic baby is this:

$$\phi(x) - \lambda \int_a^b K(x,s) \phi(s) ds = f(x)$$

↖ parameter ↗ kernel ↗ free term or right hand side

def: Fredholm Type Equation

what you get if the kernel is continuous on the square $(a,b)(a,b)$ and if $\int_a^b \int_a^b |K|^2 dx ds$ is finite.

def: Weak Singularity. ~ kernel is bounded except where $x=s$ so has the form $K = H/|x-s|^\alpha$ where H is bounded + $0 < \alpha < 1$.

def: Full Singular Equation: ~ kernel is $K = A/(x-s)'$, a full pole. Then you have to do a principal part to get something finite

$$\oint K(x,s) \phi(s) ds = \oint \frac{A(x,s)}{x-s} \phi(s) ds$$

↖ tick around the pole at x .

2.
He then gives good examples of each type of integral equation ~ Fredholm, weak singular, and singular. Even a Volterra type is included.

Note: weak sing. does not do much damage, they are about the same as for pure Fredholm.

Multidimensional ~ The idea goes into $\iint_{\Omega \times \Omega} d\mathbf{m} d\mathbf{m}' |K(\mathbf{m}, \mathbf{m}')|^2 < \infty$ and so on.

def: homogeneous: $\phi(x) - \lambda \int_a^b K(s, x) \phi(s) ds = 0 \leftarrow$

inhomogeneous: $f(x) \neq 0$

def: second kind ~ these are the ones we already saw, where ϕ appears twice, once inside the integral.

def: first kind ~ ϕ only appears in integral: $\int_a^b K(x, s) \phi(s) ds = f(x)$

Note: the multivariable theory is a "divivial" generalization of the single variable theory, so we will stick to one variable.

1.2 Successive Approx Method + Resolvent.

Restrict now to Fredholm only. Let's add some extra assumes:

#1. Assume $\int_a^b |K(x, s)|^2 ds < C$, #2 Assume $\int_a^b |f(x)|^2 dx < \infty$
 $\nwarrow$ only one integration here.

Another note, assume K in s and $f(x)$ are L^2 .

Now, the obvious thing to do is to try iterative solution like:

$$\begin{aligned}\phi_0(x) &= f(x) \\ \phi_1(x) &= f(x) + \lambda \int_a^b K(x, s) f(s) ds \\ \phi_2(x) &= f(x) + \lambda \int_a^b K(x, s) \left[f(s) + \lambda \int_a^b K(s, s') f(s') ds' \right] ds \\ \phi_2(x) &= f(x) + \lambda \int_a^b K(x, s) f(s) ds + \underbrace{\lambda^2 \iint_a^b K(x, s) K(s, s') f(s') ds' ds}_{= \lambda^2 \int ds' \left[\int ds K(x, s) K(s, s') \right] f(s')} \\ &\quad \text{"second iterated kernel"} \quad \nearrow K_2(x, s')\end{aligned}$$

Clearly you can go until the cows come home,

$$\phi_n(x) = f(x) + \sum_{m=1}^n \lambda^m \int_a^b K_m(x,s) f(s) ds \quad (K_1 = K)$$

$\nearrow$ n -th order solution.

$\nwarrow$ " m -th iterated kernel"

$$K_m(x,s) = \int_a^b K(x,t) K_{m-1}(t,s) dt$$

If it converges, you are in good shape. If not, dump it.

Almost obvious fact: $K_m(x,s) = \int_a^b K_2(x,t) K_{m-1}(t,s) dt$.
 $\nwarrow$ anything between 1 and $m-1$

So this gives the solution to your problem! You know all the K_m , also given $f(x)$, Sat city. How fast is the convergence?

def: Bunyakovsky-Schwarz Inequality

Says that: $A(x,s) = \int_a^b B(x,t) C(t,s) dt$

$$\Rightarrow |A(x,s)|^2 \leq \left(\int_a^b |B(x,t)|^2 dt \right) \left(\int_a^b |C(t,s)|^2 dt \right)$$

(In operator language I think this says: $|\int_a^b f(t)g(t)dt| \leq \|f\| \cdot \|g\|$)
 Just like in $\mathbb{R}^3$: $|\vec{F} \cdot \vec{G}| \leq |\vec{F}| \cdot |\vec{G}|$

Now he shows that you have convergence only if $|\lambda|^2 < \frac{1}{B^2}$ where $B^2 = \iint K^2 dx ds$. This is so because he shows that the solution series is bounded by a geometric series which converges if

Theorem: If $\int_a^b |K|^2 ds \leq C$, and if $\int_a^b \int_a^b |K|^2 dx ds \equiv B^2$, then the

method of successive approx works and gives the unique solution provided that

$$|\lambda| \leq \frac{1}{B}. \quad (\text{if } \lambda = 1, \text{ then } B \leq 1.)$$

$\nwarrow$ λ can be complex!

i.e. kernel must be "small".

$\nwarrow$ circle of convergence in λ plane.

He gives a nice little example of an integral equation that "solves" by this method. If the first condition is not met (the one variable boundedness), method may still work in sense, not discussed.

def: Resolvent: Rewrite series solution as

$$\phi(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(x,s) f(s) ds = f(x) + \int_a^b \left[\sum_{m=1}^{\infty} \lambda^m K_m(x,s) \right] f(s) ds$$

This is the "resolvent" $R(s,x;\lambda)$

The equation is said to have a resolvent if that candidate resolvent does in fact solve the equation, regardless of whether or not $|\lambda| < B^{-1}$ or what have you.

Practical Note - if this succ. approx method does yield a closed form solution, then almost always there was an easier way to solve the thing from the start!

1.3 Volterra Type. $\phi(x) - \lambda \int_a^x K(x,s) \phi(s) ds = f(x)$
 not b.

Theorem: If $\int |f(x)| dx < \infty$, then the method of S. Approx always works for any λ .

In fact, even if K has weak sing, the Volterra thing solves by S.A. for all λ too (but very slow convergence!).

Example: For $\phi(x) - \lambda \int_0^x e^{x-s} \phi(s) ds = f(x)$, he gets a closed form for the iterated kernel and then for the actual resolvent itself: $\Gamma(x,s;\lambda) = \exp[(\lambda+1)(x-s)]$. This leads to a general solution for any f . BVT, the equation is trivially solved by making it an ODE!

1.4 Degenerate Kernels.

def: degenerate kernel - called "separable" by others, namely the kernel can be written as: $K(x,s) = \sum_{i=1}^n a_i(x) b_i(s)$
 Sort of "factorizable". lin indep.

Then equation goes to: $\phi(x) - \lambda \sum_{i=1}^n a_i(x) \underbrace{\int_a^b b_i(s) \phi(s) ds}_{\equiv c_i} = f(x)$
 unknown constants.

Now screw around + define: $f_i = \int_a^b b_i(s) f(s) ds$
 Equation then boils down to: $\vec{c} - \lambda \underline{A} \vec{c} = \vec{f}$
 $(1 - \lambda \underline{A}) \vec{c} = \vec{f}$
 $\alpha_{ik} = \int_a^b b_i(s) a_k(s) ds$
 - you know all these things

So you solve this for the c_i then you've got the entire answer

$$\phi(x) = f(x) + \lambda \vec{c} \cdot \vec{a}(x)$$

So you might look at $|1 - \lambda A| = 0$? to locate singularities in the solution $\phi(x)$, often a scattering amplitude? I do not think however that this is the famous "Fredholm" determinant. It only applies to factorizable kernels. Maybe this is it?! Must wait till we get to page 58 to see. (I can wait).

Two nice examples given here for "degenerate" solutions.

1.5 General Case of Fredholm's Equation

The main idea is that any reasonable kernel can be approximated as closely as you want by a degenerate kernel!! Just do a double Fourier series:

$$K(x, s) = \sum_n \sum_m a_{nm} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi s}{L}\right) \quad L = b-a$$

We show that as the sums get more + more terms, the solution converges to the full K . So now we can do any old Fredholm equation by the separation method! Yay say!

Example: some fancy Dirichlet problem with an ellipse. He first does it for small ϵ + gets a kernel of three terms hence c_1, c_2, c_3 . Then he does the full solution as an infinite series.

1.6 Systems of Integral Equations "often occur in practice"

$$\phi_i(x) - \lambda \sum_{k=1}^n \int_a^b K_{ik}(x, s) \phi_k(s) ds = f_i(x) \quad \forall i=1, n.$$

An additional matrix aspect to deal with. I could run into this with isospin some day (som). There are $n \times n$ kernels, all of them have to be bounded. If they are all bounded & in the one variable sense, then the M of SA works if $|\lambda| <$ some appropriate expression.

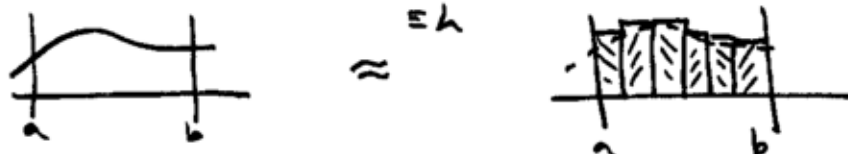
There is a trick here to get a system into a single hotshot equation by somehow laying out all the $f_i(x)$ along an elongated interval. Does not seem important and is the first unclear paragraph in 31 pages.

Comment: This book is well done. I like the good usage made of white space. (Unlike in Phys Rev. papers).

1.7 Applications of Approx. Integration Formulae.

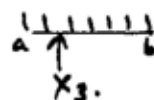
def: mechanical quadrature. - This simply means doing the integral by putting in rectangles ~:

$$\int_a^b K(x,s) \phi(s) ds \approx \underbrace{\left(\frac{b-a}{n}\right)}_{=h} \sum_{k=1}^n K(x, x_k) \phi(x_k)$$



You pick some points (n of them) called $\{x_k\}$. The integral equation, true for all x , must be true for these points too, so:

$$\phi(x_i) - \lambda h \sum_{k=1}^n K(x_i, x_k) \phi(x_k) = f(x_i)$$



But this is just n equations in n unknowns [i.e., $\phi(x_1), \phi(x_2), \dots$]. The trivial solution gives you n points from which you can interpolate a reasonable approximate solution. No integration needed!

Example: will need that discrete integral equation.

$$\mu(t) + \underbrace{(-1)}_{\lambda} \int_0^{2\pi} \underbrace{\left\{ \frac{1}{1 - .64 \cos^2\left(\frac{t+\tau}{2}\right)} \right\}}_{K(t,\tau)} \mu(\tau) d\tau = \underbrace{25 \cos^2 t + 9 \sin^2 t}_{f(t)}$$

We chop up the range into 4 (unequal) parts + gets 4 in 4, solves. Amazingly, this powerful method gets you to 1% of the exact soln - obviously we are leaning heavily on the fact that the solution is smooth. Still, this is a dynamic method, probably the one best suited to the computer. (well, except for Gaussian quadrature maybe).

1.8 Fredholm's Theorems. There are four of them + they are an analysis of the equation before you try to solve it.

def: regular ~ values of λ are "regular" if the resolvent exists (ie, usually small λ are regular)

def: characteristic values $\equiv$ eigenvalues ~ for these values of λ , the resolvent does not exist.

Fact: you can see that if $f(x) \equiv 0$ (homo. eqn.), then for λ regular there can only be a trivial solution to $\lambda K\phi = \phi$. Because, regular $\lambda \Rightarrow$ resolvent exists $\Rightarrow \phi(x) \equiv 0$ is unique solution.

Fact: $\therefore$ the homo. equation has non-trivial solutions only when $\lambda =$ an eigenvalue, i.e., only when the resolvent does not exist! These solutions are then called: characteristic fctns. (eigenfunctions).

The first two theorems are:

FRED-1: If Ω is a ^{finite} portion of the λ -plane, then there are a finite # of eigenvalues in Ω . i.e., no cluster points etc.



FRED-2: For each eigenvalue there is at least one eigenfunction; at most, $\exists$ a finite # of lin indep eigenfctns.
 $\nwarrow$ just as in quantum mechanics.

Before the other 2 theorems, need:

def: conjugate kernel: $\sim$ "conj of $K(x,s)$ " = $\overline{K(s,x)} (= K(s,x)^*)$

def: conjugate equation $\sim$ start with ϕ

$$\phi(x) - \lambda \int_a^b K(x,s) \phi(s) ds = f(x) \leftarrow \text{an equation}$$

$$\tilde{\phi}(x) - \lambda \int_a^b \overline{K(s,x)} \tilde{\phi}(s) ds = \tilde{f}(x) \leftarrow \text{the conjugate}$$

$\uparrow$ some other fctn.

note λ^* ! $\nwarrow$ $\uparrow$ one new solution

Now we have.

FRED-3: (a) if λ_0 is E.V. of equation, then $\bar{\lambda}_0$ is EV of conjugate equation

(b) if equation has N lin indep eigenfctns, then the conjugate eqn also has N lin indep ef's.

def: scalar product: $(\phi, \psi) \equiv \int_a^b \phi(x) \overline{\psi(x)} dx$

$\nwarrow$ note backwards from usual.

A quick review of linear algebraic ideas: norm, normalized

Note: $|(\phi, \psi)| \leq \| \phi \| \cdot \| \psi \|$ and $\| \phi + \psi \| \leq \| \phi \| + \| \psi \|$
 $\nwarrow$ Schwarz. $\nwarrow$ triangle

Review up to here

1. classification: Fredholm, Weak Sing, Singular equations, 1st Kind, 2nd Kind, homo, inhom
2. successive approx method - n^{th} iter kernel, resolvent, radius of convergence in $|\lambda| \leq B^{-1}$. Fa Valterra, S.A. method works for all λ .
3. Degenerate kernels. This I think is what Chew is referring to, namely $1 - \lambda A = 0$ tells you the poles.
4. General case can always be approx by degenerate case!
5. Systems of L.I.E's.
6. The method of mechanical quadrature. This also has a matrix aspect + is also called Fred. det, as we shall later see.
7. Fredholm's Theorems - regular λ , eigenvalue λ , conjugate kernel, conjugate equation.

Now let's pause + take a look at a typical Chew type equation:

For example, p 772 of CRS has:

$$F(t, t') = F_1(t, t') + \int_{-\infty}^0 dt'' F(t, t'') K(t'', t')$$

Rewrite as:

$$F(t, t') - \int_{-\infty}^0 K(t'', t') F(t, t'') dt'' = F_1(t, t')$$

How does this fit into our mould? Let $t = \text{constant}$, then:

$$\begin{array}{c} F(x') \\ \uparrow \\ \phi(x') \end{array} - \int_{-\infty}^0 dt'' K(x', t'') \begin{array}{c} \uparrow \\ \phi(t'') \end{array} F(t'') = \begin{array}{c} \uparrow \\ f(x) \end{array} F_1(x')$$

(w/ no order, no sum)

This is not a system, no sum indices in F . Look at kernel:

$$K(x'', t') = \frac{1}{(x'' - m\pi)^2} \int_{4m\pi}^{\infty} ds C(s) \left[\frac{e^{-\frac{(\lambda+1) \cosh^{-1} \left[\frac{s - t' - t''}{2\sqrt{x-t''}} \right]}}}{\lambda+1} \right]$$

Hay: when Chew says Fred. det and trace, he

means it in the real sense, not in

the degenerate kernel sense! So I must read on to page 90 at least.

not obviously
factorizable.

$$\text{Tr } K = \int_a^b K(x, x) dx$$

We now pick up the text on page 38.

date is 5/1/74.

(Moffatt)

Note: I think this idea of "conjugate" is called "adjoint" by others.

$$K\phi = \int_a^b K(x,s) \phi(s) ds$$

$$K^*\phi = \int_a^b \overline{K(s,x)} \phi(s) ds$$

↑ conjugate operator

$$K^2\phi = \int_a^b \int_a^b K(x,s') \phi(s') K(s',s) \phi(s) ds$$

↑ "squared" operator = $\int_a^b K_2(x,s) \phi(s) ds$
↑ 2nd iterate

The important property of conjugate operators is:

$$(K\phi, \psi) = (\phi, K^*\psi) \quad / \text{as in Q.M.}$$

$$(KL)^* = L^* K^*$$

With all this support, we can formulate:

FRED-4: Let's say $\phi = \lambda_0 K\phi$ has eigenfunctions ϕ_i for λ_0 .
You want to know whether $\phi - \lambda_0 K\phi = f$, the inhomo
equation, has a solution.

$$\text{has a solution} \iff (f, \psi_i) = 0 \quad \forall \psi_i \Rightarrow \psi_i - \overline{\lambda_0} K^* \psi_i = 0.$$

Seems odd. f must be orthogonal to all eigen functions (of λ_0)
but of the conjugate equation.

Now we break into a review of "Linear Vector Spaces". Note that in this Russian book, the Right side of (ψ, ϕ) gets the $*$!!!

def: conjugate system of linear equations:

If $\sum_k a_{jk} x_k = b_j$ is a system, then $\sum_k \overline{a_{kj}} y_k = g_j$ is
a conjugate system. Note: $\overline{a_{kj}} = a_{kj}^* = \overline{a_{jk}^\dagger} = (a_{jk}^\dagger)^* \leftarrow$ we
would call this a hermitean conjugate matrix.

fact: $|A|^* = |A^*|$ or $|\overline{A}| = \overline{|A|}$ The det's have c.c.
do not confuse $*$ with † . values.

clarify: use $—$ for cc, use $*$ for "Conjugation" of operator.

Do not use symbols $*$ for cc, † for conjugate, $\sim$ for transpose.

Theorem: #1: If $|a| \neq 0$, both system + conj are uniquely soluble.
 If $b=0$, then solution is 0. This is I think because $\det a \neq 0$ allows an inverse to exist.

#2: If $|a|=0$, then $ax=0$ will have nontrivial solutions. The # of these nontrivial sol is $0 < N < n$, and the conjugate equation has the same # of solutions.

#3: If $|a|=0$, $ax=b$ has a solution only if $b \perp$ all $\tilde{x}_i$ where $\tilde{x}_i$ solve $a^* \tilde{x}_i = 0$, i.e. the conjugate equation.

I think "rank of the nullspace of a " is N . Then $ax=b$ has a solution if $b \perp$ entire conjugate nullspace.

def: rank of $b \sim$ # of eigensolutions to $\phi = \lambda K \phi$.

Now Mikhlin gives quick proofs of Fredholm's four equations; let's just look at FRED-4, the most interesting:

$$\begin{aligned} \phi - \lambda_0 K \phi &= f \\ (5) \quad \psi - \bar{\lambda}_0 K^* \psi &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} (f, \psi) &= (\phi - \lambda_0 K \phi, \psi) \\ &= (f, \psi) - \lambda_0 (K \phi, \psi) \\ &= (f, \psi) - (f, \bar{\lambda}_0 K^* \psi) \\ &= (f, \psi - \bar{\lambda}_0 K^* \psi) \\ &= (f, 0) \\ &= 0! \end{aligned} \quad \left. \vphantom{\begin{aligned} (f, \psi) &= (f, \psi) - \lambda_0 (K \phi, \psi) \\ &= (f, \psi) - (f, \bar{\lambda}_0 K^* \psi) \\ &= (f, \psi - \bar{\lambda}_0 K^* \psi) \\ &= (f, 0) \\ &= 0! \end{aligned}} \right\} \text{amazing!}$$

Note: Look at (5) as if it had a degenerate kernel. Then equation would boil down to:

$$(1 - \bar{\lambda}_0 a^*) \psi = f_i$$

$\uparrow$ $\uparrow$
 ψ_i $= 0$

$$\psi_i = \int_a^b \bar{a}(s) \psi(s) ds.$$

Slight mixup of a and b when you apply to a conjugate equation, but nearly trivial.

$\nwarrow$ b is replaced by $a +$ gets a bar. So call $\bar{s}$ instead of c .

Recap: FREDHOLM'S ALTERNATIVE $\sim$ (1) The inhom is soluble for all f .
 OR (2) The homo has a nontriv solution.

If the homo has a NTSol, then so does homo*, in which case inhom has sol only if $(f, \psi) = 0$, not for all f , so (2) $\Rightarrow$ NOT (1). "Alternatively", if homo has only trivial, so does homo*, and $(f, \psi) = 0$ for all f since $\psi_i \equiv 0$, and there are sol for all f . (NOT (2) $\Rightarrow$ (1)).

1.9 Fredholm's Resolvent. Problem: how do you find the resolvent of a Fredholm equation for any regular λ ? Let's first go over some key ideas obtained so far.

Either λ is regular in $\phi - \lambda K(\phi) = f$, or it is an eigenvalue. If it is regular, then the resolvent Γ exists and you can write down the solution as $\phi = f + \lambda \Gamma(f)$. If you can write down ϕ in this way, then ϕ does not blow up for regular λ .

For λ an eigenvalue, $\phi - \lambda K(\phi) = f$ may or may not have a solution. It depends on whether or not f is $\perp$ to all the ψ of $\psi - \lambda K^* \psi = 0$. So you look at $\psi = \lambda K^* \psi$ and find out the set $\{\psi_i\}$ and check (f, ψ_i) . If $(f, \psi_i) \neq 0$ for some ψ_i , then $\phi - \lambda K\phi = f$ must blow up at λ .

So, if you set out to find the places in λ where there is no solution $\phi(x, \lambda)$, you have a chance only if λ is an eigenvalue. So how do you find the eigenvalues? Examine $\phi = \lambda K\phi$! Even if λ is an eigenvalue, the solution ϕ may still exist if f is right.

I think we should think of Λ as $\frac{1}{\lambda}$, altho. CRS call it λ . λ is hidden inside K , is maybe called $(\lambda + 1)^{-1}$... we shall see.

It seems to be toying with the resolvent. Some results are:

$$\Gamma(x, s) = \underline{K(x, s)} + \lambda \int_a^b \underline{K(x, t)} \Gamma(t, s; \lambda) dt$$

This has a striking resemblance to our multiphysics equation, by the way.

Also:

$$\int_a^b \Gamma(x, t; \lambda) \Gamma(t, s; \lambda) dt = \frac{\partial \Gamma(x, s; \lambda)}{\partial \lambda}$$

And once more,

$$\int_a^b \Gamma(x, x; \lambda) dx = \sum_{m=1}^{\infty} A_m \lambda^{m-1}$$

where A_m is called the m^{th} trace

$$A_m \equiv \int_a^b K_m(x, x) dx$$

So far all this data has little relevance to me. Things are starting to get a little fancy.

12.

what's left.

finite polynomial estimate for K

think of as $\hat{f}(x)$

So, for given λ , choose a large enough n , then you can get the resolvent P' of this equation:

So, since we have the residual, we can write the solution!

Now regions:

$$\phi(x) \equiv F(x) + \lambda \int ds \phi(s) K''(x,s)$$

$$\phi - \lambda K'' \phi = \pi$$
$$\phi - \lambda K'' \phi = F \Rightarrow [1 - \lambda R] \vec{c} = \vec{f} \quad R_{ij} = \int a_i(x) b_j(x) dx$$

$\left(\int b_i(s) \phi(s) ds \right) \left(\int b_i(s) f(s) ds \right)$
 $\underbrace{\hspace{10em}}_{\text{that make up } K''}$

$$q(x) = F(x) + \lambda \sum c_i a_i(x)$$

$$L = f(x) + \lambda \int \gamma'(x, s; \lambda) f(s).$$

So we see that the general case uses two methods. Part of the solution come from a 'resolvent' calculation based on the small kernel K' and the rest comes from a degenerate kernel solution based on K'' . As $n \rightarrow \infty$, of course $K' \rightarrow 0$ and entire solution is degenerate.

So, back to page 49, we now dig (7).

def: $D_R(\lambda) \sim$ in the general Fredholm case being discussed, this is the det of the system:

This will appear in the Cramer's Rule attempt at solving for λ .

$$D_R(\lambda) = \det(1 - \lambda R)$$

$$R = (a_i b_j) \text{ from } K''$$

$$[1 - \lambda R] \vec{f} = \vec{c}$$

$$\Rightarrow c_k = \frac{1}{D_R(\lambda)} \sum_m \Delta_{mk}(\lambda) f_m$$

Why R?

"adjoint" of $(1 - \lambda A)$

$$K'' = P + \lambda \int P' P \text{ etc etc}$$

This is what you get up top by replacing one column with $\{f_i\}$.

So we can now write the entire solution as:

$$g(x) = f(x) + \lambda \int \Gamma'(x, s; \lambda) f(s) ds + \lambda \sum_k c_k a_k(x)$$

$$c_k = \frac{1}{D_R(\lambda)} \sum_m \Delta_{mk}(\lambda) f_m$$

$$\int_0^b F(s) b_m(s) ds = \int f(s) \tilde{B}_m(s) ds.$$

In other words:

$$g(x) = f(x) + \lambda \int f(s) ds \left[\Gamma'(x, s; \lambda) + \underbrace{\left\{ \tilde{b}_m(s) \frac{\Delta_{mk}(\lambda) a_k(x)}{D_R(\lambda)} \right\}}_{k=m} \right]$$

Now claims:



$$\equiv \Gamma(x, s; \lambda)$$

this baby is your effective resolvent for the complete problem!

Claim 1: In any circle in $\mathbb{C}_\lambda$,

there are finite # eigenvalues, so finite # places where resolvent no exist is known. This $\sim$ resolvent is meromorphic in $\mathbb{C}_\lambda$

No cuts, no cluster points of poles, no crap!

Claim 3: (skip?) Recall $\Gamma = K + \lambda \int K \Gamma$. When λ is regular, Γ exists + solves this equation. We can continue the solution over the poles of Γ + make a $\tilde{\Gamma}$ solve this equation everywhere in λ .

Claim 4: Every eigenvalue is a place where Res. no exists, but only poss. In Res no exist is a pole. Thus:

$$(\lambda_0 = \text{eigenvalue}) \Rightarrow \lambda_0 = \text{pole of resolvent.}$$

ie, $\phi = \lambda_0 K \phi$ has nontrivial solution. Recall: if λ is regular, resolvent exists, and then $\phi = \lambda_0 K \phi$ can have only trivial solution.

Application: suppose you have $\Gamma(x,s;\lambda) = K(x,s) + \lambda \int K(x,t) \Gamma(t,s;\lambda) dt$ and you want to know where Γ has poles. All you need to do is look for the eigenvalues of λ . How do you find these?

- 1) examine the homog equation + hunt for eigenvalues as if it were a problem in quantum mechanics.
- 2) try to compute $D_R(\lambda)$ + then set $D_R(\lambda = \text{eigenvalue}) = 0$. Maybe we will get a way to do this soon. So far $D_R(\lambda)$ is very involved: $D_R(\lambda) \leftarrow A \leftarrow a,b; \leftarrow K'' \leftarrow P P' \leftarrow \text{etc etc.}$

Stand by to construct the full resolvent as N/D . First:

$$\begin{aligned} D_R(\lambda) &= \det(1 - \lambda A) = \det(\delta_{mk} - \lambda a_{mk}) = \det(B_{mk}) \\ &= \sum_m B_{mk} \Delta_{mk}^B(\lambda) = \sum_m (\delta_{mk} - \lambda a_{mk}) \Delta_{mk}(\lambda) \\ &= \Delta_{kk}(\lambda) - \lambda \sum_m a_{mk} \Delta_{mk}(\lambda) \quad \text{for any } k. \end{aligned}$$

Now by one way: $\frac{\partial \Delta_{mk}(\lambda)}{\partial \lambda} = \text{don't know.}$

Maybe he means:

$$\begin{aligned} \det \left[\frac{\partial}{\partial \lambda} (1 - \lambda A) \right] &= \det \left[-A - \lambda \frac{\partial A}{\partial \lambda} \right] \\ &= -\det \left[a_{mk} - \lambda \frac{\partial a_{mk}}{\partial \lambda} \right] \\ &= ? \end{aligned}$$

Well, let it go for now. After 2 pages of mess we find:

$$\underbrace{\int_a^b \Gamma(s,s;\lambda) ds}_{\equiv \delta(\lambda)} = \int_a^b \Gamma'(s,s;\lambda) ds = - \frac{D_R'(\lambda)}{D_R(\lambda)}$$

def: $\delta(\lambda) \equiv \int_a^b \Gamma(s, s; \lambda) ds$

● This baby has simple poles at the eigenvalues of the kernel K for which this Γ is the resolvent. i.e., let:

$$D_K(\lambda) = (\lambda - \lambda_0)^3 \quad \text{so } \lambda_0 = \text{zero of multiplicity } 3$$

$$D'_K(\lambda) = 3(\lambda - \lambda_0)^2 \quad - \frac{D'_K(\lambda)}{D_K(\lambda)} = \frac{-3}{(\lambda - \lambda_0)}$$

Even if the zero of $D_K(\lambda)$ has multiplicity 3, the pole of $\delta(\lambda)$ is still simple. It has a residue of -3.

Thus: $-\oint_{\gamma} \delta(\lambda') d\lambda' = \text{Residue} = -3$ (integer)

And thus: $-\oint_{\gamma} \delta(\lambda) d\lambda$ is always an integer $\Rightarrow e^{\text{integer}} = \text{integer}$.

def: $D(\lambda) = e^{-\oint_{\gamma} \delta(\lambda) d\lambda}$

● Look at the sequence here: $\lambda_0 = \text{a zero of } D_K(\lambda) \Rightarrow \lambda_0 = \text{simple pole of } \delta(\lambda) \Rightarrow \delta(\lambda) \sim \frac{1}{\lambda - \lambda_0} \Rightarrow \text{exists a } \lambda_0 \text{ such that } D(\lambda) = 1$

What is the meaning of: $\oint_{\gamma} \delta(\lambda) d\lambda$

Well in our little example we had: $\delta(\lambda) = \int \Gamma' ds - \frac{3}{(\lambda - \lambda_0)}$

So it's just a straight integral? $\oint \delta \lambda d\lambda = \oint [\int \Gamma' ds] d\lambda - 3 \ln(\lambda - \lambda_0)$

As you get near $\lambda = \lambda_0$, get $-3 \ln 0 \rightarrow +\infty$.

Then: $D(\lambda) = e^{-[\infty]} = 0!!$

So, when $D_K(\lambda)$ has a 0, so does $D(\lambda)$! The zeros of

$D(\lambda)$ can cancel the poles of $\Gamma(\lambda)$. Thus, the

representation: $\Gamma(x, s; \lambda) = \frac{\{D(\lambda) \Gamma(x, s; \lambda)\}}{D(\lambda)}$ holomorphic

If λ is small enough, then $D(\lambda)$ may be written as a series of traces:

$$D(\lambda) = \exp\left[-\sum \frac{A_n \lambda^n}{n}\right] = 1 - A_1 \lambda - \frac{A_2 \lambda^2}{2} - \frac{A_3 \lambda^3}{3} \dots \text{etc.}$$

where: $A_1 = \int_a^b K(x,x) dx$ ($K = K_1$)

Thus, if λ is very small, you can approximate: $D(\lambda) \approx 1 - d(K) \cdot \lambda$
This must be the meaning of those references!

Anyway, to solve the problem we seek explicit expressions:

$$\Gamma(x,s;\lambda) = \frac{D(x,s;\lambda) \leftarrow \text{Fredholm's first Minor}}{D(\lambda) \leftarrow \text{Fredholm's determinant}}$$

↑
full resolvent

The exact solution was given by Fredholm based on Hadamard's Theorem about upper bounds on determinants. The solutions converge in entire λ plane. They are:

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_n \lambda^n \quad \rightarrow \quad C_n = \int \dots \int dt_1 \dots dt_n \left| K(t_1, t_1) \dots K(t_1, t_n) \right|$$

$$D(x,s;\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n(s,x) \lambda^n \quad \rightarrow \quad B_n(s,x) = \int \Delta_n(x,s) dt_1 \dots dt_n$$

In principle you can get the exact resolvent from these series.

$$\Delta_n(x,s) = \begin{vmatrix} K(x,s) & K(x,t_1) & \dots & K(x,t_n) \\ K(t_1,s) & & & \\ \vdots & & & \\ K(t_n,s) & & & \end{vmatrix}$$

By the way:

$$C_1 = \int dt_1 K(t_1, t_1) = \text{trace } K$$

$$C_2 = \int dt_1 dt_2 \begin{vmatrix} K(t_1, t_1) & K(t_1, t_2) \\ K(t_2, t_1) & K(t_2, t_2) \end{vmatrix}$$

> NOTE: if Kernel is factorizable so that $K(x,y) = f(x)g(y)$, then C_2 and all higher C_n vanish and only C_1 remains. Thus:

factorizable kernel $\Leftrightarrow$ trace approx is exact

$$\begin{aligned} A_2 &= \int dt_1 K_2(t_1, t_1) = \int dt_1 \left[\int dt_2 K(t_1, t_2) K(t_2, t_1) \right] \\ &= \int dt_1 dt_2 K(t_1, t_2) K(t_2, t_1) \end{aligned}$$

So it looks like we have two methods to locate the poles of the resolvent.

● (I.) If $|\lambda| < 1/B$ where $B^2 = \int |K(t,s)|^2 dt ds$, then you have:

$$D(\lambda) = \exp \left[- \sum_{n=1}^{\infty} \frac{A_n}{n} \lambda^n \right]$$

where

$$A_n = \text{trace } K^n = \int dx_1 dx_2 \dots dx_n K(x_1, x_2) K(x_2, x_3) \dots K(x_n, x_1)$$

This is a messy double sum in λ . But, if we have very small λ so that

$$A_1 \lambda \gg \frac{A_2 \lambda^2}{2} \quad \text{i.e.} \quad \lambda \ll \frac{2A_1}{A_2}$$

Then one could say: $D(\lambda) \approx e^{-A_1 \lambda} = 1 + (-A_1 \lambda) + \frac{(-A_1 \lambda)^2}{2!} + \dots$

If in addition $|A_1 \lambda| \ll 1$, then you have: $D(\lambda) = 1 - \lambda A_1$
 $= 1 - \lambda \text{trace}(K)$

This is the "trace approx to the Fredholm determinant"

[So $\lambda \approx 1/\text{trace}(K)$ would be a 0 of $D(\lambda)$, so a pole of $\Gamma(\lambda)$]

● (II.) More generally, we can use the full Fredholm determinant method to locate poles of the resolvent:

$$D(\lambda) = 1 - c_1 \lambda + \frac{1}{2} c_2 \lambda^2 - \frac{1}{3!} c_3 \lambda^3 \dots$$

where:

$$c_1 = \int dt_1 K(t_1, t_1) = A_1 = \text{trace}(K)$$

$$c_2 = \int dt_1 dt_2 \begin{vmatrix} K(t_1, t_1) & K(t_1, t_2) \\ K(t_2, t_1) & K(t_2, t_2) \end{vmatrix} \quad \text{etc.}$$

Thus, the "trace approximation" is like taking the leading term of the full Fredholm determinant.

Now at last, how does this apply to particle physics:

$$\Gamma(x, s; \lambda) \longleftrightarrow F^\lambda(t, t')$$

↑
The Fredholm resolvent

↑
the projected
scatt amplitude.

$$(\lambda K(x, s)) \longleftrightarrow K^\lambda(t, t') \text{ or } F_1^\lambda(t, t')$$

↑
kernel

↑
kernel

↑
inhom term

↑
"parameter"

↑
projection rep # (angular momentum).

Now let's have a look at Stakgold to see what is + is not there.

Starts p191 -

def: bounded K : $\|K\|$ = least upper bound on $\frac{\|K u\|}{\|u\|} \leftarrow \mathcal{L}^2 \text{ norms}$

Recall: bounded $\Leftrightarrow$ continuous.

def: separable kernel: $\sum p_i(x) q_i(y)$ — degenerate in Russian books.

def: Hilbert-Schmidt kernel, one for which $\iint |K(x, y)|^2 dx dy < \infty$.

Note: a "Fredholm equation" must involve an equation of one of the usual 3 forms which has a H-S. kernel

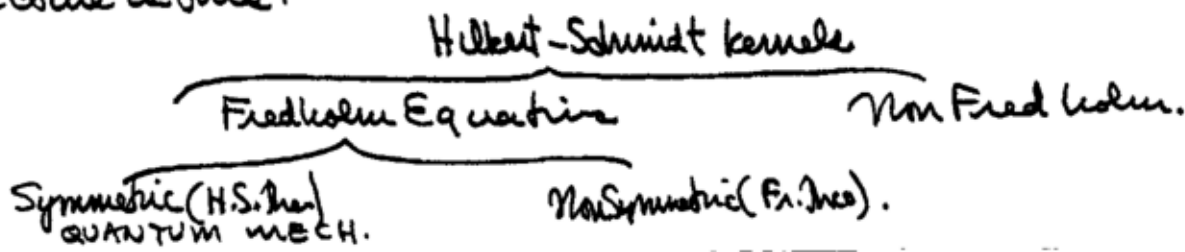
def: Neuman Series — same as "method of successive approx" !

Note that $B = \|K\|$, so $|\lambda| < 1/\|K\| \Rightarrow$ unique solution. The word "resolvent" is not mentioned here.

def: adjoint operator — called "conjugate" in Rus.

Now Stakgold goes on using symmetric kernels. But, the last section 3.9 does deal with nonsymmetric kernels. I think this is the realm of Fredholm.

So the picture is this:



Asymptotic Methods

Comments about the Method of Steepest descents.

1. References: after initial dislike, I think the discussion of Jeffries is the best because he keeps reminding the reader of the philosophy of the method: keep your contour as "low" as possible, whatever the cost. Other references are: the thick math handbook article by Olver, the book by Olver 1974 not at LBL, a book by Erdelyi; the general subject area is: asymptotic techniques.

2. My interest in the subject was to evaluate definite integrals in the limit that some parameter goes large. In particular, the HGF's. Watson's paper thoroughly answers any reasonable question you might ask. Copy retained.

3. The technique seems to work when the parameter of interest is a power or exponent. When this exponent goes asymptotic, only that piece of the contour which maximizes the exponents multiplier function $f(z)$ is relevant. If the entire contour lies on a hillside, then you simply deform the thing so that it ascends up to the high point. Only the last little piece of this ascent will matter. Watson's lemma is a special case of this, where the piece that matters is the little segment close to 0. In fact, you often reduce things to Watson form because his answer is so simply. Eg, "Laplace's Method" reduces to Watson's Lemma when you just change variable to $s = f(z)$.

4. The case I have not done in detail is that of the endpoints lying in different valleys. In this case, the best you can do is run your contour through a saddle. The saddle point will then be the highest point and will therefore control the integral. So you merely expand things around the saddle point and get an answer. It is this saddle method (called col) that Watson uses to get his main results.

By the way, you also want to come into the highest point along the line of steepest ascent to minimize the amount of contour at high altitude. This goes also for the saddle. That is why Watson goes to so much trouble locating the descent curves (which happen to be curves of constant "phase", or constant imaginary part of f).

5. The method of stationary phase is a different technique, though it is related.

6. Generally, these methods apply as a parameter goes to plus infinity. If you want your asymptotic result in some direction in the complex parameter plane, you have to check to see if your contour can be rotated down to Watson form. If there are no singularities of the integrand in the way, you are in fat city, because the great circle stuff always vanishes in our cases of interest, .

Steepest descent: relief maps.

① Harmonic functions: consider $|e^{f(z)}|$ where $f(z) \equiv \text{analytic}$. This picks off one of the harmonics $= e^{u(x,y)}$. But clearly, $\nabla^2 [e^{u(x,y)}] = 0$ because $\nabla^2 u = 0$. So at any z , $|e^{f(z)}|$ has opposite curvature in the two directions.

② If $\frac{df}{dz} = 0$, then f does not change in all directions, so neither does $|e^f|$. Thus, you are at a "flat place". But due to ①, this can only be some kind of saddle, not a grassy knoll or valley floor.

③ Suppose you are at a place where $\text{Re} f = u = \text{constant}$, or you move along a segment of z where $u = \text{const}$. Then in terms of $|e^f|$, you are following a topological contour, because $|e^f| = e^u = \text{const}$. Also called a level curve. $\uparrow$ topographical

④ Suppose you move with $\text{Im} f = v = \text{constant}$. Suppose this is the y -direction. Then $\frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$. (Cauchy-Riemann). This shows that you are moving $\perp$ to the contours, i.e., you are on the fall line, or path of steepest descent.

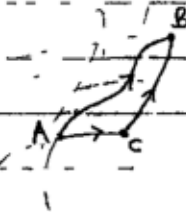
⑤ By the way, saddles are only "simple" if $f''(z) \neq 0$ at $f' = 0$.

Theorems:

- | | | |
|--------------------------------|---------------|------------------|
| ① $\text{Re} f = \text{const}$ | $\Rightarrow$ | contour motion |
| ② $\text{Im} f = \text{const}$ | $\Rightarrow$ | steepest descent |
| ③ $f'(z) = 0$ | $\Rightarrow$ | saddle |

The crudest notion of steep descent & Laplace's Method.

- ① Consider $I = \int_A^B e^{af(z)} g(z) dz$ where f, g are analytic and $a = +10,000$ say. Also, suppose the entire "relief path" lies on a simple "hillside". So assume there is a point C so that we can do $A \rightarrow C = \text{contour}$, then $C \rightarrow B = \text{ascent}$.



Let $f = u + iv$. So $u(A)$ is clearly defined; it is a label of "altitude" of path $A \rightarrow C$.

- ② The "contour" piece: $e^{af} = e^{au(A)} e^{aiv}$ so we get:

$$I_{AC} = e^{au(A)} \left(\int_A^C dz e^{aiv(z)} g(z) dz \right)$$

But $\left| \int_A^C \dots \right| \leq \int_A^C |dz| |g(z)| \leq M \cdot L = \text{presumably some finite quantity not dependent on } a!$ Thus, $I_{AC} \sim \exp[a u(A)]$. Notice that $u(A) < u(B)$.

- ③ The "ascent" piece: $e^{af} = e^{au(B)} e^{aiv(z)}$ so

$$I_{CB} = e^{au(B)} \left(\int_C^B dz e^{aiv(z)} g(z) dz \right)$$

This piece will be controlled by the contour segment near B because $e^{au(z)}$ has a huge peak there [$a = +10,000!$]. This is Laplace's Method at work. Thus, we shall get

$$\left(\int_C^B \dots \right) = e^{au(B)} \cdot \tilde{c} \left[\frac{1}{\sqrt{2\pi}} + \frac{g_1}{a} + \frac{g_2}{a^2} + \dots \right] \leftarrow \text{asym. exp.}$$

- ④ So, $I_{AC} \sim e^{au(A)}$, but $I_{CB} \sim e^{au(B)}$, so $I_{CB} \gg I_{AC}$. So only the ascent piece matters.

- ⑤ Philosophy: minimize the amount of contour at large u . In this case, it is just the top of an ascent.

direct ascent, 3

$$I_{CB} = \int_c^B dz g(z) e^{af(z)}$$

① Let $\tilde{z} = -f(z)$ so $d\tilde{z} = -f'(z) dz$ $z = f^{-1}(-\tilde{z}) = z(\tilde{z})$

$$I_{CB} = \int_{-f(B)}^{-f(c)} (-d\tilde{z}) e^{-a\tilde{z}} (g/f') = \int_{-f(B)}^{-f(c)} d\tilde{z} e^{-a\tilde{z}} (g/f')$$

② Let $\xi = \tilde{z} + f(B) = -f(z) + f(B) = \xi$ then

$$I_{CB} = e^{af(B)} \int_0^{f(B)-f(c)} d\xi e^{-a\xi} (g/f')$$

③ Now, $f(B)-f(c)$ is real because $v(B)=v(c)$, only u changes. Thus, we have Watson's Lemma!

$$g/f' \equiv F(\xi)$$

$$\int_0^\infty d\xi e^{-a\xi} F(\xi) = \int_0^\infty d\xi e^{-a\xi} \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \xi^n = \sum_{n=0}^{\infty} \frac{n! F^{(n)}(0)}{[a]^{n+1}}$$

Thus:

$$I_{CB} = e^{af(B)} \sum_{n=0}^{\infty} \frac{F^{(n)}(0) n!}{[a]^{n+1}}$$

④ Now we need way to write $F^{(n)}(0)$ in terms of f and g :

$$F(\xi) = \frac{g(z(\xi))}{f'(z(\xi))} \quad \text{so} \quad F(0) = \frac{g(z(0))}{f'(z(0))}$$

Need to know $z(\xi)$. $\xi = -f(z) + f(B) \Rightarrow f(z) = f(B) - \xi$

so $z = f^{-1}[f(B) - \xi] = z(\xi)$. Then:

$$\frac{dF}{d\xi} = F'(0) = \left(\frac{d}{dz} \left[\frac{g}{f'} \right] \cdot \frac{dz}{d\xi} \right) \Big|_{\xi=0}$$

Probably best to actually compute $F(\xi)$.

example

$$\textcircled{1} \quad I = \int_1^{5+2i} dz (z)^a = \int_A^B dz \cdot e^{a \ln(z)}$$

$\swarrow$
 f
 $\nwarrow$
 $a \ln(z)$
 $\uparrow$
 z

$\textcircled{2}$ Since $f'(z) = \frac{1}{z}$, this is never 0 in finite region, so you are always on a "hillside".

$\textcircled{3}$ Which end is "highest"? $f = u + iv$, $u = \ln|z|$. Largest at $5+2i$.

$$\textcircled{4} \quad \underline{s} = -f(z) + f(B) = -\ln z + \ln B = \ln(B/z) = -\ln(z/B)$$

$$+ e^{-s} = \frac{z}{B} \Rightarrow \underline{z(s)} = + B e^{-s}$$

$$\textcircled{5} \quad \underline{F(s)} = \frac{g(z(s))}{f'(z(s))} = \frac{1}{1/z} = z(s) = + B e^{-s}$$

$$F(0) = +B$$

$$F'(0) = -B$$

$$\underline{F^{(n)}(0)} = +B(-1)^n; \quad f(B) = \ln B.$$

$\textcircled{6}$ Thus, our Theory predicts, as $a \rightarrow +\infty$:

$$I = e^{a \ln B} (+B) \sum_{n=0}^{\infty} \frac{n! (-1)^n}{(a)^{n+1}} = +B(B)^a \left[\frac{1}{a} + O\left(\frac{1}{a^2}\right) \right]$$

$\textcircled{7}$ Compare to direct computation:

$$\int_A^B dz (z)^a = \frac{1}{a+1} z^{a+1} \Big|_A^B \approx \frac{1}{a} (B)^{a+1}$$

Concept of contour-rotation in Laplace/Watson.

① The simple Watson Lemma says

$$\int_0^{\infty} dz e^{-az} g(z) = \sum_{n=0}^{\infty} g^{(n)}(0) \cdot \frac{n!}{a^{n+1}}$$

is controlled by $z=0$ when $a \rightarrow +\infty$, so $g(z) = \sum g^{(n)}(0) z^n$
and we get result shown. Result is same for finite integration
along real axis. Also consider a weird complex contour:



② Obviously this is Laplace with $f(z) = z$, $f'(z) = 1$, $\text{never} = 0$ so
no saddles.

③ Now: suppose a is complex $a = |a|e^{i\alpha}$ if $|\alpha| < \frac{\pi}{2}$, integral
will tend to converge, so we expect this restriction.

$$\text{So } \int_0^{\infty} dz e^{-|a|ze^{i\alpha}} g(z)$$

Obvious trick is: $z = \xi e^{-i\alpha}$ $\xi = z e^{i\alpha}$ so

$$I = \int_0^{\infty} e^{-i\alpha} d\xi e^{-|a|\xi} g(\xi e^{-i\alpha})$$



Since $|\alpha| < \frac{\pi}{2}$, you can rotate contour down to real axis provided
 g has no blocking singularities. So

$$I = e^{-i\alpha} \int_0^{\infty} d\xi e^{-|a|\xi} g(\xi e^{-i\alpha}) = \sum g^{(n)}(0) (e^{-i\alpha} \xi)^n$$

$$\text{Then } \int_0^{\infty} d\xi e^{-|a|\xi} \xi^n = \frac{n!}{|a|^{n+1}} \text{ so } I = e^{-i\alpha} \sum g^{(n)}(0) e^{in\alpha} \frac{n!}{|a|^{n+1}}$$

$$I = \sum g^{(n)}(0) \frac{n!}{(a)^{n+1}}, \text{ the same result.}$$

④ Thus, we have extended result to $|\arg(a)| < \frac{\pi}{2}$.

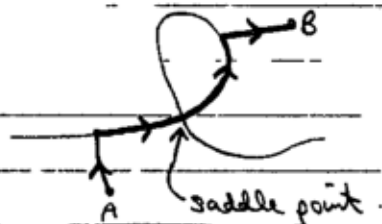
Attempt at a saddle version

① Reconsider $\int_A^B e^{af(z)} g(z) dz$ but now A, B lie in different valleys. So, let $f'(z_0) = 0$ denote the saddle location, assume $f''(z_0) \neq 0$ so a simple saddle.

② Now find steepest descent, a "pass" curve, the fastest roadway up thru the saddle. This is given by

$$\operatorname{Im} f(z) = \text{constant} = \operatorname{Im} f(z_0)$$

This equation will give something which does this:
I.e., constant phase works both ways: 90° intersect



③ Then make sure you can run your contour through the saddle, get the direction right:

fix later. ④ Near z_0 , $f(z) \approx f(z_0) + \frac{1}{2} f''(z_0) (z-z_0)^2$ but $f''(z_0)$ must of course be negative, or else you are on the wrong road. Definitions suggest defining:

$$[-f''(z_0)](z-z_0)^2 = \xi^2$$

$$\text{So } \xi = \sqrt{-f''(z_0)} \cdot (z-z_0) \Rightarrow z = z_0 + \frac{1}{\sqrt{-f''(z_0)}} \xi = z(\xi).$$

$$\text{Let } \sqrt{-f''(z_0)} = k. \text{ Then } \xi = k(z-z_0) \quad z = z_0 + \xi/k. \quad dz = \frac{1}{k} d\xi.$$

$$\text{So we end up with } \int \frac{1}{k} d\xi e^{a[f(z_0) - \frac{1}{2}\xi^2]} g(z(\xi))$$

$$= \frac{e^{af(z_0)}}{k} \int d\xi e^{-\frac{1}{2}a\xi^2} g(z(\xi)).$$

⑤ The path here is a little contour passing through $\xi=0$ in some very definite direction determined by the original roadway in the z -plane.

① If, suppose $z - z_0 = re^{i\alpha}$, and we know α . Then

$$f = ke^{i\alpha} \text{ so } \quad \begin{array}{c} \beta \\ \nearrow i\alpha \\ \text{---} \end{array}$$

Obviously direction also counts.

② Now define a real variable $s = |k|r$. $k = \sqrt{-f''(z_0)}$

so $|k| = |f''(z_0)|^{1/2}$. Then

$$df = e^{i\alpha} \frac{k}{|k|} ds \quad f = \frac{k}{|k|} e^{i\alpha} s$$

$$\text{So } e^{-\frac{1}{2}as^2} = e^{\frac{1}{2}a \frac{k^2}{|k|^2} e^{2i\alpha} s^2} \quad \text{getting messy.}$$

③ OK, too much rush, but eventually you will get:

$$e^{af(z_0)} \int_{-\infty}^{\infty} ds e^{-\frac{1}{2}as^2} = g(z_0)$$

$$= \left[e^{af(z_0)} \cdot g(z_0) \cdot \sqrt{\frac{\pi}{\frac{1}{2}a}} \cdot \frac{1}{|k|} \cdot e^{i\alpha} \right]$$

$\uparrow$ usual factor coming from highest point.
 $\uparrow$ fcn evaluated there.
 $\uparrow$ jacobian
 $\uparrow$ direction thro.

$\frac{1}{\sqrt{a}}$ is first term.

You could of course expand $g(z)$ to get higher terms.

Laurent book: "Asymptotic Methods" by Fowler Oliver

1 asymptotic expansion defined + generalized in the "series"

2 parts integration + you can get the asymptotic behaviour of any degree polynomial in that device way.

3 Watson's lemma: if $q(t)$ be complex fcn of real t with finite # of discontinuities + singularities. Assume that in small circle about $t=0$ $\exists$ power series $q(t) = \sum_{s=0}^{\infty} a_s t^s$. Then as $\lambda \rightarrow \infty$ by-term and you get the asymptotic expansion

$$\int_0^{\infty} e^{-\lambda t} q(t) dt \sim \sum_{s=0}^{\infty} \frac{a_s}{\lambda^s} \Gamma(s) \quad \text{as } \lambda \rightarrow \infty$$

Large not say λ real, but $\lambda \rightarrow \infty \Rightarrow \text{O.K.}$

If you can't do it up then, better not do it.

4 Stieltjes-Lebesgue lemma - Consider $\int_0^{\infty} e^{-\lambda t} q(t) dt$ where a, b could both be ∞ + show as $\lambda \rightarrow \infty$, a in that limit is integrated in $a(1)$. The lemma integrated $\rightarrow 0$ in that direction. Is, you get your function died up to nothing! Other condition - must satisfy that $\lambda \rightarrow \infty$.

5 Fourier integrals: again, do parts integration to get any fcn. Note in $(\frac{\lambda}{2})$ use 1 to get something is $a(x)$ say fcn, take to get in that way.

6 Stieltjes Method - $\int_0^{\infty} e^{-\lambda t} q(t) dt$. As $\lambda \rightarrow \infty$, most come from region near the minimum of $q(t)$. So use

$$p(t) = p(t_0) + p'(t_0)(t-t_0) + \dots$$

$$I = \int_0^{\infty} e^{-\lambda t} q(t) dt \sim \int_0^{\infty} e^{-\lambda t} p(t) dt$$

At the endpoint, say to get next. Notice that this is right generalized Laplace transform

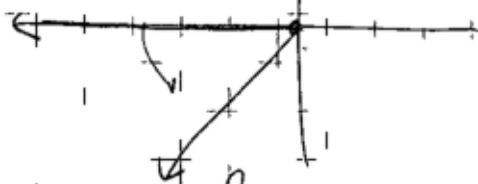
If you could get a definite minimum point and if you expand $q(t)$ and $p(t)$ about you get an asymptotic expansion. This is slightly generalized Watson's lemma.

④ Steady Phase = $-II(x) = \int_{b_1}^{b_2} x p(x) dx + \int_{a_1}^{a_2} q(t) dt$. Changes for $p(t)$ to be stationary ($\frac{d}{dt} p(t) = 0$) at t_0 . Only except input a , independent $\neq 0$ by correlation. Phase derivative is $I \approx e^{x p(x)} / (x)$ power

$$I \approx e^{x p(x)} / (x)$$

Asymptotes can get again power!

⑤ Contour Integration: well, $\int_{-\infty}^{\infty} e^{q(t)} dt$. Not the same with $|z| \rightarrow \infty$ in RHP + get same result. But we notice that ranges $(\arg(z) \pm \pi/2)$ can be extended by extending rotated contour! This must be true that $p(z)$ range is extended.



Comments: Olver proved many these claims, but he was a 1974 book called "Asymptotics and Special Functions". Wonder if he does the HFT Olver? Olver also gives

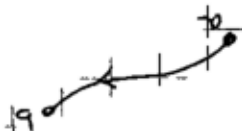
⑥ Contour Deformed Method:

$$\int_a^b e^{z p(t)} dt$$

$z = \text{complex}$
in a sector.

Changes for $\min [Re(z p(t))]$ to

occur at an Olver you get same old result. Many



⑩ Saddle point $\sim$ suppose $Re(z p(t)) = \min$ at some interior point of contour. Call this to be important question: what does $q(t) \equiv 0$ [complex derivative] at this location to which minimizes $Re(z p(t))$?

First, if $p'(t_0) = 0$, then when you break into $J_1 + J_2$, the two asymptotic expansions [see (9)] do not cancel, and you get an answer.

Why "saddle"? Plot $|e^{p(t)}|$ versus complex t . When $p'(t) = 0$, you will be at a saddle where tangent plane is flat horizontal.

Now, if $p'(t_0) \neq 0$, then you want to deform contour to make min occur at either endpoint or a saddle. Claim is that you can always do this.

So the game is to pick a good contour. If you get pieces with $\operatorname{Im}[z p(t)] = \text{constant}$, these are helpful but not necessary.

Plot $|e^{z p(t)}|$. The path $\operatorname{Im}[z p] = \text{const}$ will be a line of steepest descent out of the saddle or endpoint!

We do give good examples!

descent meets $\Gamma(z)$

$$z^i = \int_{-\infty}^{\infty} z^i t^2 dt \quad \text{Why not say } = \int_{-\infty}^{\infty} z^i t^2 dt$$

Then $f(t) = \int_{-\infty}^{\infty} z^i t^2 dt$ and $f'(t) = t^i$. Oh, path is "pullout" e^{∞}

You have to change variables z central point is not singular

Why not suggest: $t = z \cdot x$? Then $dt = z dx$ and

$$z^i = \int_{-\infty}^{\infty} z^i dx \cdot e^{-z} = \int_{-\infty}^{\infty} z^i \cdot z \cdot dx \cdot e^{-z} = \int_{-\infty}^{\infty} z^{i+1} dx \cdot e^{-z}$$

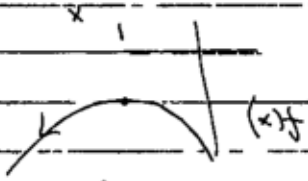
Make that z approach outside. We are now interested in

$$\int_{-\infty}^{\infty} dx \cdot \exp[-z + \ln x] z^i = \int_{-\infty}^{\infty} dx \cdot e^{-z} x^i$$

So now $f(x) = x - \ln x \Rightarrow f'(x) = 1 - \frac{1}{x} = 0$ at $x=1$.

③ Choose $z \rightarrow +\infty$, so $\theta = 0$. Then integral runs $0 \rightarrow +\infty$ and

$$f(x) = x - \ln x$$

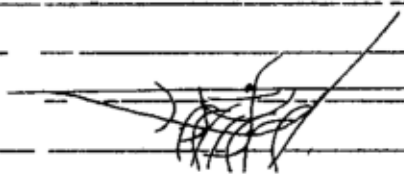


So $f(x)$ has saddle at $x=1$

$$z f(x)$$

$$f(1) = 1 - \ln 1 = 1, \text{ so } e^1$$

$$f(x) = +\infty$$



So perhaps real path is steepest descent? $\ln f(x) = \text{const} \Rightarrow \ln x = \ln x$. Certainly true on real axis.

④ So central point is now at $x=1$.

$$\int_0^{\infty} dx e^{-z[x - \ln x]} \approx \int_{-\infty}^{\infty} dx e^{-z[x - \ln x]} \approx \int_{-\infty}^{\infty} du e^{-z[u+1 - \ln(u+1)]}$$

Why not change to $u = x-1$ then

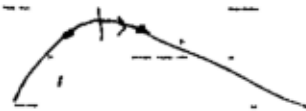
$$\ln(1+u) \text{ near } u=0 \approx u - \frac{u^2}{2} + \frac{u^3}{3} \dots$$

$$1+u \approx 1+u$$

$$S_0 = \int_{-\infty}^{\infty} du e^{-z \left[\sqrt{1+u} - \left\{ u - \frac{u^2}{2} + \frac{u^3}{3} - \dots \right\} \right]}$$

$$= e^{-z} \int_{-\infty}^{\infty} du e^{-z \left[\frac{u^2}{2} - \frac{u^3}{3} + \dots \right]}$$

how do you set scale?



Seems to depend on epsilon. But, if $z \rightarrow \infty$, then extend $-\infty, \infty$.

$$\int_{-\infty}^{\infty} du e^{-z \frac{u^2}{2}} = \sqrt{\frac{\pi}{z}} = \sqrt{\frac{2\pi}{z}}$$

$$\text{Show: } z \cdot \sqrt{\frac{2\pi}{z}} \text{ is answer.}$$

⑤ More terms?

$$e^{-z \frac{u^2}{2}} \left[1 - \frac{u^2}{3} \right]$$

u^{k+1}

$$e^{-z \frac{u^2}{2}}$$

odd terms cancel So

$$\int_{-\infty}^{\infty} du e^{-z \left[\frac{u^2}{2} + \frac{u^4}{4} \right]} = \int_{-\infty}^{\infty} du e^{-z \frac{u^2}{2}} \left[1 + \frac{u^2}{2} \right]$$

⑥ Fancy way to get full expansion:

$$\psi(z) = \frac{d}{dz} \ln z! = \int_0^{\infty} \left\{ \frac{e^{-t}}{t} - \frac{e^{-t(z+1)}}{1-e^{-t}} \right\} dt$$

$$\psi'(z) = \int_0^{\infty} \frac{e^{-tz} t dt}{e^t - 1} = \sum_{n=0}^{\infty} a_n \left(\int_0^{\infty} e^{-tz} t^n dt \right) \frac{n!}{[z]^{n+1}}$$

Notice that this guy seems well behaved as $z \rightarrow +\infty$.

Can easily use Watson here.

$$\frac{t}{e^t - 1} = \sum a_n t^n \text{ So}$$

$$\Rightarrow \psi(z) = \sum_{n=0}^{\infty} a_n \frac{n!}{[z]^{n+1}}$$

So if this series is uniformly convergent, you can integrate

$$\psi'(z) = \frac{1}{z} - \frac{1}{z} \cdot \frac{1}{z^2} + \sum_{n=2}^{\infty} \frac{B_n}{z^{n+1}} \quad \text{Bernoulli}$$

$$\Rightarrow \psi(z) = \ln z + \frac{1}{z} + \sum_{n=2}^{\infty} \frac{B_n}{(-n)} \frac{1}{z^n}$$

$$\Rightarrow \ln \Gamma(z) = z \ln z - z + \frac{1}{z} \ln z + \sum_{n=2}^{\infty} \frac{B_n}{(-n)(-n+1)} \frac{1}{z^n}$$

$$\Rightarrow \Gamma(z) = z \cdot e^{-z} \cdot z^{\frac{1}{z}} \cdot \exp \left[\sum_{n=2}^{\infty} \frac{B_n}{(-n)(-n+1)} \frac{1}{z^n} \right]$$

$$\begin{aligned} & a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \dots \\ & \approx 1 + a_1 z^{-1} + \left(a_2 + \frac{1}{2} a_1^2 \right) z^{-2} + \dots \end{aligned}$$

Comments: the gamma function does not submit directly to the usual method of descents to yield up an asymptotic expansion. You can quickly get the first term by changing variables in such a way that the saddle appears at a finite point, but this does not give lower terms.

The correct method to use involved dealing with the derivative of the PSI function. This derivative ψ' may be written as a simple integral which at once admits Watsons Lemma.

Baileman p.16: $\psi'(z) = \int_0^{\infty} dt \left[\frac{e^{-t}}{t} - \frac{e^{-tz}}{(1-e^{-t})} \right]$

$$\left[\psi'(z) = \int_0^{\infty} dt \cdot \frac{t}{(1-e^{-t})} \cdot e^{-tz} \right]$$

$$\frac{t}{1-e^{-t}} = \frac{e^t t}{e^t - 1} = e^t \cdot \left[\sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{t^n}{n!} \right] \quad \text{Bernoulli}$$

$$\Rightarrow \psi'(z) = \int_0^{\infty} dt \cdot e^{-t[z-1]} \cdot \sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{t^n}{n!}$$

$$= \sum_n \frac{B_n}{n!} \times \frac{n!}{(z-1)^{n+1}} = \frac{1}{z-1} - \frac{1}{2} \frac{1}{(z-1)^2} + \dots$$

$$\Rightarrow \psi(z) = \ln(z-1) + \frac{1}{z-1} + \dots \quad \text{etc.}$$

non-Euclidean Geometry

1/23/74

Digression on Non Euclidean Geometry

paths

Executive (1974)



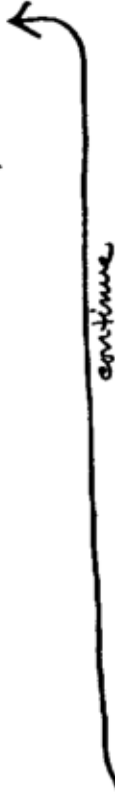
Thurman Wick (1968)



Wick (1964)



Martin & Sparrmann (1970)



mass shell "hint"



<Differential geometry>

{ O'Neill's book (excellent!)
Gruenberg (not so)

hyperbolic metric
surface representations
{ $\mathbb{H}$ not $\mathbb{D}$ }
Riemannian geom.
intrinsic, etc.

$K=+1, K=-1$

<non-Euclidean geometry>

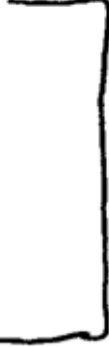
{ Moschovaki's book (A-1!)
(exposed @ #2.5)
Poincaré model
all formulas
Euclid (358e!)



Question remains: on what hy. surface does that triangle exist?!



Somerfeld resolves problem.



Investigation into Hyperbolic Geometry.

The starting point of this diversion was a "velocity space" diagram in Wick's paper, copied into Martin and Spearmans. We know that the following is true:

$$R_y(w) = R_z(\sigma') R_y(\theta) R_z(x) R_y(\theta) R_z(\sigma)$$

In words: if you boost in some direction, then do a z -boost, then it is possible to unwind with θ' and σ' so you end up where you started except for a rotation. On the mass shell diagram, where you cannot see the rotation,

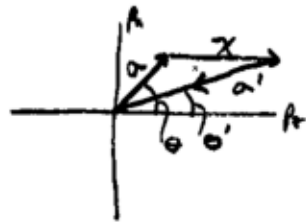


Here you can see clearly, however, that once the point $\cdot$ is reached, θ' and σ' are uniquely set to get you back.

If we combine (σ, θ) and (σ', θ') into one line, the picture looks like:



If this figure is projected down into the (p_x, p_z) plane, we see: still, the angle w is not apparent, because $(m, 0, 0) \rightarrow (m, 0, 0)$.



Notice that the actual lengths of the lines σ and σ' are known, namely they are $\sinh \sigma$ and $\sinh \sigma'$. Thus we are at once able to conclude the law of sines except for the X part, i.e.,

$$\sinh \sigma \sin \theta = \sinh \sigma' \sin \theta'$$

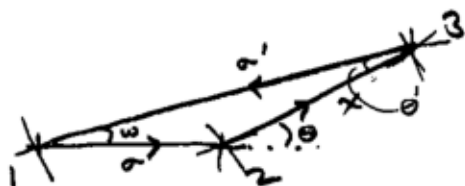
The length of the segment X is not $\sinh X$, or anything easy, so there is really nothing else we can do with that picture. Also, I was hoping that the law of sines + cosines would apply to that S on the hyperbolic surface, but that seems also to fail since X is not a geodesic, nor are the arc lengths on that surface anything simple.

Of course still we know that the series of operations shown above is a $R_y(w)$ because it leaves the $(m, 0, 0)$ unchanged, as Wick stated. Of course all these rules

can be obtained by direct matrix multiplication (and I have done this). The question is this:

→ Why do the rules resemble exactly the laws of sines + cosines on a hyperbolic surface? ←

It is very helpful to make a velocity picture. Here, the cond system is moving and distance is relative velocity. Thus, the operations listed produce these results:



Now ω has a clear meaning. It is the direction between the velocities of frames 2 and 3 as observed from (rest) frame (of) 1.

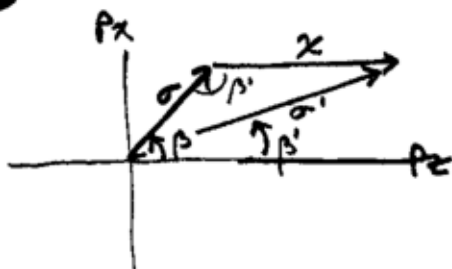
Now, however, you cannot simply do vector addition on these velocities because velocities do not add that way. This means that this Δ is not sitting in a Euclidean plane; there is some kind of fancy metric going on. In fact, it must be a hyperbolic metric because the result we know is:

$$\left\{ \begin{array}{l} \cosh x = \cosh a \cosh a' - \sinh a \sinh a' \cos \omega \\ \frac{\sinh a'}{\sinh \omega} = \frac{\sinh a}{\sinh \omega'} = \frac{\sinh x}{\sinh \omega} \end{array} \right\}$$

It is as if that triangle really were on some kind of hyperbolic surface. I have been unable so far to explain why this is so; of course I have derived the formulas shown. It may be connected to the famous hyperbolic plane noted in O'Neill's book. Or, it may really tie into the hyperbolas I keep drawing. No way to find out unless I look up Smorfeld's or Cho's reference.

preliminary results:

Problem 1. Write equations relating to this picture: ($p_y = 0$)



$$\text{ch}\sigma' = \text{ch}\chi \text{ch}\sigma + \text{sh}\chi \text{sh}\sigma C_\beta \quad \text{note: } C_\beta = -C_{\beta'}$$

$$\text{sh}\sigma' S_{\beta'} = \text{sh}\sigma S_\beta$$

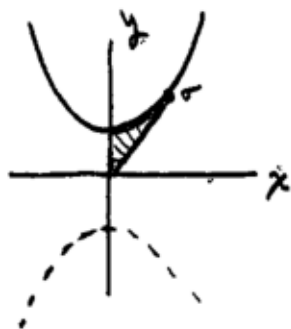
$$\text{sh}\sigma' C_{\beta'} = \text{sh}\chi \text{ch}\sigma + \text{ch}\chi \text{sh}\sigma C_\beta$$

$$\text{ch}\chi = \frac{\text{ch}\sigma \text{ch}\sigma' - \text{sh}\sigma \text{sh}\sigma' C_\beta C_{\beta'}}{\text{ch}^2\sigma - \text{sh}^2\sigma C_\beta^2}$$

The lengths of σ and σ' are really $\text{sh}\sigma$ and $\text{sh}\sigma'$, but the length of the χ segment is not $\text{sh}\chi$. Rather:

$$\begin{aligned} \text{length}\chi &= \text{sh}\sigma' C_{\beta'} - \text{sh}\sigma C_\beta \\ &= \text{sh}\chi \text{ch}\sigma + \text{sh}\sigma (\text{ch}\chi - 1) C_\beta \end{aligned}$$

on the hyperbola itself



1. The hyperbola is chosen with $m=1$ so that

$$y^2 - x^2 = 1$$

$$\cosh^2 \sigma - \sinh^2 \sigma = 1$$

$$y = \cosh \sigma$$

$$x = \sinh \sigma$$

2. The interpretation of σ is clear. (neglect sign)

$$\text{area} = \frac{1}{2} \int_0^\sigma r^2 d\theta = \frac{1}{2} \int_0^x (x dy - y dx) = \frac{1}{2} \int_0^\sigma (\sinh^2 \sigma - \cosh^2 \sigma) = \frac{1}{2} \sigma$$

3. The length along the arc to point σ is very non trivial!

$$\text{length} = \int_0^\sigma \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^x \sqrt{1 + \frac{x^2}{x^2+1}} dx = \text{elliptical integrals}.$$

Euclid's Elements dominated geometrical thought for 2000 years (he was 325 B.C.). But no one could ever prove the parallel axiom from the other axioms. The idea that a geometry is really a set of axioms was clarified by Hilbert's particular ordering (1902). These axiom sets are

1. connection - 2 pts det. a unique line
2. order - betweenness
3. congruence - of angles and segs.
4. continuity - lots of points
5. parallels - exactly one parallel...

In 1763 there were 28 "proofs" of (5.). Kluegel showed that they were all fallacious. The first advance was made by Jesuit Saccheri ^(quadrilateral) ~ 1790 ~ who showed that $\alpha + \beta + \gamma \leq 2R$, based on the first four axioms. Legendre improved his proof of this, but made an error in his equality proof. (p31)

The breakthrough that 5. was nonprovable came in the work of Lobachevski (1826), Bolyai Jr (1831) and Gauss. But was not accepted for several decades.

Hyperbolic geometry is the geometry resulting from the replacement of 5. with 5a.

5a. There are at least two parallels.

There are several models of the hyperbolic geometry. Klein's disc has the problem that angles are non euclidean; thus, Poincaré's model is easier to deal with. The book goes on to derive many theorems of hyperbolic geometry within the Poincaré "model", but claims that all the results are general.

As far as trigonometry goes, the crucial step is the definition of length as the ln of a certain cross ratio of segments. In particular, this yields the famous hyperbolic law of cosines (valid in any hyperbolic model).

Elliptic geo uses 5b: any 2 lines intersect. However, set 2. of the "absolute" axioms must be modified.

law of cosines for spherical triangle



$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos \gamma$$

Here, (α, β, γ) are angles.
and (a, b, c) are the sides expressed in radians.

Note: suppose all 3 sides had imaginary lengths. Then
let.

$$C = ic$$

$$B = ib$$

$$A = ia$$

$$\cos a = \cos(-iA) = \cosh A$$

$$\sin a = \sin(-iA) = -i \sinh A$$

Then the law of cosines would be:

$$\cosh C = \cosh A \cdot \cosh B - \sinh A \cdot \sinh B \cdot \cos \gamma$$

But this happens to be the law of cosines in hyperbolic geometry? Why?

Look at $K = k_1 k_2 = 1$, say, for spherical geometry. If all lengths become imaginary including radii of curvature, then the gaussian curvature changes sign!

$$K = (ik_1)(ik_2) = -K_1 K_2 < 0 \Rightarrow \text{hyperbolic metric.}$$

Resolution of The Hyperbolic Problem.

1. Martin & Speereman err when they suggest that the mass shell is the hyperbolic surface which causes the $\cosh$ formula.
2. The real reason is that the actual triangle lies on a sphere of imaginary radius (so the rotation angles can be imaginary). The imag radius causes K to be negative, so it is the sphere that becomes the hyperbolic surface.
3. All the trigonometry of hyperbolic geometry carries over into this realm, because the h.g. is $K = -1$. Or, you can use the spherical formulae with imaginary angles.

We call it, with Minkowski, the *light cone*. The interior is called the *forecone* and the *aftercone*, depending on whether $t < 0$ or $t > 0$.

All four-vectors leaving the origin which lie outside of the light cone are called *space-like*, those which lie inside of it, *time-like*. Thus the vector $\mathbf{r}$ in the equatorial plane of the light cone is space-like, whereas all permitted velocities leaving the origin are time-like.

The sequence of all four-dimensional positions assumed by a moving material point is called its *world line*. The world line of a point at rest is parallel to the t -axis. All world lines passing through the origin lie in the forecone for $t > 0$, in the aftercone for $t < 0$.

We consider the *element of a world line*

$$ds = \sqrt{\sum_{i=1}^4 dx_i^2}.$$

is the distance between two neighboring world points, it is Lorentz-invariant (see p. 213). The same applies to Minkowski's *intrinsic time*

$$\begin{aligned} \frac{ds}{ic} &= \sqrt{dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)} \\ &= dt \sqrt{1 - \frac{v^2}{c^2}} = dt \sqrt{1 - \beta^2}. \end{aligned} \quad (17)$$

We will now define the *four-vector of the velocity* along a world line. The term

$$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, ic = (\mathbf{v}, ic)$$

could not be a permissible definition, since dt has no invariant meaning. This does not apply, however, to

$$\mathbf{V} = \frac{dz}{d\tau}, \frac{dy}{d\tau}, \frac{dx}{d\tau}, ic \frac{dt}{d\tau} = \frac{dt}{d\tau} (\mathbf{v}, ic). \quad (18)$$

The square of its length is

$$\mathbf{V} \cdot \mathbf{V} = \frac{dx^2 + dy^2 + dz^2 - c^2 dt^2}{d\tau^2} = -c^2, \quad (18a)$$

is in fact an invariant which, furthermore, has the same value for all velocity vectors $\mathbf{V}$. The *four-vector of the acceleration* should be defined correspondingly by

$$\mathbf{W} = \frac{d\mathbf{V}}{d\tau} = \frac{d^2x}{d\tau^2}, \frac{d^2y}{d\tau^2}, \frac{d^2z}{d\tau^2}, ic \frac{d^2t}{d\tau^2}. \quad (18b)$$

It is, in the four-dimensional sense, perpendicular to the four-vector of the velocity; differentiation of (18a) with respect to τ leads to:

$$\mathbf{V} \cdot \mathbf{W} = 0. \quad (18c)$$

H. The Addition Theorem for Velocities of Different Directions

Velocities of the *same* direction were to be combined in such fashion that their *angles* were added. Since in their elementary meaning angles denote arcs on the unit circle, their addition is equivalent to the joining of arcs on a circle which in our case, it is true, has the radius i instead of 1. The formulas of *plane* trigonometry could be applied to the joining of these arcs.

In order to combine velocities with *different* directions it is necessary to pass from the circle to the sphere, i.e. from the formulas of plane to those of *spherical* trigonometry, and for a sphere of radius i , not of radius 1. Combination of the velocities v_1 and v_2 to form the resultant v is therefore equivalent to the combination of the angles γ_1 and γ_2 to the resultant angle γ , i.e. the construction of a spherical triangle with the sides γ_1 , γ_2 , and γ . If α is the angle of inclination of v_2 relative to v_1 , then α appears as the external angle between the sides γ_1 and γ_2 in the spherical triangle. We then have by the cosine law (see Problem I.4):

$$\cos \gamma = \cos \gamma_1 \cos \gamma_2 - \sin \gamma_1 \sin \gamma_2 \cos \alpha. \quad (19)$$

This is the desired *generalized addition theorem*. For $\alpha = 0$ we obtain $\cos \gamma = \cos(\gamma_1 + \gamma_2)$; $\gamma = \gamma_1 + \gamma_2$, i.e. the earlier Eq. (15a). In view of the relation $\cos \gamma = (1 - \beta^2)^{-1/2}$ etc. (19) is equivalent with the rather untransparent formula

$$\beta^2 = \frac{\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 \cos \alpha - \beta_1^2\beta_2^2 \sin^2 \alpha}{(1 + \beta_1\beta_2 \cos \alpha)^2}. \quad (19a)$$

which was given already by Einstein. In Problem III.2 it will be proved analytically by application of the Lorentz transformation.

The introduction of our sphere of radius i may seem an arbitrary trick; actually, it is merely an expression of the fact that the arcs γ_1 , γ_2 which we must combine are imaginary, according to (11a).

We shall mention one more interesting result, which may be read off directly on Fig. 40: In the theory of relativity the sequence of differently oriented velocities is *not exchangeable*; the result of the combination of v_1 and v_2 differs from that of the combination of v_2 and v_1 . Though the *magnitude* of the resultant is the same, the *direction* differs. The difference in the two directions increases as the velocities increase; in fact, as we shall know, it is equal to the spherical excess of the spherical triangle formed in our construction.

In Fig. 40 the angle α between v_1 and v_2 has been chosen equal to $\pi/2$

to simplify the drawing and the arc γ_1 corresponding to v_1 has been placed on the equator of the sphere. The extension of the arc γ_2 then passes through the northpole N . If, on the other hand, starting from the same point A , we first record γ_2 (denoted by γ'_2 on the figure), perpendicular to the equator and with its extension also passing through N , we must draw through the endpoint B' of γ'_2 a great circle perpendicular to the meridian AB' and measure off on it $\gamma'_1 = \gamma_1 = AB$. The point A' located in this manner does not coincide with C ; instead, the connecting arcs AC and $C'A'$ enclose a certain angle ϵ . In view of the equality of the two triangles ABC and $A'B'C'$ we have here $\angle BAC = \angle B'A'C'$ and $\angle ACB = \angle A'C'B'$. If we call

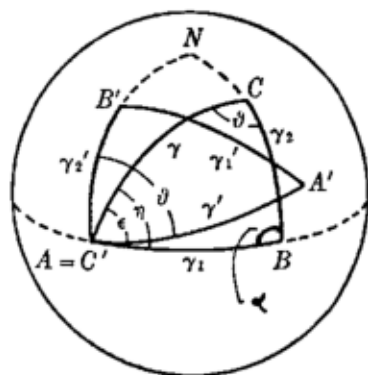


FIG. 40. Combination of two differently directed velocities v_1 and v_2 to form the resultant v , corresponding to the circular arcs γ_1 , γ_2 , and γ on a sphere of radius i . For convenience in representation the angle between v_1 and v_2 has been set equal to $\pi/2$. The figure shows the non-commutative character of the components: $v_1, v_2 = ABC \neq v_2, v_1 = C'B'A$; the angle ϵ between AC and $C'A'$ is equal to the spherical excess of the triangle ABC (and that of the triangle $A'B'C'$ which is congruent to it).

these two angles η and ϑ , we see that the right angle at A is formed by η , ϑ , and ϵ in the following manner:

$$\frac{\pi}{2} = \eta + \vartheta - \epsilon.$$

Hence

$$\epsilon = \eta + \vartheta - \pi/2 = \eta + \vartheta + \pi/2 - \pi. \quad (20)$$

ϵ thus is in fact the spherical excess of our right spherical triangle ABC and the congruent triangle $A'B'C'$. (The same applies for a general spherical triangle.)

The limiting case $\gamma_1 = \gamma_2 = \pi/2$, where the two triangles ABC and $A'B'C'$ become equal to the same spherical octant, is particularly simple. Here the resultants are evidently perpendicular to each other and, in view of $\eta = \vartheta = \pi/2$, the spherical excess is also $\pi/2$.

J. The Principles of the Constancy of the Velocity of Light and of Charge

Einstein in 1905 expressly added the first of these principles to the principle of relativity as an empirical postulate. It states that the velocity

of propagation of light is independent of the state of rest or motion of the emitting body. This principle is already included in the original formulation of our world geometry insofar as we have demanded the universal validity of the Maxwell equations. Like the velocity, the spherical propagation of the light is invariant in the transition from $x_1 \cdots x_4$ to $x'_1 \cdots x'_4$. The Lorentz transformation does not change the light sphere into a light ellipsoid, but leaves it a light sphere. (This does not apply to the wave-length of the light, which is not invariant but is known to depend on the frame of reference of the observer: Doppler effect.)

In the earlier but long since discarded theory of the universal ether, the independence of the light wave from the state of motion of the emitting body was readily understood: once transferred to the ether, it propagates itself in accord with the (elastic or electromagnetic) properties of this medium. Constancy of the velocity of light was here equivalent with field action. The same does not apply for a mechanical emission theory such as that surmised by Newton. Here a transfer of the velocity of the emitting body to the emitted light particles seems almost unavoidable.¹ We may say: The constancy of the velocity of light is today the only valid remnant of the ether concept. If at present we should speak of an ether, we would have to assign a separate ether to every frame of reference, i.e. speak e.g. of a primed and an unprimed ether. We now regard Lenard's "absolute ether (Uräther)" merely as a freak and the Aristotelian and scholastic "quintessence" (the fifth element, added to fire, water, air, and earth) as an historical curiosity. Thus in parts I and II, we have almost never spoken of the "ether", but used instead the not readily misinterpreted word "vacuum".

The principle of constant charge is as important as that of the constancy of the velocity of light. The charge is the same for every frame of reference. This is not obvious, but follows from the Maxwell equations if we can claim their universal validity for all frames of reference. On the other hand the principle of the constancy of mass with change of the system of reference, formerly regarded as obvious, cannot be upheld, as we shall see presently. The charge is an absolute invariant with respect to Lorentz transformations; mass and, as we shall also see, energy, are not.

Summarizing the content of this and the preceding section we may say: From the standpoint of the Maxwell equations the theory of relativity is obvious. A mathematician whose eyes had been trained by Klein's Erlangen program could have read from the form of the Maxwell equations its transformation group along with all its kinematic and optical consequences.

¹ The fact that Newton's emission theory could in a sense, experience a resurrection in the present theory of the light quanta rests solely on the addition theorem of the theory of relativity according to which effectively $c + v = c$ (c = velocity of light quanta, v = velocity of the emitting body).