

Phil Lucht's old
Math II Binder

Infinite Series

ODE's

Complex Variables

Infinite Series

1. Theorems on infinite series.
2. The meaning of various kinds of convergence.
3. Interchanging: Limit with Integration.
a. example given; FESR-application
4. Interchanging: Integration with Another Integration.
5. Facts about how you can find out whether a SUM is Carlson in a parameter.
6. Convergence of: $\sum_{n=1}^{\infty} z^n/n^p$
7. Convergence of: $\phi(a, c; x) = \overline{H}(a; c; x)$
8. Some standard series.
9. Sum calculation by Method of Piecewise Contours.
10. How do you bound this series tail: $\sum_{n=N}^{\infty} (n)^a z^n$
11. Asymptotic Series (Asymptotic expansions)
12. Infinite series as analytic function.

B. Simple Sums

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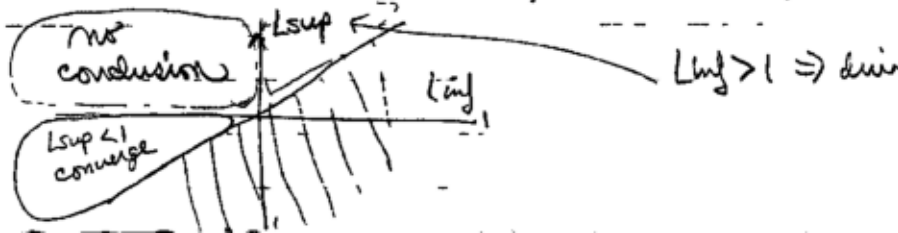
1. References: most of these are given in Buck for real variables in his chapter on convergence. Many however are generalized to complex variables and are stated in Spiegel's book. In the library there are books specializing in infinite series some giving theory and advanced tests, others stating various series.
2. Plan: below I state in no particular order any theorems I might be able to use. Hopefully all qualifiers are included.
3. Power Series Radius: The domain of convergence of a complex power series is always a circle in the z plane of some radius R . In order to find the value of R , you must use the various tests given below. Right at the radius $|z|=R$, you do not a priori know whether series converges or not. When a function $f(z)$ is represented in some region by a series, and you write an expansion about a point a , then series converges in circle which is as large as possible about a without hitting a singularity of the function.
4. Simple Divergence Test: if a_n does not approach 0, then series must diverge.
5. Cauchy convergence: the partial sums S_n form a sequence in the space C . With the usual metric, this space is complete. Thus, for infinite series the concepts of convergence and Cauchy convergence coincide.
6. Ratio test: let $L = \lim |a_{n+1}/a_n|$. If L is greater than 1, series diverges. If L is less than 1, series converges. If $L=1$, no conclusion. (If $a(n)$ were the coefficients of a power series instead of the actual terms, then this test would tell you the radius of convergence: $R = 1/L$.) Here, convergence is absolute.
7. Root Test: this time, let $L = \lim (|a(n)|)^{1/n}$. Then test is same as ratio test. It is this test which really shows that power series has a circle of convergence. If a_n are power series coefficients, then radius is $R = 1/L$ again.
8. Integral test: this test only applies to a real, positive definite sequence a_n . Actually, to a positive definite tail of a series, if you like. Test says that $\sum a_n$ converges if $\int a(x)dx$ converges. Also, sum diverges if integral diverges. Sometimes integrals are easier to calculate than sums.
9. Raabe's Test: sometimes when $L=1$ in the ratio test and the test fails, Raabe's test will succeed. It says to define $L' = \lim (n(1 - |a_{n+1}|/|a_n|))$. Now, if L' is greater than 1, series converges absolutely. If L' is less than 1, series may diverge or it may conditionally converge. If $L' = 1$, test fails.
10. Gauss's Test: this is a more sophisticated version of Raabe's test. See page 142 of Spiegel for details.
11. Absolute versus Conditional Convergence: if a series converges absolutely then it must converge. The reverse of course need not be true. If a series converges, but does not absolutely converge, it is called conditionally convergent. Thus, a convergent series is either AC or CC.
12. Rearrangements: if a series converges absolutely, then you can rearrange it however you like. Thus, if two series converge absolutely, so does their sum and Cauchy product, see Buck p 159 to define sum and product. This is the great strength of the concept of absolute convergence. Note that all power series (including of course hypergeometric function) converge absolutely inside their radius.

13. Alternating Series Test: Let the terms a_n be real and monotonic decreasing and approaching 0 in the limit. Then the alternating sign series converges, but no conclusion may be drawn about the series $\sum a_n$ itself.
14. The bracket theorem: this is a particular form of rearrangement where you insert brackets as you like but you do not change the order on paper. Then clearly, the brackets do define a rearrangement, so we expect brackets to have no effect on an absolutely convergent sequence. However, even if series converges only conditionally, brackets do not hurt.
15. Cauchy product: even if one series is AC and second is only CC, the Cauchy product converges. If both are AC, then Cauchy product series is convergent.
16. Double sums: Suppose two series A and B are AC. Then the double sum series is also AC. You can then rearrange as you like. Thus, the Cauchy product is also AC. See Buck for more details.
17. Uniform convergence: consider a series where each term carries an additional parameter like z . (Example: series of functions $f_n(z)$). With this series f_n we associate a sequence of partial sums $F_n(z)$. If this sequence converges uniformly on some domain of z , then series "converges uniformly". See page 140 or notes for meaning of "uniformly".
18. Weierstrass M test: if a series $f_n(z)$ is such that $|f_n(z)|$ is less than a convergent series of reals M_n for all z in chosen region, then series converges uniformly in that region. This is clearly the easiest test to check for uniform convergence.
19. term-by-term calculus: if a series is uniformly convergent, then you can integrate or differentiate term by term, but there are certain qualifiers. For integration to be defined, a function must be continuous. So if the terms in your uniformly convergent series are continuous, you can prove that the sum of the series is also continuous. The only other requirement is of course that the integration domain (or contour) lie inside region of uniform continuity in z . Also, in order to differentiate term by term, you have to show that the resultant series is uniformly convergent before you are justified.
20. Comparison tests: see Spiegel page 141 theorem 10.

Addendum on convergence tests: These tests often require some limit L . For example, in the ratio test, L is the limit of this sequence: $r_n = |a(n+1)/a(n)|$. BUT, suppose this real sequence r_n has no limit! If the sequence is bounded from above, then it is bounded and must have at least one cluster point, but if it has two cluster points, then there is no limit. Example: $r_n = \cos(\pi n) = 1, -1, \dots$. In this case the various tests must be reformulated. Here is the reformulation:

Let $L_{\sup} = \limsup r_n$ and $L_{\inf} = \liminf r_n$. Even if there is an infinity of cluster points for sequence r_n , these two limits must exist if r_n is bounded. Or, perhaps $L_{\sup} = \infty$. The test then reads: if $L_{\sup} < 1$, series converges. if $L_{\inf} > 1$, then series diverges. In words, if all cluster points are below 1, then convergence, and if all cluster points are above 1, then diverges. But if some are above and some below, no conclusion.

Picture:



Convergence of a sequence of functions $f_n(x)$ to a function $f(x)$.

1. We of course have in mind that $f_n(x)$ is the partial sum of n terms in some series so the question of whether the sequence converges is the same as the question of whether the series converges.

2. Think of x as a single real variable for illustration. There are basically three kinds of convergence: pointwise, "in the mean", and uniform convergence.

3. Pointwise convergence. (Buck page 180) Let all these functions $f_n(x)$ be defined on some domain D . Let E be any open piece of this domain. Pointwise convergence means that you pick a particular x in E and you show that the sequence converges for that particular x . You show that this is true for any point x in E . Then your series converges pointwise.

This is the weakest form of convergence. As shown in the simple example on page 182 of Buck, you can have pointwise convergence, but the $\sup \text{ norm } \|f - f_n\|_{\sup}$ can still be infinite. This is because the trouble can jam up against one of your boundaries.

4. Convergence in the Mean. (Buck, page 195) This means that $\int |f - f_n|^2$ goes to 0. I.e., the integral of the difference squared over the interval must vanish to have convergence in the mean.

I can imagine that you could have f_n converging to f in the mean, but not pointwise due to some outstanding point (of zero measure under integration). This same outstanding point would ruin uniform convergence also because the $\sup$ would always depend on this one bad point. So:

mean	is	pointwise
mean	is	uniform

5. Uniform Convergence. This is the most important and significant convergence idea. Uniform convergence means (Buck, page 182) that $\|f - f_n\|_{\sup}$ goes to 0. I.e., the maximal difference between the two functions (at whatever point that may be) must vanish.

If f_n converges uniformly, then it has the Cauchy Property $\|f_n - f_m\|_{\sup}$ goes to zero if m, n larger than some M . (etc)

Most importantly: if a series converges uniformly, then you can integrate it term by term. (page 187)

I presume that uniform convergence allows you to interchange the procedure of summation of series with any other procedure, such as limit taking.

limit and integral: uniformity

① When are these equal

$$\lim_{y \rightarrow y_0} \int_a^b dx f(x, y) = \int_a^b dx \lim_{y \rightarrow y_0} f(x, y)$$

Assume both limits exist. Then question is:

$$\lim_{y \rightarrow y_0} \left[\int_a^b dx f(x, y) \right] = ? \int_a^b dx f(x, y_0)$$

② As usual, examine error:

$$\left| \int_a^b dx f(x, y) - \int_a^b dx f(x, y_0) \right| \leq \int_a^b dx |f(x, y) - f(x, y_0)|$$

Now suppose y suff. close to y_0 :

$$|f(x, y) - f(x, y_0)| < \epsilon \quad \forall x \in [a, b] \quad \text{"uniformly"}$$

$$\text{Then: } \leq \int_a^b dx \epsilon = \epsilon(b-a) = \epsilon'$$

Thus, as $y \rightarrow y_0$, difference of two sides $\rightarrow 0$.

Theorem: if $\lim_{y \rightarrow y_0} f(x, y) = f(x, y_0)$ uniformly for $x \in [a, b]$,

then

$$\lim_{y \rightarrow y_0} \left[\int_a^b dx f(x, y) \right] = \int_a^b dx f(x, y_0)$$

Note: $b-a =$ finite interval only.

generalization to more general interval

① Now consider

$$\lim_{y \rightarrow y_0} \int_a^b dx \cdot h(x) \cdot f(x, y)$$

Assume that $f(x, y) \rightarrow f(x, y_0)$ uniformly over $[a, b]$. Then

$$\left| \int_a^b dx \cdot h(x) \cdot f(x, y) - \int_a^b dx \cdot h(x) \cdot f(x, y_0) \right|$$

$$\left| \int_a^b dx \cdot h(x) \cdot [f(x, y) - f(x, y_0)] \right|$$

$$\leq \int_a^b dx \cdot |h(x)| \cdot |f(x, y) - f(x, y_0)|$$

$$\leq \int_a^b dx \cdot |h(x)| \cdot \epsilon$$

Assume that $\int_a^b dx |h(x)|$ converges to $\#$. Then you're ok.

Theorem: if $f(x, y) \rightarrow f(x, y_0)$ uniformly for $x \in [a, b]$

and if $\int_a^b dx |h(x)| < \infty$

then

$$\lim_{y \rightarrow y_0} \int_a^b dx \cdot h(x) \cdot f(x, y) = \int_a^b dx \cdot h(x) \cdot f(x, y_0)$$

Valid even if interval is infinite.

Example of Theorem.

① consider $f(y) = \int_1^\infty dx: x^{-2} \cdot \frac{1}{(x^2+y^2)}$ as $y \rightarrow +\infty$.

Naively, you look at fixed x and take $y \rightarrow \infty$ inside to get

$$f(y) \sim \frac{1}{y^2} \int_1^\infty dx: x^{-2} = \frac{1}{y^2} \text{ as } y \rightarrow \infty.$$

But of course $x \rightarrow \infty$ also and not clear that you can make this argument.

② To apply previous Theorem, we try this

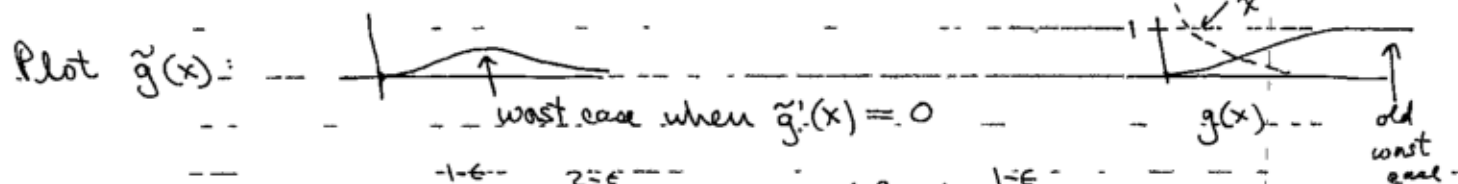
$$f(y) = \frac{1}{y^2} \int_1^\infty dx: x^{-2} \cdot \frac{1}{\left[\frac{x^2}{y^2} + 1\right]}$$

But fails because $|f(x,y) - f(x,y_0)| = \frac{1}{y^2} \cdot \frac{1}{\left[\frac{x^2}{y^2} + 1\right]}$ For fixed y , you want this $< \epsilon \forall x$; but as $x \rightarrow \infty$, $\frac{1}{\left[\frac{x^2}{y^2} + 1\right]} \approx 1$.

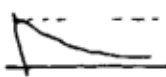
③ But now suppose we borrow $x^{-\epsilon}$ from $h(x)$ so that

$$\tilde{h}(x) = x^{\epsilon-2} \quad \tilde{f}(x,y) = \frac{x^{\epsilon}}{x^2+y^2+1}$$

$$\text{Then } |\tilde{f} - f| = x^{\epsilon} \left(\frac{x^2/y^2}{(x^2/y^2+1)} \right) = \frac{x^{2-\epsilon}}{x^2+y^2} = \tilde{g}(x)$$



$$\tilde{g}'(x) = (x^2+y^2)^{-1-\epsilon} (2-\epsilon)x - x^{2-\epsilon} \cdot 2x = -\epsilon(x^2+y^2)^{-1-\epsilon} x^{1-\epsilon} < 0.$$

So  worst case is at $x=1$ since $\tilde{g}$ is decreasing.

$$\tilde{g}(1) = \frac{1}{1+y^2} \text{ can be made as small as you like.}$$

④ Thus, any small amount of "borrowed power" makes $\tilde{f}(x,y)$ "uniform".

Moreover, $\int_1^{\infty} dx x^{2+\epsilon} < \infty$. Thus, we apply theorem and conclude naive result.

⑤ Explicit calc:

$$\begin{aligned} \int_1^{\infty} dx x^{-2} \frac{1}{(x^2+y^2)} &= \left(-\frac{1}{y^2 x} - \frac{1}{y^3} \tan^{-1}\left(\frac{x}{y}\right) \right) \Big|_1^{\infty} \\ &= -\frac{1}{y^3} \cdot \frac{\pi}{2} + \underbrace{\left(\frac{1}{y^2} + \frac{1}{y^3} \tan^{-1}\left(\frac{1}{y}\right) \right)}_{\approx \frac{1}{y^2}} \end{aligned}$$

There is the explicit $\frac{1}{y^2}$ term with coeff. = 1.

Comment: in integrals of this form, there would be no question about the validity of the "naive" result if the integral were finite. You could then take y to $\inf$ inside the integral because it is trivial to get uniformity on a finite interval. The real question is what happens at the high end of the x -integration. We see from the above example that as long as $h(x)$ decays faster than x^{-1} you are OK. The amount of power "borrowed" can be as small (but finite) as needed to make the argument work.

The FESR example.

① Dispersal in DHS reads:

$$E(y) = \frac{2y}{\pi} \int_0^{\infty} \frac{\text{Im}F(x) dx}{x^2 - y^2}$$

This example differs from the previous one on two counts: well, mainly the minus sign in the denom which means that the "worst case" x will be at $x=y$.

I.e, even borrowing power, it may be that you do not have uniformity.

② First, take to principle part:

$$\frac{1}{x-y \pm i\epsilon} = \mathcal{S}(x-y) i\pi \pm \mathcal{P}\left(\frac{1}{x-y}\right)$$

So

$$\text{Im}F(y) = 2y \cdot \frac{\text{Im}F(y)}{2y} = \text{Im}F(y) = \text{trivial}$$

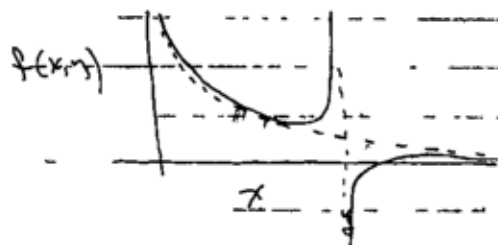
$$\text{Re}F(y) = \frac{2y}{\pi} \int_0^{\infty} \frac{\text{Im}F(x) dx}{x^2 - y^2}$$

③ Assume at hand (only place where naive result may be wrong),

$$\text{Im}F(x) \sim x \quad \text{then}$$

$$f(y) = -\frac{2y}{\pi} \int_0^{\infty} dx \frac{x}{x^2 - y^2} = -\frac{2}{\pi} \frac{1}{y} \int_0^{\infty} dx \frac{x}{x^2 - y^2} \left[\frac{x}{\left(\frac{x^2}{y^2} - 1\right)} \right]$$

Now look at $f(x,y)$ versus x to hunt for worst case:



Clearly, worst case at $x=y$ except for pole!

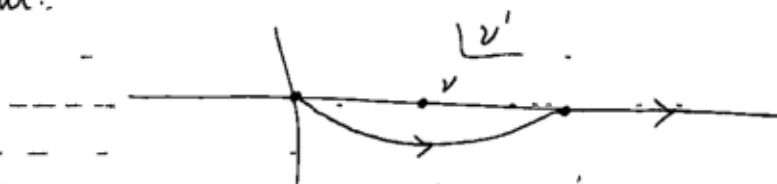
④ But pole is a local effect. Assume this piece:

$$\int_{y-\epsilon}^{y+\epsilon} dx \cdot x \frac{1}{\left(\frac{x^2}{y^2} - 1\right)} \approx y \cdot \frac{1}{2y} \int_{y-\epsilon}^{y+\epsilon} dx \frac{1}{\left(\frac{x^2}{y^2} - 1\right)} = 0.$$

⑤ Another point of view:

$$F(v) = \frac{2v}{\pi} \int_0^{\infty} \frac{dv' (v')^{\alpha}}{(v')^2 - v^2}$$

Deform contour:



Well, no more time for this. I think the point is clear. If $\alpha < -1$ then you are OK at the high end of x integration as in last example and this is what is important. The contribution from the pole (half a contour) will be $\text{Im} F = v^{\alpha}$ and since $\alpha < -1$ this term is below v^{-1} . The high endpoint contribution is v^{-1} precisely, and it is this thing which must vanish in FESR so no fixed pole.

On the Interchange of Order of Integration.

1. I am finding that I finally have to worry about this. The context is as follows:

In evaluating unknown integrals of special functions, a standard procedure is to elevate the special function into one of its integral representations, and then to "do" the original integration by moving it to the right. Of course this involved order interchange and you have to make sure the interchange is allowed.

2. The precise theorem about order interchange is given in Buck page 207. The main idea is simply stated: in order that interchange be allowed, each of the inside integrals must be uniformly convergent in its "parameter" over the entire relevant range of integration. Normally, the place you run into trouble is at the extremes of the integral ranges, usually when it is infinity.

3. Here is a beautifully simple illustration of the problem. In general, when you do it one way, the thing converges, but when you go the other way, it diverges.

But in this example it converges both ways, but gives different answers:

$$\int_0^{\infty} dy \cdot y \cdot e^y \cdot \underbrace{\int_0^{\infty} dx \cdot e^{-yx}}_{\frac{1}{y}} = \int_0^{\infty} dy \cdot e^{-y} = 1.$$

$$\int_0^{\infty} dx \cdot \int_0^{\infty} dy \cdot y \cdot e^{-y(1+x)} = \int_0^{\infty} dx \cdot \frac{1}{(1+x)^2} = \tan^{-1} x \Big|_0^{\infty} = \frac{\pi}{2}$$

Note: $\int_0^{\infty} dy \cdot y \cdot e^{-y(1+x)}$ is uniformly convergent $\forall x \in [0, \infty]$.
 $\int_0^{\infty} dx \cdot e^{-yx}$ is not u. c. $\forall y \in [0, \infty]$.

Here we see very clearly that one of the "internal integrals" is not uniformly convergent over the range of the external integration.

Comments on Uniform Convergence.

1. Theorem: if $F(j,n)$ is the summand of a series in n , and if $\tilde{F}(j,n)$ is the limit behavior as $n \rightarrow \infty$, then from the form of $\tilde{F}(j,n)$ alone you cannot make any conclusions about whether series converges uniformly in j .

Example: let $F(j,n) = \frac{1}{n^2} + j \frac{1}{n^3}$. Then $\tilde{F}(j,n) = \frac{1}{n^2}$, independent

of j . But, $|F(j,n) - \tilde{F}(j,n)| = |j| \cdot \frac{1}{n^3}$, so series is not

uniformly convergent as $|j| \rightarrow \infty$ (I.e., when you say $\exists N$

$\Rightarrow \forall n > N$, $|F - \tilde{F}| < \epsilon$, N increases with $|j|$.) But if

$F(j,n) = \frac{1}{n^2}$, then series would be uniformly conv. in $|j| \rightarrow \infty$.

The point is, obviously, that the lower terms which you do not show in $\tilde{F}$ are possibly going to cause non-uniform convergence.

2. Application: it you are wondering whether the addition theorem $P = \sum_n PP$ is uniformly convergent as $|j| \rightarrow \infty$, you will learn nothing by looking at the large $|j|$ behavior of the summand.

Detail on the Carlson method of continuing "through" an infinite sum.

① First, suppose $f(j) = \sum_{m=0}^{\infty} F(j, m)$ $j=0, 1, 2, \dots$

Assume $f(j)$ is Carlson; assume $F(j, m)$ is Carlson for fixed m . Question: when is RHS Carlson?

$$F(j, m) \text{ Carlson} \Rightarrow |F(j, m)| < O(e^{+\pi|j|})$$

This means: $\lim_{|j| \rightarrow \infty} |F(j, m)| < C_m e^{+\pi|j|}$ as $|j| \rightarrow \infty$ in closed RHP.

② Now define: $S(j) = \sum_{m=0}^{\infty} F(j, m)$

$$\text{Then } |S(j)| = \left| \sum_m F(j, m) \right| < \sum_m |F(j, m)| < \sum_m C_m e^{+\pi|j|}$$

$$\Rightarrow |S(j)| < e^{\pi|j|} \left(\sum_{m=0}^{\infty} C_m \right)$$

need uniform convergence in j under to say $\lim_{j \rightarrow \infty} \sum = \sum \lim_{j \rightarrow \infty}$

If $\left(\sum_{m=0}^{\infty} C_m \right)$ converges, then $S(j)$ is Carlson!

③ Does $\sum C_m$ converge? A quick answer is: yes, if $\sum_m F(j, m)$ converges absolutely, and uniformly $\forall j$ in RHP!

④ Theorem: Let $f(j) = \sum_{m=0}^{\infty} F(j, m)$ for $j=0, 1, \dots$. Then, if

(a) $f(j)$ is Carlson in j

(b) $F(j, m)$ is Carlson for each fixed m

(c) $\sum_m F(j, m)$ converges uniformly for j in RHP, and absolutely

then equation is true for complex j .

⑤ Example: first-kind Legendre addition sums extend to complex j .

⑥ Good test case: can you continue same theorems to complex m' ?

$$e^{-\pi\phi} P_{nm}^j(z) e^{-\pi\phi} = \sum_{n'=0}^{\infty} e^{-i\omega n'} P_{nm}^j(z_1) P_{nm'}^j(z_2)$$

Assume $P_{nm}^j(z)$ and $P_{nm'}^j(z)$ are Carlson in m' (they are). If $|\phi| < \pi$, LHS is Carlson in m' . Now

$$P_{nm}^j(z_1) P_{nm'}^j(z_2) \rightarrow |n|^{m+m'-1} \exp\left(-\frac{\pi m}{z} \ln \left| \frac{z-1}{z+1} \right| \left| \frac{z_2-1}{z_2+1} \right| \right)$$

Let be:

(m' now role of j)
(n role of m)

$$\sum_{n=0}^{\infty} \exp[(m+m'-1) \ln |n| - \frac{1}{2} |n| \ln n]$$

ratio

Question: consider, is this $< C_n C_n^{+\pi|m'|} \forall n$ RHP?

Does this sum converge uniformly $\forall m'$ in RHP? No matter how large m' is, it does converge. But a fixed tail is not uniformly small!

Thus, cannot continue to complex m' !

Find conditions of convergence for: $\sum_{n=1}^{\infty} \frac{z^n}{n^p}$

① The ratio is: $r_n = \left| \frac{a_{n+1}}{a_n} \right| = |z| \cdot \left[\frac{n}{n+1} \right]^{\operatorname{Re}(p)} \rightarrow |z|$

For $|z| < 1$ this thing converges absolutely. The real problem is the case $|z| = 1$. The ratio test is inconclusive, because $L = 1$.

② Raabe says:
$$L' = \lim_{n \rightarrow \infty} \left[n \left(1 - \left| \frac{a_{n+1}}{a_n} \right| \right) \right]$$

$$= \lim [n(1 - r_n)] = \lim \left[n \left(1 - \left\{ \frac{n}{n+1} \right\}^{\operatorname{Re}(p)} \right) \right]$$

Now work on this: let $\alpha = \operatorname{Re}(p)$:

$$\left(\frac{n}{n+1} \right)^\alpha = \left(\frac{1}{1 + \frac{1}{n}} \right)^\alpha = \left(1 + \frac{1}{n} \right)^{-\alpha} \approx 1 - \alpha \frac{1}{n}$$

$$[n(1 - \left(\frac{n}{n+1} \right)^\alpha)] \approx [n(\alpha \frac{1}{n})] = \alpha \Rightarrow L' = \alpha = \operatorname{Re}(p)$$

Conclusion: $\operatorname{Re}(p) > 1 \Rightarrow$ absolute convergence when $|z| = 1$
 $\operatorname{Re}(p) \leq 1 \Rightarrow$ no conclusion.

So from Raabe we learn: $|z| = 1$ and $\operatorname{Re}(p) > 1 \Rightarrow$ absolute convergence. But this is obvious from comparison/integral test.

③ Of course $a_n = \frac{z^n}{n^p}$. If $|z| < 1$, then $a_n = n^{-p} e^{i n \arg z} \rightarrow 0$ for all p , converges even if $\operatorname{Re}(p) < 0$.

If $|z| = 1$, $a_n = n^{-p} (e^{i\phi})^n$

If $\operatorname{Re}(p) < 0$, $a_n \not\rightarrow 0 \Rightarrow$ divergence, $|z| = 1$ $\operatorname{Re}(p) = 0$ marginal.

④ The case we are now still missing is $|z| = 1$ and p in range $0 \leq \operatorname{Re}(p) \leq 1$. Here we can use the fancy Dirichlet test of Buck p.165.

$$\sum_{n=1}^{\infty} \frac{z^n}{n^p} \quad \text{let } a_n = \frac{1}{n^p}, \quad b_n = z^n$$

(i) if $\operatorname{Re}(p) > 0$ (but not $= 0$), $a_n \rightarrow 0$ so ✓

$$(ii) \sum |a_{n+1} - a_n| = \sum \left| \frac{1}{(n+1)^p} - \frac{1}{n^p} \right|$$

In this series, $A_n = \left| \frac{1}{(n+1)^p} - \frac{1}{n^p} \right|$

$$\approx \left| \frac{1}{n^p} \right| \cdot \left| (-p) \frac{1}{n} \right| \approx |p| \cdot \frac{1}{n} \cdot n^{-\operatorname{Re}(p)}$$

$$\approx |p| \cdot n^{-1-\operatorname{Re}(p)}$$

Since $\operatorname{Re}(p) > 0$, this series does converge, so ✓

$$(iii) \text{ Let } |B_N| = \left| \sum_{n=1}^N b_n \right| = \left| \sum_{n=1}^N z^n \right|$$

$$= \left| \sum_{n=1}^{\infty} z^n - \sum_{n=N+1}^{\infty} z^n \right| = \left| \frac{z}{1-z} - \frac{z^{N+1}}{1-z} \right|$$

$$= \frac{|z|}{|1-z|} |1 - z^N|$$

Now, ~~also~~ $|1 - z^N| \leq |1| + |z^N| = 1 + |z|^N$

assume $|z| > 1$. Then: $|B_N| \leq \frac{2}{|1-z|}$

As long as we avoid $z=1$,

$|B_N|$ is bounded for all N . ✓

Conclusion: for $|z|=1$ but $z \neq 1$ and $\operatorname{Re}(p) > 0$

(the main result here) $\sum_{n=1}^{\infty} \frac{z^n}{n^p} = \text{convergent}$

⑤ For $z=1$, $\sum_{n=1}^{\infty} \frac{1}{n^p} = \text{convergent only if } \operatorname{Re}(p) > 1 = \zeta(p)$

convergence of $\phi(a, c; x) = {}_1F_1(a; c; x)$

$$\textcircled{1} \quad \phi(a, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{(x)^n}{n!} = \sum_{n=0}^{\infty} A_n$$

$$r_n = \frac{A_{n+1}}{A_n} = \frac{\Gamma(a+n+1) \cdot \Gamma(c+n)}{\Gamma(a+n) \cdot \Gamma(c+n+1)} \cdot \frac{x}{(n+1)!} \cdot (n)!$$

$$= \frac{(a+n)}{(c+n)} \cdot \frac{x}{(n+1)}$$

$$\lim_{n \rightarrow \infty} r_n = \frac{x}{n} = 0. \quad \text{Also, } |r_n| \rightarrow 0$$

Therefore, this series converges absolutely in entire x -plane!
i.e., $L = 0 < 1$

Unlike $F(a, b, c; z)$ which has $L = 1$ so Raabe needed.

Some Standard Series:

$$\textcircled{1} \quad \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^z \quad |z| < \infty$$

$$\textcircled{2} \quad \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{n!} z^n = \Gamma(a) \cdot (1-z)^{-a} \quad |z| < 1$$

$$\textcircled{2}' \quad \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a} \quad |z| < 1$$

$$\textcircled{3} \quad \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \cdot \frac{z^n}{n!} = \frac{\Gamma(a)}{\Gamma(c)} \cdot \Phi(a, c; z) \quad |z| < \infty$$

$$\textcircled{3}' \quad \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} = \Phi(a, c; z) \quad |z| < \infty$$

$$\textcircled{4} \quad \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = F(a, b, c; z) \quad |z| < 1$$

$$\textcircled{5} \quad \sum_{n=0}^{\infty} \frac{1}{(c)_n} \frac{z^n}{n!} = {}_0F_1(c; z) = \text{see note } \mathbb{J}_{c-1} \quad |z| < \infty$$

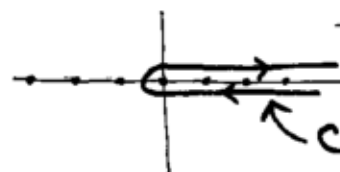
Calculation of sums by piecewise continuation of contours.

1. In order to explain what this title means, consider this sum:

$$S = \sum_{n=0}^{\infty} z^n$$

This is some function of z , $S(z)$. Suppose we did not know how to add up this series. Our first approach would be to do a SWT. This yields:

$$S(z) = -\frac{1}{2i} \int_C \frac{dn z^n}{\tan \pi n}$$

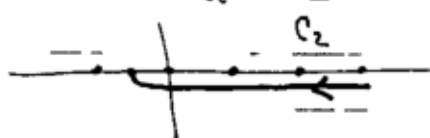


We would like to simply open the contour and run it vertically. However, if z is at all complex, one of the tails will diverge and you are screwed.

2. So, we try a piecewise analysis. Break the contour into two contours C_1 and C_2 as shown. Thus:

$$S = S_1 + S_2$$

$$S_i = -\frac{1}{2i} \int_{C_i} \frac{dn z^n}{\tan \pi n}$$



$$S_1 = -\frac{1}{2i} \int_{-\frac{1}{2}+i\epsilon}^{\infty+i\epsilon} \frac{dn z^n}{\tan \pi n} = S_1(z)$$

$$S_2 = +\frac{1}{2i} \int_{-\frac{1}{2}-i\epsilon}^{\infty-i\epsilon} \frac{dn z^n}{\tan \pi n} = S_2(z)$$

3. Now recall rule that

$$a = \int_C dz f(z)$$

$$\Rightarrow \bar{a} = \int_{\bar{C}} d\bar{z} \bar{f}(\bar{z})$$

$$\text{Thus: } S_2(z) = -\frac{1}{2i} \int_{-\frac{1}{2}+i\epsilon}^{\infty+i\epsilon} dn \left[\frac{z^n}{\tan \pi n} \right] = -\frac{1}{2i} \int_{-\frac{1}{2}+i\epsilon}^{\infty+i\epsilon} dn \frac{\bar{z}}{\tan \pi n}$$

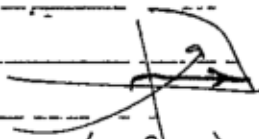
$$\text{Thus, } S_2(\bar{z}) = S_1(z) \Rightarrow \boxed{S_2(z) = \overline{S_1(\bar{z})}} \checkmark$$

Thus, we need only compute $S_1(z)$, since we can then get S_2 from this little formula.

4. Calculation of S_1 :

$$S_1(z) = -\frac{1}{2i} \int_{-\frac{1}{2}+i\epsilon}^{\infty+i\epsilon} dm \, z^m \cot \pi m \quad \checkmark$$

$$z^n = |z|^n e^{i \frac{2\pi}{n} n}$$

Choose great circle  where $|z| > 0$ i.e. $0 < \arg z < \pi$.
Then $z^n \sim e^{-i \frac{2\pi}{n} n} = e^{-|z| \sin \frac{2\pi}{n} n} = e^{-|z| n \sin \frac{2\pi}{n}}$

So this exp decays, $|\cot \pi n| \rightarrow 1$ does nothing.

Therefore, when $\arg(z)$ is positive, we are justified in rotating the contour to the vertical direction. Thus:

$$S_1 = -\frac{1}{2i} \int_{-\frac{1}{2}}^{-\frac{1}{2}+i\infty} dm \, z^m \cot \pi m \quad \checkmark$$

5. Now change variables to $m = n + \frac{1}{2}$, then

$$S_1 = -\frac{1}{2i} \int_0^{+i\infty} dm \, z^m \, z^{-\frac{1}{2}} \underbrace{\cot \pi (m - \frac{1}{2})}_{= \cot(90^\circ - \pi m)} \quad \checkmark$$

$$= -\frac{1}{2i} \int_0^{+i\infty} dm \, z^m \tan \pi m \quad \checkmark$$

$$\text{So } S_1 = +\frac{1}{2i} z^{-\frac{1}{2}} \int_0^{+i\infty} dm \, z^m \tan \pi m \quad \checkmark$$

6. Next, change to $m = ix$ $\tan \pi(ix) = i \tanh(\pi x) \quad \checkmark$

$$\text{Get: } \frac{1}{2i} z^{-\frac{1}{2}} \int_0^\infty dx \, z^{ix} i \tanh(\pi x) \quad \checkmark$$

$$= \frac{1}{\sqrt{2}} \frac{i}{2} \int_0^\infty dx \, \tanh(\pi x) e^{ix [\ln z]}$$

① Now change to $y = \pi x$

$$= \frac{1}{\sqrt{z}} \frac{i}{2\pi} \int_0^\infty dy \ln(y) \in i \frac{1}{2} [\ln z] \checkmark$$

$$= \frac{1}{2\pi} \frac{1}{\sqrt{z}} \int_0^\infty ds \ln(s) \in \left[\frac{-i \ln z}{\pi} \right] \checkmark$$

$$= \frac{i}{2\pi} \frac{1}{\sqrt{z}} \left[\beta \left[\frac{-i \ln z}{2\pi} \right] - \left[\frac{-\pi}{-i \ln z} \right] \right] \checkmark$$

GR p 356.

where $\beta =$ incomplete beta fun.

$\beta(x)$ is RA, so $S_1(z)$ is not RA.

② Thus we have found that:

$$S_1(z) = \frac{1}{\sqrt{z}} \frac{i}{2\pi} \beta \left(-\frac{i}{2\pi} \ln z \right) + \frac{1}{2} \frac{1}{\ln z} \frac{1}{\sqrt{z}}$$

③ Then

$$S_2(\bar{z}) = -\frac{i}{2\pi} \beta \left(+\frac{i}{2\pi} \ln \bar{z} \right) + \frac{1}{2} \frac{1}{\ln \bar{z}} \frac{1}{\sqrt{\bar{z}}}$$

10. Now, an important point: the beta function is like the gamma function in that it tends to be simple and real if its argument is real and positive. If the real part of its argument is positive, then beta has a simple integral representation, just like the gamma. However, as expected, notice that our S_1 will be "simple" in this sense only if $\arg(z)$ is positive, whereas S_2 is simple if $\arg(z)$ is negative. However, the beta function has analytic continuation to all argument, so actually both these functions are defined for all z , except where the beta may have singularities.

11. Add the pieces together and what do you get:

$$\sqrt{z} S = \frac{i}{2\pi} [\beta(x) - \beta(-x)] + \frac{1}{\ln z}$$

$$x = -\frac{i}{2\pi} \ln z$$

$$= \frac{1}{2} i \operatorname{csch} \pi x + \frac{i}{2\pi} x + \frac{1}{\ln z}$$

$$\rightarrow = \frac{1}{2} i [-i \operatorname{csch} (-\frac{1}{2} \ln z)] = -\operatorname{csch} [\ln \sqrt{z}]$$

$$= \frac{1}{2} \frac{-1}{\sinh [\ln \sqrt{z}]}$$

12. What is $\text{sh}[\ln \sqrt{z}]$? Let $x = \text{sh}[\ln \sqrt{z}]$. Then

$$\sqrt{z} = x + \sqrt{x^2 + 1}$$

$$z = x^2 + x^2 + 1 + 2x\sqrt{x^2 + 1}$$

$$\left(\frac{z - 2x^2 - 1}{2x} \right)^2 = x^2 + 1$$

$$z^2 - 2z(2x^2 + 1) + (2x^2 + 1)^2 = (2x)^2(x^2 + 1) = 4x^2(x^2 + 1)$$

$$z^2 - 2z(2x^2 + 1) + 4x^4 + 4x^2 + 1 = 4x^4 + 4x^2$$

$$z^2 - 2z - 4zx^2 = -1$$

$$z^2 - 2z + 1 = 4zx^2$$

$$\frac{(z-1)^2}{4z} = x^2$$

$$x = \frac{z-1}{2\sqrt{z}}$$

So, ~~sh~~

$$\boxed{\text{sh}[\ln \sqrt{z}] = \frac{z-1}{2\sqrt{z}}}$$

13. So far,

$$\sqrt{z} S = \frac{\sqrt{z}}{1-z} + \frac{1}{z} \left[\frac{z-1}{2\sqrt{z}} \right] + \frac{1}{\ln z}$$

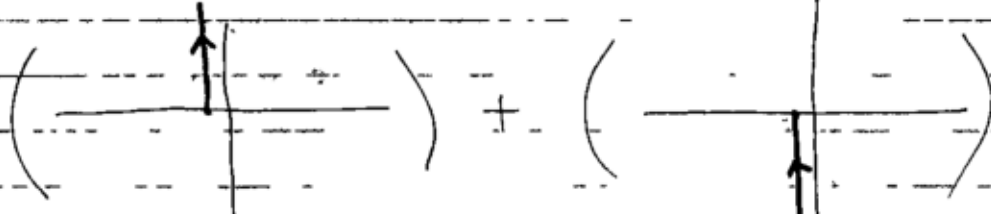
$$= \frac{\sqrt{z}}{1-z} + \frac{1}{2\sqrt{z}} - \frac{1}{2\sqrt{z}} + \frac{1}{\ln z}$$

oops.

$$\Rightarrow S = \frac{1}{1-z}$$

$$\Rightarrow \boxed{S = \frac{1}{1-z}} \leftarrow \text{finally}$$

14. So after many errors which I finally corrected, we get out the desired answer that we knew all along. Needless to say, this decomposition of the simple ~~piece~~ function in question here into two messy pieces is not too enlightening:

$$\frac{1}{1-z} = \left[\frac{1}{2\sqrt{z}} \left\{ \frac{i}{\pi} \beta\left(-\frac{i}{2\pi} \ln z\right) + \frac{1}{\ln z} \right\} \right] + \left[\frac{1}{2\sqrt{z}} \left\{ \frac{-i}{\pi} \beta\left(\frac{i}{2\pi} \ln z\right) + \frac{1}{\ln z} \right\} \right]$$


In fact, the decomposition is amazingly complicated involving a special function which is so special it doesn't even have a name!

But, the method is what is important. Suppose you have a convergent sum which you do not know the sum of. Then this method is the only alternative that I know of!

15. Here is the interpretation of what I have done here. Consider just the "upper" contour. This upper contour actually generates one half the series sum S ? Seems that way since it picks up half the residue at each pole. Is this really true? No, because this leaves out the rest of the contour which covers along the real axis between the poles. This integral is highly "improper" because it cuts through lots of poles. You might think that such an integral would be real, being a real contour integral of a real function, but this is not the case. Consider:

$$\int_{-1}^1 \frac{dx}{x} = \ln x \Big|_{-1}^1 = \ln(-1) = i\pi$$

16. So we continue with the interpretation. Consider just the upper contour above the real axis. Does this upper contour integral converge for all z ? That is, for z inside the unit circle (where the sum converges). I think the answer is yes. The upper contour is made of two pieces. The "poles" piece is just half the original sum which we know converges. The "Cauchy" piece is the sum of lots of little segments as we have said. A typical segment is:

$$\int_{N-\frac{1}{2}}^{N+\frac{1}{2}} dm \frac{z^m}{\tan \pi m} = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} dm \frac{z^m}{\tan \pi m} \right) (-1)^N z^N = (f(z)) \cdot z^N$$

So yes, this Cauchy piece also converges because z^N is sitting there. In fact, you know that the upper contour converges because as you go off to the right a little above the real axis, the $\tan$ factor simply oscillates at finite values, and the z^N does expo decay.

Thus, both contours separately converge for z in the unit circle, which is to say, for $\ln|z|$ smaller than 0, ie, negative.

17. So each contour we know separately converges. We know how to get one from the other, so the whole problem is reduced to evaluating the upper contour. When $\arg(z)$ is positive, we know that GC vanishes so we can replace the upper contour with a semivertical contour and we can evaluate that to get result. When $\arg(z)$ is negative, then semivertical contour diverges and GC diverges, but we know that the upper contour itself is still converging. So, when z is in UHP, we identify the upper contour with a certain function $\beta(z)$, say. When z goes to lower HP, we don't know what the upper contour is, but we know what $\beta(z)$ is. Since the upper contour converges for all z , and since contour = $\beta(z)$ in the UHP, we must also have contour $= \beta(z)$ in the LHP. If two analytic functions agree in the upper half plane, then they have to have the same analytic continuation!

18. So once more, let's review the general method.

- a) you want to evaluate an infinite sum. Can't do it directly. So,
- b) you SWT the thing and break into two contours. Where the sum converges, each of these contours going off to the right will also converge.
- c) you relate the lower contour to the upper, so only have to do the upper.
- d) you find a region of z such that the upper quarter GC vanishes.
- e) then you rotate the upper contour and replace with a known, convergent real integral. Look up the integral and you have the upper(z) function.
- f) get lower(z), add them, and you are done!

19. Explicit statement of this technique.

Let $f(z) = \sum_{m=0}^{\infty} h(m, z)$, convergent for $z \in \mathcal{R}$.

Assume: (a) $h(m, z) = \overline{h(\bar{m}, \bar{z})}$, i.e., h is analytic in both variables, and h is real when m, z are real.

(b) $h(m, z)$ is not singular when $m = -\frac{1}{z} + is$ for $s > 0$. We don't want to run over any poles or branch points!

Then: $f(z) = -\frac{1}{2i} \int \frac{dn h(n, \bar{z})}{\tan \pi n} = f_1(z) + f_2(z)$.

But: $f_2(z) = \overline{f_1(\bar{z})}$ due to assumption a. So need only compute $f_1(z)$.

(c) assume that upper quarter great circle vanishes for $z \in \mathcal{R}_1 \subset \mathcal{R}$, where $\mathcal{R}_1$ has some measure.

Then, for $z \in \mathcal{R}_1$, Wick rotate to get

$$f_1(z) = \frac{i}{2\pi} \int_0^{\infty} ds h(s) h(i\frac{s}{\pi} - \frac{1}{z}, z).$$

Look this up, then answer is

$$f(z) = f_1(z) + \overline{f_1(\bar{z})}.$$

20. Application:

$$W_k(z) = z^k \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(2k+n)} z^n$$

$$\text{so } f(z) = z^k W_k(z) = \sum_{n=0}^{\infty} h(n, z)$$

$$\text{where } h(n, z) = \frac{\Gamma(n+1)}{\Gamma(2k+n)} z^n$$

Check condition: (a) as $n \rightarrow \infty$, $h(n, z) \rightarrow (n)^{1-2k} (z)^n$

$$\text{Now } z^n = |z|^n e^{i n \arg z} = |z|^n e^{-i n \arg z}$$

When $|z| < 1$, get expo decay $e^{-\text{Re} \cdot \ln(1/|z|)}$, so good convergence $|z| < 1$. When $|z| = 1$, also converges when $k = 1, 2, \dots$ Also real analytic ✓.

(b) $n = -\frac{1}{2} + is$ causes no trouble.

$$(c) z \in \text{UHP} \quad z^n = e^{n \ln z} = e^{-\text{Re} \ln |z|} e^{-i \text{Im} n \arg z}$$

$\Rightarrow \arg z > 0 \Rightarrow$ convergence so GC = 0 ✓.

$$h(i\frac{s}{\pi} - \frac{1}{2}, z) = \frac{\Gamma(i\frac{s}{\pi} + \frac{1}{2})}{\Gamma(2k + i\frac{s}{\pi} - \frac{1}{2})} z^{-1/2} e^{s[\frac{i}{\pi} \ln z]}$$

$$\text{So: } f_+(z) = \frac{i}{2\pi} z^{-1/2} \int_0^{\infty} ds \, h(s) \frac{\Gamma(i\frac{s}{\pi} + \frac{1}{2})}{\Gamma(2k + i\frac{s}{\pi} - \frac{1}{2})} e^{-s[\frac{i}{\pi} \ln z]}$$

But where do you look this up?! Flip lower Γ to get

$$(-1)^{2k+1} \frac{i}{2\pi} z^{-1/2} \int_0^{\infty} ds \, sh(s) \Gamma(i\frac{s}{\pi} + \frac{1}{2}) \Gamma(-i\frac{s}{\pi} - 2k + \frac{3}{2})$$

Flip the other way:

$$f(z) = \frac{i}{2\pi} z^{-1/2} \cdot \pi \cdot \int_0^\infty ds \frac{\text{sh}(s)}{\cosh^2(s)} \cdot \frac{1}{\Gamma(\frac{1}{2} - i\frac{s}{\pi}) \Gamma(2k - \frac{1}{2} + i\frac{s}{\pi})}$$

Q1. Try other way use Euler:

$$f_1(z) = \frac{i}{2\pi^2} z^{-1/2} \int_0^\infty ds \text{sh}(s) \Gamma(i\frac{s}{\pi} + \frac{1}{2}) \Gamma(-i\frac{s}{\pi} - 2k + 3/2)$$

Convergence?

$$\Gamma(z) = e^{-z} z^{-z} \Gamma(z)$$

$$\text{so } e^{-(i\frac{s}{\pi} + \frac{1}{2})} (i\frac{s}{\pi} + \frac{1}{2}) \ln(i\frac{s}{\pi} + \frac{1}{2}) e^{-(\frac{1}{2} - 2k + 3/2)} (-\frac{1}{2} - 2k + 3/2) \ln(-\frac{1}{2} - 2k + 3/2)$$

$$\Gamma(x+a) \Gamma(-x+b) \rightarrow \left(\sqrt{2\pi} e^{-x} x^{x+a-1/2} \right) \left(\sqrt{2\pi} e^{-(-x)} (-x)^{-x+b-1/2} \right)$$

$$= 2\pi e^{-x} x^{x+a-1/2} (-1)^{-x} (-1)^{-x+b-1/2} x^{-x} x^{x-b-1/2}$$

$$= 2\pi x^{-a-b+1} (-1)^{-2x} = 2\pi e^{(a+b-1)\ln x + i\pi x}$$

$$\Gamma(x+a) \Gamma(-x-b+1) \rightarrow 2\pi (x)^{a-b} e^{-\pi i \ln(x+b)}$$

$$= (x)^{a-b} e^{-\pi i \ln x + \pi i \ln b}$$

$$\text{So } a = \frac{1}{2} \quad -b+1 = -2k + 3/2 \Rightarrow -b = -2k + \frac{1}{2} \quad b = 2k - \frac{1}{2}$$

$$a-b = \frac{1}{2} - 2k + \frac{1}{2} = 1-2k \quad x = i^{3/\pi}$$

$$\text{So } (i\frac{s}{\pi})^{1-2k} e^{-\pi i \ln i\frac{s}{\pi}} = s^{1-2k}$$

Now $\text{sh}(s) \sim e^s$, so overall is

$$s^{1-2k}$$

Comment: in this example, I have successfully replaced the undoable sum with an integral. Unfortunately, this is an integral which is not very easy to do. It is doable if I use Euler expansions for the two gamma functions. But this is a lot of work.

It suddenly occurs to me that there is another possible way which is an induction proof. This integral business should only be used as a last resort.

Series: how do you bound a tail $\sum_{n=N}^{\infty} (n)^{\alpha} z^n$?

① First:

$$\text{Tail } N \leq \sum_{n=N}^{\infty} (n)^{\alpha} |z|^n \quad \alpha = \text{Re}(a).$$

② Converges if $|z| \leq \sigma < 1$, because $L \rightarrow |z|$.

③ Consider quantity $(n^{\frac{\alpha}{n}})$. As $n \rightarrow \infty$, this $\rightarrow 1$. Let

$n^{\frac{\alpha}{n}} = 1 + \epsilon$. For any $\epsilon > 0$, we can find an $M \geq n > M \Rightarrow n^{\frac{\alpha}{n}} < 1 + \epsilon$. Let's begin our tail at such a point M , call it N .

④ Now we'd like to have some real $x < 1$ such that:

$$(n)^{\alpha} |z|^n < (x)^n \quad \text{i.e.} \quad n^{\frac{\alpha}{n}} |z| < x$$

Clearly, for fixed $|z| < 1$, we can take n large enough so that $n^{\frac{\alpha}{n}} |z| = x < 1$. Call this N !

So given $|z|, \alpha$, choose $N \geq$:

$$N^{\frac{\alpha}{N}} |z| = x < 1.$$

$$\text{Then: Tail } N \leq \sum_{n=N}^{\infty} (x)^n = \frac{x^N}{1-x}.$$

$$\downarrow n^{\frac{\alpha}{n}} |z| < x, n > N$$

⑤ State result cleanly:

Given a complex # a , and $z \Rightarrow |z| < 1$ [or $|z| \leq \sigma < 1$], it is possible to find an N large enough so that $x \equiv |z| (N^{\frac{\alpha}{N}})^{\alpha} < 1$ for $n \geq N$ [$\alpha = \text{Re } a$]. Therefore, we can say, for such an N ,

$$\sum_{n=N}^{\infty} (n)^{\alpha} z^n < \frac{(x)^N}{1-x}$$

$$\text{where } x = |z| (N^{\frac{\alpha}{N}})^{\alpha} < 1.$$

Asymptotic Series.

- References: Smith "Math Methods for S and E"
Whittaker and Watson "best book"
Jeffries big book.

2. Here is the standard example:

$$f(x) = \int_x^{\infty} dt \, t^{-1} e^{-xt} \, dt$$

$$= \underbrace{\bar{x}^1 - \bar{x}^2 + 2! \bar{x}^3 - 3! \bar{x}^4 + \dots}_{S_n(x)} + (-1)^n (n-1)! \bar{x}^n + (-1)^n n! \int_x^{\infty} e^{-xt} dt / t^{n+1}$$

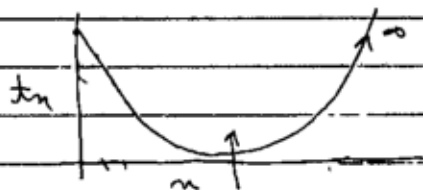
$$\text{Ratio} = \frac{\bar{x}^{n-1} (n!)}{\bar{x}^n (n-1)!} = \frac{n}{\bar{x}} \rightarrow \infty \text{ for any fixed } x \rightarrow \text{divergent series}$$

But

$$|f(x) - S_n(x)| = \left| n! \int_x^{\infty} e^{-xt} dt / t^{n+1} \right| \leq n! \int_x^{\infty} \frac{1}{t^{n+1}} dt = \frac{n!}{n} x^{-n} = \frac{(n-1)!}{x^n}$$

but

Thus, the error is less in magnitude than the last term included in $S_n(x)$. Clearly, this is the catch. The series diverges, but if you take 105 terms, the error is less than the 105th term. The idea is that you never really consider the entire infinite series which actually doesn't even exist (converges nowhere). If you can show an error condition like this, and if the first few terms temporarily converge, then you can make a good approximation with these terms. How small can you make the error for some given value of x ? Look where series has "minumum".



$$|t_n| = P(n) \bar{x}^n$$

$$\frac{\partial |t_n|}{\partial n} = 0 = P(n) (-n) \bar{x}^{n-1} + \bar{x}^n P'(n)$$

$$\Rightarrow P(n) n = P'(n) x$$

$$\Rightarrow n \approx \psi(n) \cdot x$$

$$P(n) \approx e^{(n-\frac{1}{2}) \ln n} \approx n^{n-\frac{1}{2}} \Rightarrow P'(n) \approx \ln n \cdot n^{n-\frac{1}{2}} \Rightarrow \psi(n) \approx \ln n$$

so $n \approx \ln x$ as $n \approx x$. So (from best) $\approx P(x) \bar{x}^x$ well, forget it.

Anyway, you can compute the turn-around point to get best ideal error. In practice you get within desired error in say three terms.

Infinite Series as Analytic Functions:

12

1. Suppose you have an infinite series of analytic functions $f_n(z)$ which you add up to get some total function $f(z)$. Suppose the series converges uniformly over some region in the z -plane. Then if the individual terms are analytic in that region, then the function which is the sum must be analytic in that region.
2. Example: Consider the series $1 + z + z^2 + \dots = 1/(1-z)$. This series converges uniformly over the inside of the unit disc. Thus, in the interior of this ~~the~~ disc, the sum simply can not have any singularities. No branch points, no poles. The reason is that the individual terms are analytic in the entire z -plane! Of course you cannot count the branch point at $z=1$ since this is not in the disc.
3. Application: you are not going to get a series of regge poles terms which produce the normal threshold cuts. Reason is: if you have such a series, then there can be no branch points in the convergence region.

Simple Sum.

① From GR p1 we have:

$$\left\{ \begin{array}{l} \sum_{j=1}^b (1) = b \\ \sum_{j=1}^b (j) = \frac{b(b+1)}{2} \\ \sum_{j=1}^b (j)^2 = \frac{b(b+1)(2b+1)}{6} \\ \sum_{j=1}^b (j)^3 = \frac{b^2(b+1)^2}{4} \\ \text{etc.} \end{array} \right.$$

② Consider this frequent sum.

$$S = \sum_{j=j_1-j_2}^{j_1+j_2} \underbrace{(Aj^3 + Bj^2 + Cj + D)}_{f(j)}$$

Let $j-j_1 = q$ so get

$$S = \sum_{q=-j_2}^{j_2} \underbrace{(aq^3 + bq^2 + cq + d)}_{f(q+j_1)}$$

But now drop odd terms:

$$S = \sum_{q=-j_2}^{j_2} (bq^2 + d) = b \left[\sum_{q=-j_2}^{j_2} (q^2) \right] + d(2j_2+1)$$

But. $\sum_{q=-j_2}^{j_2} (q^2) = \frac{1}{3} j_2(j_2+1)(2j_2+1)$ on either type lattice (d and d)

So:

$$S = (2j_2+1) \left[\frac{b}{3} j_2(j_2+1) + d \right]$$

$$b = [3j_1 A + B]$$

$$d = [Aj_1^3 + Bj_1^2 + Cj_1 + D]$$

is solution to

$$S = \sum_{j=j_1-j_2}^{j_1+j_2} [Aj^3 + Bj^2 + Cj + D]$$

Example: $\sum (2j+1) \Rightarrow A=B=0, C=2, D=1 \Rightarrow b=0, d=(2j_1+1) \Rightarrow S = (2j_2+1)(2j_1+1) \checkmark$

ODE's

1. Consider:

$$\frac{d^2u}{dx^2} = f(u)$$

Here is a possible way to solve the thing:

$$\text{Let } y = \frac{du}{dx} \text{ so } \frac{dy}{dx} = f(u) = \frac{dy}{du} \cdot \frac{du}{dx} = y \frac{dy}{du} \Rightarrow y \frac{dy}{du} = f(u).$$

$$\Rightarrow y dy = f(u) du \Rightarrow \frac{1}{2} y^2 = \underbrace{\int f(u) du}_{\equiv F(u)} + \underbrace{\text{const.}}_{\equiv A/2} = F(u) + A/2$$

$$\Rightarrow y = \pm \sqrt{2F(u) + A} \quad \text{but} \quad = \frac{du}{dx} \text{ so } \frac{du}{\pm \sqrt{2F(u) + A}} = dx$$

$$\Rightarrow x = \int \frac{du}{\pm \sqrt{2F(u) + A}} + B = x(u; A, B).$$

Then you would invert this result to get $u = u(x; A, B)$. Thus A and B are the two undetermined constants which must be controlled by the boundary conditions which accompany the original equation.

Comments: Note that you cannot have any first derivative present in the original equation or this method fails. Note also that you have to be able to do two indefinite integrals. If these cannot be done analytically, you cannot get a closed form solution at least by this method. Numerically of course you are guaranteed a solution.

Example: $f(u) = -\omega^2 u \Rightarrow F(u) = -\frac{1}{2} \omega^2 u^2$

$$\Rightarrow x = \int \frac{du}{\pm \sqrt{A - \omega^2 u^2}} + B = \frac{1}{\omega} \int \frac{dx}{\pm \sqrt{A/\omega^2 - x^2}} + B = \pm \frac{1}{\omega} \left[\sin^{-1} \left(\frac{u}{A'} \right) \right] + B$$

$$\Rightarrow \pm (x+B') \omega = \sin^{-1}(u/A') \Rightarrow u(x) = A' \sin(\pm \omega x + \phi) \quad \checkmark$$

This was a linear example. Now for a non-linear example:

Example 2: $f(u) = K e^{au} \Rightarrow F(u) = \frac{K}{a} e^{au} \Rightarrow \frac{du}{dx} = \sqrt{\frac{2K}{a} e^{au} + A'}$

and so on.

some standard ODE's

① elliptic integrals K, E, K', E'

$$\frac{d}{dx} \left[x(1-x^2) \frac{du}{dx} \right] - xu = 0$$

K, E, K'

$$(1-x^2) \frac{d}{dx} \left(x \frac{du}{dx} \right) + xu = 0$$

E, E', K'

② Bessel $\ddot{u} + \frac{1}{x} u' + \left(1 - \frac{\nu^2}{x^2}\right) u = 0$

plus, see table of ODE's, p 971. GR.

③ Legendre: $(1-x^2) u'' - 2xu' + \left[\nu(\nu+1) - \frac{\mu^2}{1-x^2}\right] u = 0$

④ Gegenbauer:

$$u'' + \frac{(2\lambda+1)x}{(x^2-1)} u' - \frac{n(2\lambda+n)}{(x^2-1)} u = 0. \quad C_n^\lambda(x)$$

$$(x^2-1)u'' + (2\lambda+1)xu' - n(2\lambda+n)u = 0$$

⑤ Chebyshev

$$(1-x^2)u'' - xu' + n^2 u = 0$$

$$(x^2-1)u'' + xu' - n^2 u = 0$$

⑥ Hermite

$$u'' - 2xu' + 2nu = 0$$

⑦ Laguerre

$$xu'' + (\alpha - x + 1)u' + nu = 0$$

⑧ Hypergeometric: $x(1-x)u'' + [c - (a+b+1)x]u' - abu = 0$

⑨ Whittaker

$$u'' + \left(-\frac{1}{x} + \frac{\lambda}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right)u = 0$$

On the Limit-Point and Limit-Circle Business: Weyl's Theorem.

1. I am reading this in Stakgold page 297. $Lu = \lambda u$ $L = S-L$.
2. Part I of the theorem is uncomplicated. The claim is simply this: if at any particular eigenvalue λ , both solutions to the ODE are of finite s-norm on the interval, then at any other eigenvalue, the same is true. The proof, given on page 298, is just an exercise in inequalities.
3. Part II of the official theorem says this: given any λ not on the real axis, there must be at least one s-norm solution. I.e., either there is one solution, or there are two solutions of s-norm. If there are two solutions, then by part I there are two finite s-norm solutions at all λ . Thus, the actual system itself is characterized by this dichotomy:
4. The dichotomy: one of these situations must prevail for the standard Sturm-Liouville Hermitian (self-adjoint) system with a singular point taken as one of the interval endpoints:
 - a) for any λ , both solutions are of finite s-norm (circle) (point)
 - b) for any λ , there exists only one (or maybe no) solutions of f s-norm.

That is, if for one λ you can find two finite s-norm sols, then you are in category a) according to part I. On the other hand, suppose for some λ there is only one solution. Then at no other λ can there be two solutions because then that would have to be also true at λ . Thus, if you find a λ where there is one solution, you know that at every other λ there is either 1 or NO solutions.

5. I think this should be presented as a series of corrolaries. Well, we know that if you are in situation b), and for λ off the real axis, then you know there is precisely one s-norm solution. If you then take λ onto the real axis, then all you can say is: either there is one fns, or none at all. I think there will be one fns if $\lambda =$ isolated eigenvalue.
6. The way you test a singular SL system to see which case you have is simple: pick a trivial value of λ and solve for the two solutions. Then check their s-norms. If they are both finite s-norm, you are a). If one of them fails to be s-norm (or if both fail), then you are b).
7. Wronskian facts: (time out) From Abels formula page 60, we know that the Wronskian of two solutions of $Lu=0$ (~~XXXXXX~~) $= C \exp(-m(x))$. Therefore, if $W(x_1) \neq 0$, then $W(x) \neq 0$ for all x in interval. Conversely, if $W(x_0) = 0$ at any one point, then $W(x) = 0$ throughout interval. In this vanishing case, the two solutions must be dependent. In the nonvanishing case, the solutions are independent. However, since $m(x)$ is determined by $dm/dx = a_1/a_0$, we see that the wronskian can depend on x , in this second case.
8. Another fact is this: for S-L systems, $a_0' = a_1$. In this case, $m(x) = \ln a_0(x)$. Thus, the wronskian of any two solutions is $W = C/a_0(x)$. Later, $a_0(x)$ is called $-p(x)$. Still, W is x -dependent in general.
 But, $p(x)W(u,v;x) = C = \text{constant!}$
9. The bilinear conjunct $J(u,v)$ reduces to $-p(x) W(u,v;x)$. Thus, it is this wronskian that you want to vanish at your endpoints to get a truly self-adjoint system. Of course, if $p(b) = 0$, you have a slight problem. Or if endpoint is at infinity.

Still more data is needed before we can do the circle-point stuff.

10. Much use is made of a special-purpose "unmixed boundary condition" at the endpoint a . It is:

$$B_a u = q_1 u(a) + q_2 u'(a)$$

If one takes a boundary condition of this form at each end, then easy to show that the wronskian of a ~~solution~~ u and the CC of another solution v , namely $\bar{v}$, will vanish at the endpoints. But this is the bilinear conjunct in this case, so the point is this: those special unmixed boundary conditions give you full self-adjointness. at regular points

We have said that $W(u, \bar{v}) = 0$ at both ends. We need not assume that u and v satisfy $Lu=0$. These u and v are just two functions on the interval. The operator L is Hermitian if BC's are satisfied.

11. By the way, throughout all this, p and q are supposed to be real, and the weight s is positive real.

But $W(u, v) \neq 0$ does imply u, v are independent.

12. Note that $W(u, v) = 0$ does not imply that u and v are dependent! That is only true if u and v are solutions of $Lu=0$. In what follows, we usually speak of solutions of $Lu = s\lambda u$ which is a different equation. I want to throw the idea that if wronskian vanishes at the ends then it vanishes throughout and u and v must be dependent. Not generally true! The interesting object is always $p(x) W(u, v; x)$, not W itself.

13. I wish now to point out equation (4.80a). It says: if λ lies off the real axis, then the wronskian objects $p(a) W(u, v; a)$ cannot be zero at both ends at once. Here, u solves $Lu = s\lambda u$. However, these wronskian-like objects do have clean limits even if endpoints are singular.

14. We now start part II of Weyl's Theorem. Consider $Lu = \lambda su$. Let Q and V be two independent solutions chosen to match the very special boundary conditions at regular endpoint a , see p 299. You can certainly find such a Q and V because only one BC on each. Clearly $u = Q + mV$ is also a solution of equation.

Now, the big problem is : how do you put a boundary condition at b which is a singular point? You cannot simply "do it" because $p(b) = 0$ maybe, which goofy things up, or maybe solution u diverges at inf. The trick is this: rewrite the BC in its wronskian form, then take the limit b_0 goes to b . You can take limit of the wronskian object.

Since both Q and V satisfy the usual wronskian=0 BC at the left end a , we see from 4.80a that in general neither Q nor V could possibly satisfy our desired condition at end b . This was noted in 13 above that both wronskians cannot vanish for the same solution.

Thus, any candidate which seeks to satisfy our new condition at b must truly be a linear combination, ie, $m \neq 0$. (Except if $\lambda = \text{real}$).

Now all we are doing is taking over the old BC to this singular case. The question now is: what choices for m allow u to satisfy this right-end boundary condition?

$$\lim_{b_0 \rightarrow b} p(b_0) W(u, \bar{v}; b_0) = 0$$

15. We continue. To find the allowable values of m , we insert $u = Q + m\psi$ into the boundary condition at b_0 before taking limit. We at once find that:
 - a) if λ off real axis, m lies on a certain circle
 - b) if λ on real axis, this circle degenerates to real m axis.
16. Notice that $\lambda = \text{real}$ is very special because we have been assuming that all other functions like p and q and s are real. Thus, λ real means that solutions are real and this means that wronskians like $W(\psi, \bar{\psi}; x) = 0$. I.e., wronskian of function with itself. Also, $\lambda = \text{real} \Rightarrow$ quantization
17. The expression for the radius when λ not real is given in 4.84. This shows clearly that as you take b_0 to the limit b , the radius is monotonically decreasing! Thus, in the limit, you get either a circle of viable values for m , or you get exactly one allowed point value for m , in the case that the circle shrinks to a point.
18. What is the big advantage of having a solution solve this fancy boundary condition? The reason is shown in bottom of page 300. If $\lambda \neq \text{real}$, then this magic solution (or class of solutions) is of finite s-norm. This is the big payoff.
19. But there is even more payoff: if you get a circle in the limit, then from 4.84 you see that not only is $u = Q + m\psi$ of finite s-norm, but ψ itself has finite s-norm. Thus, if you get the circle case by applying the magic boundary condition, then you have found two finite s-norm solutions. But by part I of this theorem, you must be in situation a) of 4. So we identify: "circle case" $\equiv$ "case a)"
20. On the other hand, if you get a point, then $r=0$ and 4.84 says that ψ must not have finite s-norm. In this case (point), there is precisely one finite s-norm solution. But this paragraph is only if λ not real.
21. If $\lambda = \text{real}$, we do know that any m on real axis allows $u = Q + m\psi$ to satisfy the fancy boundary condition. In fact, any solution u will satisfy this BC. In this case, we get no information on whether any solutions are of finite s-norm. Maybe none, maybe one, maybe two.

Limit-Point Analysis.

1. You have a singular point b , but you have determined that you are in the limit point case. What do you do next?

First, you want to construct a greens function. To do this, you choose χ on the left because this satisfies your left side BC, and you choose $u = Q + m\chi$ on the right, say for b_0 . The specific greens is then given in 4.93. You find that this greens has only poles, hence usual discrete spectrum since we have not yet gone singular. Now, as you take the limit b_0 goes to b , you know that m goes to its limit. It may happen that as you do this, the poles of our meromorphic $p(\lambda)$ may move together to form a cut. I.e., in the limit, $m_p(\lambda)$ may have poles and cuts. It could have only poles or only cuts.

2. We don't really have to think of this limiting procedure however. There is a well defined system for the desired greens function. On the left side you use some solution which satisfies your regular left end BC. On the right side, you use the single finite s -norm solution which you know must exist. Thus, 4.94a specifies g completely.

3. Once you have the greens g , you contour integrate it to find your completeness statement. It will have the form 4.95.

4. Comment on this completeness relation. In the regular S-L theory, all your solutions are of finite s -norm. You impose two unmixed boundary conditions at regular points a and b . This causes λ to "quantize". I.e., only a certain denumerable set of λ will give solutions to the system. I.e., spectrum is discrete, eigenfunctions are orthogonal etc etc.

Compare this to the singular case. In the limit-circle case, there are finite s -norm solutions for all λ . ~~I.e., there is no quantization of eigenvalues.~~

In the limit point case, there is exactly one finite s -norm solution for every λ off the real axis. ~~Again, you do not get quantization. In fact, you lose that old result that eigenvalues must be real. This is because the singular equation is not truly self adjoint.~~

Nevertheless, for $\lambda = \text{real}$, you might still get that quantization effect. I think you do. In fact, when $\lambda = \text{real}$, you get the mixed spectrum situation. Then the g contour picks up all the discrete eigenfunctions, and it picks up the continuum of possibly non finite s -norm eigenfunctions.

If spectrum is purely discrete, eigenfunctions will be orthogonal.

5. Examples given by Stakgold of limit point:

p305 Bessels functions on 0 to 1; pure discrete spectrum
p308 Mellin transform equation on 0 to ∞ : pure continuum
p284 Fourier sine transform (4.58a): pure continuum
other cosine and fourier integral transforms are done.
p294 exercise 24 gives only example with mixed spectrum.

Thus, when $\lambda \neq \text{real}$, you cannot get an eigenfunction u satisfying both bc's. When $\lambda = \text{real}$, you will get eigenfunctions if
a) point: if your particular $u(x; \lambda)$ happens also to solve $Ba u = 0$
b) circle: if one of your $u(x; \lambda)$ " " " " " "
Thus, spectrum is still quantized. $\downarrow$

Any solution u
on circle will
satisfy BC at b .
But, only when
 $\lambda = \text{real}$, can
this same sol'n
satisfy BC
at a . See
(4.80a) with $q = u$.

Comment on machine solution to singular eigenvalue problem.

1. For λ to be a true eigenvalue, your $u(x;\lambda)$ must satisfy both boundary conditions. In the singular case at b , we still have a boundary condition so conceptually the same as regular case.

Now, suppose you start with a function $f(x)$ which is tied down at the regular end a . Then you pick some λ and start integrating out toward b . If the λ you picked is an eigenvalue, then you know your solution will satisfy condition at b . But then you know it will be of finite s -norm. Thus, your solution should heel down toward the axis, ie, it must approach zero.

2. If your solution blows up, then you know it is not of finite s -norm and therefore it cannot be solving the condition at b . Therefore your chosen λ was not an eigenvalue.

Limit Circle Analysis

1. Taken from page 310 of Stakgold.
2. In the limit point case, we set up a defining system for the greens function by using the single finite-s-norm solution on the right. For the circle case, there is no unique such solution on the right: the solution corresponding to any point-on-the-circle will do. So, the idea is this: pick a nice λ_0 off the real axis and pick any particular solution "on the circle". Thus, you will have ψ on the left and u on the right. These are independent, and we may choose to normalize so that their product is the greens at λ_0 .
3. The scene suddenly shifts and we consider the inhom equation with some f tacked on: $L_{\lambda_0} v = sf$. How shall we apply BC's to this equation? The left end is usual unmixed BC. The right end is now going to get $p(b) W(v, u; b) = 0$ as its condition. Here, u is one of the circle solutions chosen above. Given these two boundary conditions, the inhom equation has only one solution: that shown in 4.115. This one solution, called v , turns out to be of finite s-norm.
4. Now as a special case of P3 above, choose $f = (\lambda - \lambda_0) sw$ where w is the solution to come. We have just computed the greens up in P1, so the solution given as integral over greens is as in 4.120. BUT this is an integral equation with Hilbert-Schmidt kernel. (ie, HS means bounded operator). If you choose λ_0 real, then kernel is symmetric. From previous theory, then, you know that the eigenvalue equation $Lw = \lambda sw$ has a pure discrete spectrum with orthogonal eigenfunctions.
5. To summarize, in the circle-limit situation, the eigenvalue equation always gives you a discrete spectrum of real eigenvalues with usual orthogonality and completeness. Greens function is easily defined and then constructed. The actual results (greens; system of orthogonal functions) depends parametrically on which point you choose on the circle.
6. Example: Bessel p 313. Here, the points on the limit circle are parametrized by real number A . Then 4.123 shows the A -dependent greens at $\lambda=0$. The Bessel equation is regular at 1, but singular at 0, so at 0 we impose the wronskian singular BC 4.125 (which again is A -dependent). The differential system is shown equivalent to a real symmetric hs integral system, so we know eigenfunction data. If you choose $A = 0$, your eigenfunctions are the $J_0(\sqrt{\lambda}x)$. Of course the boundary condition $w(1) = 0$ puts special λ_k inside. ie, you get quantization of eigenvalues on the real axis.
7. Other examples of limit-circle:
Hanekl transforms, Hermites, Legendres, a K-transform, etc.

Application of All this Theory to the Radial QM Equation.

1. As shown say on page 350 of Messiah,

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2mr^2} + V(r) \right] y_l(r) = E y_l(r)$$

$$P(r) = \frac{\hbar^2}{2m}$$

$$Q(r) = \frac{l(l+1)\hbar^2}{2mr^2} + V(r)$$

$$\|y_l(r)\|^2 = \int_0^\infty |y_l(r)|^2 dr \Rightarrow s(r) = 1.$$

The BC at $r=0$ is shown.

$$y_l(0) = 0.$$

2. Is this limit point, or is it limit circle? Cannot trivially solve this thing even with $E=0$. However, we can assume the potential $V(r)$ drops off fast so that at large r the equation is roughly:

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} \right] y_l(r) = E y_l(r).$$

$$E=0 \Rightarrow y_l(r) = A + br$$

To test for finite-s-norm, we only need the asymptotic behavior of the solutions anyway. At our test point $E=0$, we find that neither solution is finite s-norm. Thus, we are in limit-point case.

3. Aside: it had to be limit-point because that is the only case which allows a continuous segment of the spectrum.
4. Now, you could construct the E -greens by taking a $Y(0) = 0$ solution on the left, and the unique decaying solution on the right (for E not on the real axis. I.e., e^{-kr} .) Then you could contour integrate this greens to get the completeness relation and the spectrum. You would get the mixed result. The discrete eigenfunctions will be of finite s-norm. I have not proved this in the theory, but seems that if you consider E just above one of these real eigenvalues, you do know there is one finite s-norm solution. As you drop down onto the isolated eigenvalue, this remains true. Not so on the cut. In fact none of the eigenvalues on the cut will be of finite s-norm.

Discussion of the General Singular Problem.

1. In all this, we are always restricting ourselves to differential operators L which are of the standard formally self-adjoint form. But, as page 269 of MM shows, you can always cast your ODE in this form. Let x be the variable, and (a,b) the range.
2. Now, a point $x=a$ is a singular point if
 - 1) $c = \text{infinity}$, plus or mins, or
 - 2) $p(c) = 0$.In either case, you should arrange to have point c be one of your interval endpoints for your scalar product. Usually, we let $c = b$.
3. Weyl's Theorem says:

Look at your equation. Pick a simple value of eigenvalue λ for which you can trivially solve the equation. Find all solutions (two). Are these solutions of finite s -norm over interval of interest? If they both are, then you know that : for any value of λ , all solutions are of finite s -norm. This means that in regards to the singular point of interest, you are in the "limit-circle case."

Suppose one or both of your solutions are not of finite s -norm. Then you must be in the "limit-point Case". Then you know that for any λ off the real axis, there is only one solution of finite s -norm.
4. Quick example: consider page 302 Stogdole, Legendre equation, but choose instead the interval 1 to inf, so you have two singular points. For $\lambda = 0$ you know that $u = \text{constant}$ is a solution. But this solution is clearly not of finite s -norm on the interval to inf . Therefore, the point infinity is a limit-point situation. I'm not sure about the class of the point $z=1$, have to test the other solution.
5. Greens functions: assume $b = \text{singular point}$. Then we define the Greens function problem by $Lg - \lambda s g = \delta(x-y)$ with these very important boundary conditions:
 - 1) $B_a(g) = 0$. (Some simple condition at the regular end.
 - 2) the s -norm of g must be finite.You can imagine solving for this greens function. The contour-completeness relation is then valid, namely:
$$\frac{1}{2\pi i} \oint g(x; \lambda) d\lambda = -\frac{\delta(x-y)}{s(y)} = -\sum_n \overline{\phi_n(x)} \phi_n(y) + \int d\lambda \phi(x; \lambda) \overline{\phi(y; \lambda)}$$

[However, if equation has only limit-circle endpoints (singulaire), then the continuous spectrum is not there and you have only discrete spectrum and usual orthogonal functions of weight s .
6. Thus, we associate continuous spectra (or mixed) with limit-point situations.
7. Thus, the reason you have a "mixed spectrum" in the $O(2,1)$ UIRs is that the related differential system has a limit-point singularity at $z=\text{infinity}$. In $O(3)$, the differential system does not have this singular point, the other singular points are circle-limit cases, so you get a pure discrete spectrum with orthogonal eigenfunctions.

Review of Spectral Theory.

① Consider $L = -\frac{d}{dx} \left[p(x) \frac{d}{dx} \right] + q(x)$

This diff. op. is formally self-adjoint, i.e., $L = L^*$. Now, usually you also assume that $T(g, h)|_a^b = 0$ as in $\langle g | h \rangle = \text{usual integral}$. In this case,

a) $Lg = \delta(x-s) \Rightarrow g(x|s) = g(s|x)$

b) symmetric ~ hermitian ~ self-adjoint

② So, given a Hermitian operator L , consider $Lw(x) = \{\lambda s(x)\} w(x)$. ← weight

What do you know: there are of course some BC's with finite interval +

a) eigenvalues are real + denumerable (i.e., "spaced"). no sing. points.

b) eigenfunctions are orthogonal with weight $s(x)$.

c) green's function for $Lg - \lambda sg = \delta(x-s)$ is easily calculated. Let a, b be two ends. $f_a, f_b =$ two "side-solutions". Then

$$g(x|s; \lambda) = - \frac{f_a(x; \lambda) \cdot f_b(s; \lambda)}{C(\lambda)}$$

$$\text{where } C(\lambda) = p(x) W[f_a(x; \lambda), f_b(x; \lambda)]$$

d) You can show that f_a, f_b both analytic in λ . Thus, $C(\lambda)$ is analytic, but may of course have zeroes.

e) by considering $Lu - \lambda su = f$, you find

$$u = \int_a^b g(x|s; \lambda) f(s) ds \quad . \text{ of course.}$$

$$\Rightarrow u = \sum_n \frac{\langle f, \phi_n \rangle}{\lambda_n - \lambda} \phi_n(x)$$

Then set $f = \delta \Rightarrow u = g$. Thus:

$$g(x|s; \lambda) = \sum_n \frac{\overline{\phi_n(x)} \phi_n(s)}{\lambda_n - \lambda}$$

Also:

$$\frac{\delta(x-s)}{s(x)} = \sum_n \overline{\phi_n(x)} \phi_n(s).$$

f) last two lines combine to say:

$$-\frac{1}{2\pi i} \oint g(x|s; \lambda) d\lambda = + \sum_n \phi_n(x) \overline{\phi_n(s)} = + \frac{\delta(x-s)}{s(x)}.$$

Thus: large circle integral of green's function yields $\delta(x-s)$!

③ Now allow $b \rightarrow +\infty$ (to go to "singular problem").
Consider:

$$Lg - \lambda g = \delta(x-s) \quad \text{on } (0, \infty)$$

Consider solutions which are green's functions of this type

a) $g(0|s) = 0$ ie, the boundary condition at the a-end

b) $\int_0^\infty |g(x|s)|^2 dx < \infty$ the BV at the b-end.

Now, solve for $g(x|s; \lambda)$. You will find that this has cuts and poles in λ .

Claim: $-\frac{1}{2\pi i} \oint g(x|s; \lambda) d\lambda = \frac{\delta(x-s)}{s(x)}$ still!

Thus, you pick up cuts and poles in λ , ie, mixed spectrum.

$$= \underbrace{\sum_n \phi_n(x) \overline{\phi_n(s)}}_{\text{discrete}} + \underbrace{\int d\mu(\lambda) \phi_\lambda(x) \overline{\phi_\lambda(s)}}_{\text{continuous}}$$

Note that the $\phi_\lambda(x)$ functions will not themselves be normalizable so do not use them to expand a normalizable function..

④ Thus, we know how to write completeness $\delta(x-s)$ for a hermitian equation $Lg = \lambda g$, where $L = \text{hermitean}$.

really, $\phi_\lambda(x)$ is one function, $\lambda \in \mathbb{C}$. Certain points are added up.

Discussion of Legendre Equation as limit-point idea.

$$\textcircled{1} \quad -[(1-x^2)u']' - \lambda u = 0 \quad 1 < x < \infty \quad s(x) = 1$$

$$p(x) = (1-x^2) \Rightarrow x = \pm 1 \text{ are singular (limit-circle).}$$

$$q(x) = 0 \quad x = \infty \text{ is limit-point.}$$

$\textcircled{2}$ Show that $x = \infty$ is limit point. Easy. Choose $\lambda = 0$.
Then $u_1 = 1$ is a solution. But $\int_1^\infty 1 \cdot dx = \infty$,
so not of finite s-norm. Thus: limit-point case.

$\textcircled{3}$ Where is limit-point? $A = - \frac{W(\phi, \bar{\psi}; \infty)}{W(\psi, \bar{\psi}; \infty)}$ says p299.

Here, well, choose $2 < x < \infty$ for now. Then

$$\psi(2, \lambda) = d_1$$

$$\psi'(2, \lambda) = d_2/p(2)$$

I presume you could choose $\psi(x, \lambda) = \overset{p}{P}_\lambda(x)$ because some d_1, d_2 would work. I.e., $d_1, d_2 \neq 0$ for obscure point $x=2$. Then, could choose $\phi(x, \lambda) = \overset{q}{Q}_\lambda(x)$. Then your magic solution is:

$$u = Q_\lambda(x) + m P_\lambda(x).$$

$$\text{Now; } W[\psi, \bar{\psi}; x] = W[P_\lambda, \bar{P}_\lambda; x] = ?$$

$$\text{Well, } W[P_n(z), Q_n(z)] \cdot \frac{1}{(z^2-1)} = -1.$$

$\textcircled{4}$ I am having trouble. It should come out that $m=0$.
Then

$$u = Q_\lambda(x) = \text{only function of finite s-norm. up to } \infty.$$

$\textcircled{5}$ But, no reason why a matrix element of group $O(2,1)$ must be of finite s-norm. In discrete basis, $P_\lambda(x)$ are ones.

But at least we see why only one solution of Legendre equation is well behaved at ∞ . Namely $Q_\lambda(x)$

Notes on Differential Equations

General Comments: I have now at my disposal several books:

1) Stakgold, 1967 nice

2) Margenau + Murphy, 1 sketchy

3) Titchmarsh, on loan, "Eigenvalue-expansions" — but poorly organized, no index

Here I plan to do reading mostly in Stakgold, but other pieces may appear.

Begin: we start in Section 1.4 of Stakgold. Here: $L = a_0 d^2 + a_1 d + a_2$.

Theorem 58: If $I = (a, b)$ and $Lu = f \quad \forall x \in I$, and if $a_0(x) \neq 0$ in I , then the "initial values" $u(x_0) = \alpha$ and $u'(x_0) = \beta$ determine a unique solution, $x_0 \in I$.

singular point — place where $a_0(x)$ vanishes.

formal adjoint of L : L^* : by parts integration one may show:

$$\int_a^b v(Lu) dx = \int_a^b u(Gv) dx + [\text{stuff}]_a^b \quad \text{"Green's Formula"}$$

Given $L = a_0 d^2 + a_1 d + a_2$, you can of course find what G is, as well as "stuff".

$$G = a_0 d^2 + (2a_0' - a_1) d + (a_0'' - a_1' + a_2) \equiv L^*, \text{ the "adjoint of } L"$$

$$\text{stuff} = a_0(vu' - uv') + (a_1 - a_0')(uv) = J(u, v), \text{ the "bilinear conjunct of } u+v"$$

$$\text{Thus: } v(Lu) - u(L^*v) = \frac{d}{dx} J(u, v) \quad \text{"Lagrange's Identity"}$$

remark: if we had a real scalar product $\langle v | u \rangle = \int_a^b v u dx$, and if $J(u, v)|_a^b = 0$, we could say:

$$\langle v | Lu \rangle = \langle L^*v | u \rangle = \langle u | L^*v \rangle.$$

Then L^* really looks like a hermitian adjoint.

self-adjoint differential operator. We have:

$$L = a_0 \partial^2 + a_1 \partial + a_2$$

$$L^* = a_0 \partial^2 + \{a_1 + 2(a_0' - a_1)\} \partial + \{a_2 + (a_0' - a_1)'\}$$

In the special case that $a_0'(x) = a_1(x) \forall x \in I$, we have $L = L^*$. Then L is called self-adjoint. Because $a_0' = a_1$, the general second-order self-adjoint diff op linear is:

$$L = 2(a_0 \partial) + a_2 = a_0 \partial^2 + a_0' \partial + a_2 \quad L = L^*$$

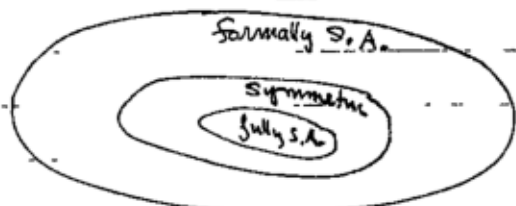
$$J(u, v) = a_0(uv' - uv') \quad \text{in this case}$$

significance of self-adjointness. Suppose $Lg = \delta(x-s)$ with $B_1(g) = 0, B_2(g) = 0$ has solution $g(x|s)$, "green's function". Consider $L^*h = \delta(x-s)$ with $B_1^*(h) = 0$ and $B_2^*(h) = 0$. These * BC's are chosen such that $J(g, h)|_a^b = 0$, this is how $B_i^*(h)$ are concocted. Then:

$$h(x|s) = g(s|x)$$

If $L = L^*$, then $g(x|s) = g(s|x)$ = symmetric! If $B_i^* = B_i$, same BC's, and if $L = L^*$, then fully self-adjoint.

symmetric - hermitean: suppose you have a formally S.A. operator $L = 2(a_0 \partial) + a_2$. Let D = domain of solutions, twice diff, satisfying any # of BC's. Then if $J(u, v)|_a^b = 0 \forall u, v \in D$, then "system" is "symmetric". By allowing too many BC's, you can have $D \neq D^*$, so symmetric but not fully self-adjoint.



(rough only)

selfadjoint form ~ from page 269 MM, suppose L is not S.A.
 Mult thru by a certain function:

$$Lu = f \quad \text{h.} \quad [a_0 x^2 + a_1 x + a_2] u = f \cdot h = f'$$

$$[\tilde{a}_0 x^2 + \tilde{a}_1 x + \tilde{a}_2] u = f'$$

Want to choose h st $\tilde{a}_1 = \tilde{a}_0'$ i.e. $(ha_1) = (ha_0)'$. To find h , we must solve:

$$(a_0' h' + (a_0') h - a_1 h = 0 \quad (a_0) h' + (a_0' - a_1) h = 0$$

But this is usual first order LDE, $h = C \int dx \frac{(a_0' - a_1)}{a_0}$, because
 $h' = h \cdot (a_0' - a_1) / a_0$

The Sturm-Liouville Equation: $Lu = \lambda w u$, where L is
 a self-adjoint, i.e., $L = \partial(p\partial) - q$, where we replace
 a_0 with p , and a_2 with $-q$. w is called the "weight",
 we could get rid of it. The classical functions are all
 solutions of S.L. equations. $\lambda = \text{eigenvalue}$, $l(l+1)$
 is Legendre, eg.

The S.L. theory: The S.L.E. is an eigenvalue equation (eg, Schrödinger)
 but it is also the Euler equation which stabilizes a certain
 action functional: $\Lambda(u) \equiv \int dx (p u'^2 + q u^2)$ subject to
 $\int w u^2 dx = 1$. In fact, the stationary values are the
 eigenvalues, $\lambda_i \equiv \int dx (p u_i'^2 + q u_i^2)$. Thus, there is the
 possibility of a variational search for S.L. solutions.

modifications of SLT: if range (b, a) goes ∞ , eigenvalues go to
 possible continuous or mixed spectrum $\rightarrow$ changes needed.
 If solution u can be complex, still OK with $\int w u^* u = 1$.
 If p, q, w go complex, SLT is not valid! But
 λ can be complex.

Stakgold + SLT: He treats it on pp 268-283, but he does not
 refer to it as "S.L.T." He calls it "The regular B.V. problem",
 Ch 4 on diff. operators.

He uses $s(x)$ instead of $w(x)$, and p, q . Restrictions:

1) (a, b) finite

2) $-p, p', q, s$ all real, continuous in (a, b) ; λ complex

3) p, s are pos def.

This is the definition of the "regular" problem.

The SLE, Stokgold: see (4.23): $Lu = \lambda su$. Again, he never refers to this as the SLE, but that's what it is. He proves lots of facts:

a) eigenvalues are real, due to fact that L is symmetric.

I.e., in (4.25) we restrict to real, unmixed BV's at a, b .

These force $W(u, \bar{v}; a) = W(u, \bar{v}; b) = 0$, so that $\bar{v}(u, x) \big|_a^b = 0$,

so we really are dealing with $L = \text{hermitean}$.

b) orthogonal ef's with weight s .

Greens for $Lu=0$. Consider $Lg - \lambda sg = \delta(x-s)$. How do you solve for g ? Let $w(x, \lambda)$ be solution "at the a-end" with $w(a) = -\alpha_2$, $w'(a) = \alpha_1$. Let $z(x, \lambda)$ be sol at the b-end. These w, z are analytic in λ . Then:

$$g(x|s; \lambda) = - \frac{w(x, \lambda) z(s, \lambda)}{C(\lambda)}$$

where you get $C(\lambda)$ from

$$W[w, z] = C(\lambda)/p(x).$$

So: g as fn of λ is analytic except when C has a zero.

Now what about $Lu = \lambda su = f$. Has immediate green's fn solution, if λ not an ev: $u(x) = \int_a^b g(x|s; \lambda) f(s) ds$.

Also, can write as $u(x) = \sum_n u_n \phi_n$ where $u_n = \langle f, \phi_n \rangle / (\lambda_n - \lambda)$, so

$$u(x) = \int_a^b g(x|s; \lambda) f(s) ds$$

$$\text{But } g(x|s; \lambda) = \sum_n \frac{\phi_n(x) \bar{\phi}_n(s)}{\lambda_n - \lambda}$$

$$\Rightarrow u(x) = \sum_n \frac{\phi_n(x) \langle f, \phi_n \rangle}{\lambda_n - \lambda}$$

complex up to 5th.

Executive: so far we have looked into the "regular" or Sturm Liouville problems.

Now, if you take a Sturm-Liouville or regular type differential operator, you generate the "singular" case by allowing one (or both) of the interval endpoints to go to infinity, or maybe p vanished at one endpoint. This takes you into the realm of "singular" differential equations. The analysis of Sturm Liouville was generalized by Hermann Weyl, in particular the theorem on page 297 of Stakgold. This gets you into all this business of limit circle case and limit point case. Stakgold makes a very relevant observation on page 304 concerning eigenfunction expansions. If, he says, you can equate a delta function to a large circle integral of your Greens Function, then it is possible to write the delta function as a discrete sum of terms relating to the discrete part of the spectrum, and another integral over some terms having to do with the continuous spectrum. The continuous part comes from cuts in the greens function, the discrete part comes from poles. This is what Vilenkin is talking about when he writes the "expansion theorem" for $F(x)$ on the B_{mn}^1 . The point above is, of course, that if you know the delta function, then you can always write a projection/expansion pair. Stakgold does this for lots of examples, like Hankel Transforms, etc.

4 However, I am a little uncertain about Stakgold's analysis because he never says why we expect the great circle greens integral to be equal to the delta function. I would like to see a chapter in some other book on this subject. As references on this subject, Stakgold gives Hellwig and Naimark, both in German only, along with Titchmarsh Friedman, and Yosida.

Chapter 9 Regular Singular Points

1. Euler's equation. This is a warmup exercise. Euler's equation is simply $z^2 w'' + pzw' + qw = 0$ where p and q are constants. A basis of two independent solutions is given simply by $w = z^a$ where the a are the two solutions of the indicial equation $a^2 + (p-1)a + q = 0$. If these solutions happen to coincide then the second solution is $z^a \ln z$. Typically, logarithms always arise when something coincides.
2. Circuit matrix: Given a second-order DE there will be two solutions. Assume that these solutions are analytic in a disc except maybe at the center. Without loss of generality, put this potential singularity at $z=0$. Then you can take each solution and analytically continue it one turn CC about this origin to get two (possibly different) new solutions. Of course each of these must be a linear combination of the original two. The 2×2 matrix which describes this linear combination is called the circuit matrix.
3. Simple branch point: a solution $f(z)$ has a simple branch point at $z=0$ if the function goes into a multiple of itself when you continue around the branch point once.
4. A general circuit theorem: consider n -th order DE with n indep solutions. There will be an $n \times n$ circuit matrix describing rotation about origin. The eigenvectors of this matrix give linear combinations of the starting solutions (used to get the matrix) which go into themselves under such a continuation. The eigenvalue is a . Then the "diagonal" solution has the form $f(z) = z^b g(z)$ where $b = \ln a / 2\pi i$ and where $g(z)$ has no branch at $z=0$ (single valued). Since every matrix has at least one eigenvector, you are guaranteed to always have at least one solution which has a simple branch point.
5. Circuit theorem for $n=2$: (Theorem 4). Your circuit matrix is 2×2 . There are two eigenvalues corresponding to two exponents called a and b . If these are different, then the two solutions can be chosen as having the form $z^a f(z)$ and $z^b g(z)$, where f and g are single valued in the disc. If $a=b$ you might still be able to diagonalize the circuit matrix and have both solutions of the form $z^a f$ and $z^a g$. More likely, you will have one of the form $z^a f$ and the other of the form $z^a f (h(z) + \ln z)$ where f is same f , h is SV. Again, notice the log coming in in a degenerate case.
6. Regular Singular Point: Consider general equation $w'' + p(z)w' + q(z)w = 0$. If at $z=z_1$ p and q are both analytic, then z_1 is called an ordinary point. If at $z=z_1$ p and/or q are singular, then z_1 = singular point. If the singularity in p is at worst a simple pole, and in q at worst a double pole, then z_1 = regular singular point.
7. Significance of regular singular point. If $z=0$ is regular singular point (we take $z=0$ only for convenience) then you can multiply through by z^2 to get an equation resembling Euler's equation: $z^2 w'' + zP(z)w' + Q(z)w = 0$ where P and Q are analytic and may thus be power-series-expanded at $z=0$. The significance of this fact is contained in the next theorem:
8. Regular Singular Point Theorem: From $P(z)$ and $Q(z)$ compute P_0 and Q_0 , the two constants. Next, solve the Euler indicial equation for two values of v (see 1 above, replace p with P_0 etc.). Then at least one solution has the form $w(z) = z^v (1 + c_1 z + c_2 z^2 + \dots)$. If the two roots do not differ by integer, then both indep solutions have this form where v takes on the two values.

9. Singular points at infinity: This requires a special test. You change from z to $t=1/z$. Then the DE is regular singular at infinity if the transformed equation is regular singular at the origin. It turns out that you can tell by doing power series expansions of p and q in powers of $1/z = t$. Then for infinity to be regular singular point, p must have at least a first order zero in t , and q must have at least a second order zero in t .

Suppose some DE is regular singular at infinity. Expand p and q in this way:

$$p = p_1 t + p_2 t^2 + \dots$$

$$q = q_2 t^2 + q_3 t^3 + \dots$$

The indicial equation is then $v(v-1) + (2-p_1)v + q_2 = 0$. Solve this for your v and then by 8 equation will have one or two independent solutions of the form $z^{-v} (1 + a_1 z^{-1} + a_2 z^{-2} + \dots)$ depending on whether difference of the exponents is an integer or not.

10. The integer difference theorem. Suppose your two roots a and b differ by an integer. Then choose a as the larger root. You may then take as your independent solutions these forms:

$$w_1 = z^a (1 + \dots \text{usual})$$

$$w_2 = z^b (1 + \dots) + C w_1 \ln z$$

Of course the two power series are different, this just gives the form. Note that you can choose C as you like.

11. Canonical basis: forgot to mention that if your roots are unequal and do not differ by integer, then your two standard forms as in 5 will diagonalize the little 2×2 circuit matrix and such a basis is called canonical.

12. Fuchsian Equations: these are (homogeneous linear) DE's having at worst regular singular points in the extended complex plane, ie, including infinity. By doing linear fractional variable changes, you can move these regular singular points wherever you like.

For $N=2$ you can move singular points to 0 and infinity. Then equation boils down to Euler equation with elementary solutions. (N = number of sing points of Fuchsian equation).

For $N=3$, you first move points to 0, 1 and infinity. Then you can introduce 5 parameters by looking at the singular points 0 and 1. Thus, you fiddle a little to find that p and q must have very special form in terms of these 5 constants. This leads to Riemann's DE. You can reparametrize in terms of the six exponents (two at each regular singular point) and then the sum of the six exponents = 0 to give only 5 parameters. Every Riemann DE can be reduced to a hypergeometric form. If the 6 exponents take on certain values in terms of 3 parameters, you get exactly the hypergeometric with no changes.

13. Apply to hypergeometric equation: at each regular point you compute the roots by looking at various indicial equations:

$$z=0$$

$$v_1=0$$

$$v_2=1-c$$

$$\text{diff} = 1-c$$

$$z=1$$

$$v_1=0$$

$$v_2=c-a-b$$

$$\text{diff} = c-a-b$$

$$z=\text{inf}$$

$$v_1=a$$

$$v_2=b$$

$$\text{diff} = a-b$$

Clearly, the usual solution $F(a,b,c,z)$ corresponds to a $v=0$. ~~THEXXXXXXXXXX~~ Ie, this solution is the $z=0$ $v_1=0$ case.

Bateman calls this u_1 . Its partner is u_5 . So, $z=0$ gives u_1 and u_5 . Then $z=1$ roots gives Bateman's u_2 and u_6 . Finally, $\text{inf} = z$ gives u_3 and u_4 .

Each pair forms a viable canonical basis of simple branch point functions.

Analysis of Legendre ODE.

① The equation is:

$$w'' + p(z)w' + q(z)w = 0$$

where $p(z) = \frac{-2z}{(1+z)(1-z)}$ $q(z) = j(j+1) - \frac{[\mu^2 + \nu^2 - 2\mu\nu z]}{(1+z)^2(1-z)^2}$

② Points $z=1, z=-1$ are regular singular points.

~~W.D~~

~~(1-z)w' +~~

③ Consider $z=+1$. Then

$$(z-1)^2 w'' + (z-1)P(z)w' + Q(z)w = 0$$

$$P(z) = +2z \frac{\cancel{1}}{(z+1)} \quad Q(z) = j(j+1)(z-1)^2 - \frac{[\mu^2 + \nu^2 - 2\mu\nu z]}{(1+z)^2}$$

Now: $P(1) = \underset{\uparrow}{\cancel{0}}_0 \quad Q(1) = -\frac{[\mu^2 + \nu^2 - 2\mu\nu]}{4} = -\underset{\uparrow}{\cancel{0}}_0 \frac{(\mu-\nu)^2}{4}$

④ Indicial is: $a^2 + (p_0 - 1)a + q_0 = 0$

$$\Rightarrow a^2 \cancel{1} - \frac{1}{4}(\mu-\nu)^2 = 0 \quad a(\cancel{a}) = \frac{1}{4}(\mu-\nu)^2$$

$$a = \frac{+1 \pm \sqrt{1 + (\mu-\nu)^2}}{2} \Rightarrow \boxed{a = \pm \frac{1}{2}(\mu-\nu)}$$

⑤ New $w_1(z) = z^{\pm \frac{1}{2}(\mu-\nu)} (1+z)^{\dots}$

⑥ Instead, let $z = \frac{1-\xi}{2}$ then: $d\xi = -\frac{1}{z} dz$

$$\frac{\partial}{\partial \xi} = -2 \frac{\partial}{\partial z}$$

$$\frac{\partial^2}{\partial \xi^2} = 4 \frac{\partial^2}{\partial z^2}$$

$$(1-z^2) = 4 \cdot \left(\frac{1+z}{2} \right) \left(\frac{1-z}{2} \right) = 4\xi(1-\xi)$$

$$\frac{1+z}{2} = \xi(1-\xi)$$

$$\text{I get: } j(j+1) + \cancel{\frac{1}{4}(1-\xi)} \cancel{\frac{\partial^2}{\partial \xi^2}} \rightarrow 2$$

$$2\xi = 1 - 2$$

$$2 = 1 - 2\xi$$

$$\cancel{\frac{1}{4}(1-2\xi)} \left(\cancel{\frac{\partial^2}{\partial \xi^2}} \right) = \frac{1}{4\xi(1-\xi)} [\mu^2 + \nu^2 - 2\mu\nu(1-2\xi)]$$

Repeat:

$$\left(-j(j+1) + \xi(1-\xi) \frac{\partial^2}{\partial \xi^2} + (1-2\xi) \frac{\partial}{\partial \xi} - \frac{1}{4\xi(1-\xi)} [\mu^2 + \nu^2 - 2\mu\nu(1-2\xi)] \right) w = 0$$

Mult by $\xi(1-\xi)$:

$$w \left[\xi^2(1-\xi)^2 \frac{\partial^2}{\partial \xi^2} + \xi(1-\xi)(1-2\xi) \frac{\partial}{\partial \xi} - \frac{1}{4} (\mu^2 + \nu^2 - 2\mu\nu(1-2\xi)) + \xi(1-\xi)j(j+1) \right] w = 0$$

OK, $\xi=0, \xi=1$ are singular.

① Look at $\xi=0$:

$$\xi^2 \frac{\partial^2}{\partial \xi^2} + \frac{(1-2\xi)\xi}{(1-\xi)} \frac{\partial}{\partial \xi} + \frac{\xi}{(1-\xi)} j(j+1) - \frac{1}{4} \frac{(\mu^2 + \nu^2 - 2\mu\nu(1-2\xi))}{(1-\xi)^2} j w$$

$$\begin{cases} \xi^2 w'' + \xi P(\xi) w' + Q(\xi) w = 0 \\ P(\xi) = \frac{(1-2\xi)}{(1-\xi)} \\ Q(\xi) = j(j+1) \frac{\xi}{1-\xi} - \frac{1}{4} \frac{[(\mu-\nu)^2 + 4\mu\nu\xi]}{(1-\xi)^2} \end{cases}$$

$$P(0) = 1$$

$$Q(0) = -\frac{1}{4}(\mu-\nu)^2$$

$$\Rightarrow a^2 = \frac{1}{4}(\mu-\nu)^2$$

$$\Rightarrow a = \pm \frac{1}{2}(\mu-\nu)$$

So one solution is

$$\xi^{\pm \frac{1}{2}(\mu-\nu)} \left[1 + a\xi + \dots \right]$$

not relevant.

QA20 -

Friedman, ch 3, greens functions

① If $Lg(x,t) = \delta(x-t)$, then $L^{-1} = \int g(x,t) dt$

ie, $(L^{-1})u = \int g(x,t)u(t)dt$. Thus: "greens function is the kernel of the inverse operator which is an integral operator."

② Laurant Schwartz is credited with inventing the theory of Distributions to justify + understand delta functions. French reference to a 1950-51 article by Schwartz.

③ Understand all delta fcn equations as if they were integrated $-\infty$ to ∞ .

Test functions = linear space $\rightarrow \infty$ diff, but vanish outside some interval.

$\phi_n \rightarrow 0$ means all derivs $\rightarrow 0$ and all vanish outside ^{same} interval.

$F(\phi)$ = continuous functional of ϕ , obvious continuity property.

"distribution" = continuous linear functional in space of test functions.

$$F(\phi) = \int_{-\infty}^{\infty} \delta(x) \phi(x) dx = \phi(0)$$

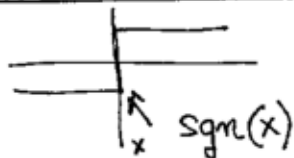
$\nwarrow$ specific "symbolic function"

$$F(\phi) = \int_{-\infty}^{\infty} \delta(x) \phi(x) dx = \text{some general distribution}$$

$\nwarrow$ general symbolic function.

$\delta(x)$ has meaning only when mult by a TF and integrated.

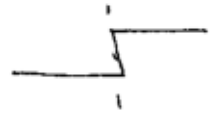
④ Symbolic derivatives: consider derivative of $|x|$. It is:



$$\int_{-\epsilon}^{\epsilon} \frac{d \operatorname{sgn}(x)}{dx} dx = \operatorname{sgn}(x) \Big|_{-\epsilon}^{\epsilon} = 2.$$

Thus $\frac{d}{dx} \operatorname{sgn}(x) = 2\delta(x)$.

ie, δ 's let you handle discontinuities.

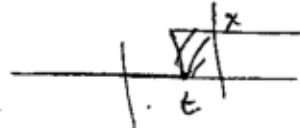
Also:  $\frac{d}{dx} \Theta(x) = \delta(x)$.
 $\nwarrow$ Heaviside $H(x)$.

⑤ Example of finding greens:

$$\frac{d^2}{dx^2} g(x, t) = \delta(x-t) = \frac{d}{dx} \Theta(x-t)$$

$$\Rightarrow \frac{d}{dx} g(x, t) = \Theta(x-t) + C(t)$$

$$\text{Then } g(x, t) = \int_0^x \Theta(x-t) \cdot dx + C(t)x + D(t)$$

$= x-t$: 

So: $g(x, t) = (x-t)\Theta(x-t) + \alpha(t)x + \beta(t)$ ✓

⑥ OK, this chapter goes on to show how you can find the greens for various kinds of BC's. Always we are merely solving:

$$Lg = \delta(x-t).$$

Chapter 4 "eigenvalue problems of ODE's"

① Like Stakgold, he opens with lots of examples. All is "regular theory". Approx methods etc.

② Then page 213 $\Rightarrow$ "the spectral representation"

Let $Lu_k = \lambda_k u_k$. Let $u(x) = \sum \alpha_k u_k(x)$

Then $Lu(x) = \sum \alpha_k \lambda_k u_k(x)$

Generalize concept to: $\nwarrow$ same function.

$$f(L)u(x) \equiv \sum f(\lambda_k) \alpha_k u_k(x)$$

③ Example: $f(t) = \frac{1}{\lambda - t}$

$$\Rightarrow \frac{1}{\lambda - L} u(x) = \sum \frac{1}{\lambda - \lambda_k} \alpha_k u_k(x).$$

But $\frac{1}{\lambda - L} = (\lambda - L)^{-1}$ = expressible in terms of greens.

Let $(\lambda - L)g(x|\xi; \lambda) = \delta(x - \xi).$

Then $\frac{1}{\lambda - L} u(x) = \int g(x|\xi; \lambda) u(\xi) d\xi.$

④ Go back to $\frac{1}{\lambda - L} u(x) = \sum \frac{\alpha_k u_k(x)}{\lambda - \lambda_k}.$

Integrate over circle $\sim$ infinite circle. Then:

$$\frac{1}{2\pi i} \oint \frac{\alpha_k u_k(x)}{\lambda - \lambda_k} d\lambda = - \sum_k \alpha_k u_k(x) = -u(x)!$$

Therefore:

$$\frac{1}{2\pi i} \oint \left[\frac{1}{\lambda - L} u(x) \right] d\lambda = -u(x). \quad \frac{1}{2\pi i} \oint \frac{1}{\lambda - L} d\lambda = I$$

But replace with:

$$\begin{aligned} \frac{1}{2\pi i} \oint \left[\int g(x|\xi; \lambda) u(\xi) d\xi \right] d\lambda &= -u(x). \\ &= \int u(\xi) \cdot d\xi \cdot \left[\frac{1}{2\pi i} \oint g(x|\xi; \lambda) d\lambda \right] = -u(x). \end{aligned}$$

Therefore:

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint g(x|\xi; \lambda) d\lambda = -\delta(x - \xi).$$

⑤ Now go back to chapter two, "The spectral theory of operators". Lots of matrix jarg. Then

"Spectral representation of L " means you represent all vectors in space as lincomb. of eigen functions of L . I.e., these should be complete.

So spectral rep exists $\iff$ completeness.

⑥ What means $f(L)$? $x = \sum \xi_i x_i$

$$Lx = \sum \lambda_i \xi_i x_i$$

$$L^2 x = \sum \lambda_i^2 \xi_i x_i \text{ obviously.}$$

So $f(L)x = \sum f(\lambda_i) \xi_i x_i$

when $f(t)$ is any polynomial, certainly. But these functions are dense in your space so you can carry idea over to any continuous function

~~$$\frac{-1}{-L + \lambda} = \sum_{n=0}^{\infty} (\lambda L)^n$$~~

$$\sum_{n=0}^{\infty} L^n = \frac{1}{1-L}$$

$$\sum_{n=0}^{\infty} (aL)^n = \frac{1}{1-aL} = \frac{a^{-1}}{a^{-1}-L}$$

So $\sum_{n=0}^{\infty} (L/\lambda)^n = \frac{\lambda}{\lambda-L}$

Then $\frac{1}{\lambda-L} = \lambda^{-1} \cdot \sum_{n=0}^{\infty} \left(\frac{L}{\lambda}\right)^n$

Thus, no question as to inverse function.

⑦ Page 249 gives a summary of "spectral representations". He is definitely thinking of

$$\delta(x-\xi) = \dots$$

when he says "spectral rep."

⑧ Not clear who first used $\delta(x-\xi)$ version. Probably Laurent Schwarz, but the theory was there earlier.

Books discussing mixed-spectrum ODE analysis.

- ① Dunford + Schwartz is just too much. I'm sure they have it (1971), but I would never refer someone there because no one could read it. Too much — too general for my purposes.
- ② Birkhoff on ODE's — no mention of anything but discrete stuff.
- ③ Morse + Feshbach : no
- ④ Courant - Hilbert : no, lots on partial D.E.'s.
- ⑤ Friedman : "Principles + Techniques of Applied Math". Here it all is!
Very nice book. What is the history of this stuff? 1956, 6 printings!
- ⑥ Ince : "ODE's" (1926). References Sturm as 1836.

I gather that Titchmarsh did all this stuff in his 1945 book, but not in modern notation.

Complex Variables

Contents of Complex Variables Section.1. Wad containing several basic theorems

- a) continuing off a segment
- b) uniqueness of a continuation
- c) meaning of analyticity in closed domain and boundary off-continuation
- d) analytic continuation as union of functions
- e) good example of gamma function
- f) Schwartz Reflection and $T(s,t)$ application: real analyticity
- g) non-analyticity of $f^*(z)$ and $f(\bar{z}^*)$, analyticity of $f^*(\bar{z}^*)$ in D^*
- h) Swarts in two complex variables

2. Wad of material on Carlson Theorem and Phragmén-Lindelöf

- a) logic of Carlson theorem for our purposes
- b) general reading conclusions
- c) notes from Titchmarsh 1939
- d) notes from Hille 1962
- e) exp. fctns.
- f) the "sum theorem"

3. Case studies of special functions to locate cuts

- a) $(z^2 - 1)^{1/2}$
- b) eight pages about square roots of gamma functions

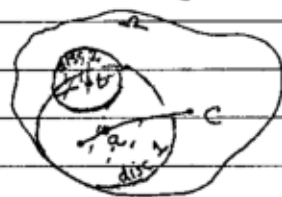
4. Comparison of adding two discs and computing total disc via real analyticity.5. The meaning of Principle Value6. Deformation of cuts in from infinity when possible.7. Example of contour pinching.8. How to shift the contour in delta function integral and have same thing.9. How to change variables in a complex integration; also, how to complex conjugate.10. Wad on the mapping $z = \text{ch}(v)$. Related details.11. Solution to the famous "cut problem".12. Theorem about when you can drop great circles.13. The inverse mapping theorem.

Complex Variables

do not know out! (7/76)

A. Theorem: let $f(z)$ be analytic on open domain Ω . Let C be a contour of finite length all of whose points are a finite distance from the boundary of Ω . Then, if we are given $f(z)$ only along C , we can compute $f(z)$ everywhere in Ω . Let Ω be simply connected.

proof: since we know f everywhere on C , and since f is analytic all over Ω , including points on C , we therefore know all derivatives of f at points on C . This is so because we are always free to compute a derivative by moving along the curve C . We never need information from points not on C .



Now draw an open disc about any point on C . Let this disc brush the boundary. Since disc $\subset \Omega$, we know we can write $f(z)$ for $z \in \text{disc}$ as the "usual" power series about center:

$$f(z) = f(a) + f'(a)(z-a) + \frac{1}{2!} f''(a)(z-a)^2 + \dots$$

this is a theorem in the book. (or for finite disc!)

But since $a \in C$, we know $f(a), f'(a)$, etc. Thus, we explicitly know $f(z)$ everywhere in the disc. This is a continuation; later we will show that the continuation is unique.

Now comes the easier problem of continuing $f(z)$ from this original disc to all of Ω . Clearly we can get to any $z \in \Omega$ by a series of overlapping discs, so the problem is merely how to continue between two discs.

Let $B \in \text{first disc}$. We know $f(z)$ near b , so we know all derivatives at b :

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(z-a)^n}{n!}$$

$$f^{(1)}(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)(z-a)^{n-1}}{(n-1)!}$$

$$f^{(2)}(z) = \sum_{n=2}^{\infty} \frac{f^{(n)}(a)(z-a)^{n-2}}{(n-2)!}$$

$$f^{(2)}(z) = \sum_{n=2}^{\infty} \frac{f^{(n)}(a)(z-a)^{n-2}}{(n-2)!}$$

$$\Rightarrow f^{(n)}(b) = \sum_{n=n}^{\infty} \frac{f^{(n)}(a)(b-a)^{n-n}}{(n-n)!}$$

Explicit formula for all derivatives at $z=b \in \text{disc 1}$.

Now make disc centered at b which goes beyond disc 1. Everywhere in disc 2 we have:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)(z-b)^n}{n!}$$

Thus, we have cont'd from one disc to the next.

✓ B. Theorem: Suppose in Theorem A we know that $f = k$, constant, along C . Then if $f(z)$ is analytic in Ω , $f(z) \equiv k$ everywhere in Ω .

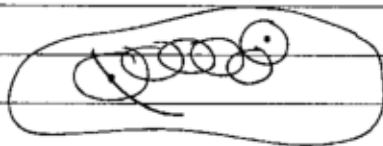
proof: as in Theorem A, we extend $f = k$ to the first disc. This is done by observing that:

$$f(a) = k \quad f'(a) = 0$$

Thus: $f(z) = k$ in disc 1.

Then we keep overlapping discs until we find that $f(z) = k$ for any $z \in \Omega$:

Eg, second disc has



$$f(b) = k \quad f'(b) = 0 \Rightarrow f(z) = k \text{ in disc 2.}$$

✓ C. Subtheorem: if $f(z)$ analytic in Ω + $f(z) = 0$ along C , then $f(z) \equiv 0$ throughout Ω .

✓ D. Theorem (uniqueness of continuation): The function $f(z)$ found via prescription in A. as the "continuation of curve C " is unique.

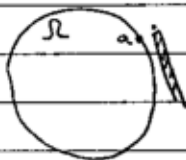
proof: suppose $f(z)$ and $\tilde{f}(z)$ are two different continuations, both analytic in Ω , both agreeing with initial data on C . Then $g(z) \equiv f(z) - \tilde{f}(z)$ is analytic in Ω and $g(z) = 0$ along C . By above sub theorem, $g(z) \equiv 0$ throughout $\Omega \Rightarrow f(z) \equiv \tilde{f}(z)$. But then $f = \tilde{f}$, which contradicts assumption they were different. Thus, two different continuations do not exist. Thus, $f(z)$ is unique.

E. Application: if we know $T(s, t)$ exactly up in the regge region, we could continue to find $T(s)$ in all $1/s$, including resonances. But $T(s) = (-s)^\alpha$ is not exact. $T(s)$ must also include poles that are obscured by $(-s)^\alpha$. So by itself, $(-s)^\alpha$ does not contain info on the s poles.

Note that here, although C appears to lie on the boundary of Ω [the cut], we can assume analyticity on the sheet that falls thru the cut at that point. Thus, continuation via Theorem A is feasible.
(But see later).

✓ F. Theorem: Suppose $f(z)$ is analytic in a closed domain Ω . Then in fact $f(z)$ must be analytic in a larger closed domain containing Ω . This is "the first extension", called $\Omega^{(1)}$.

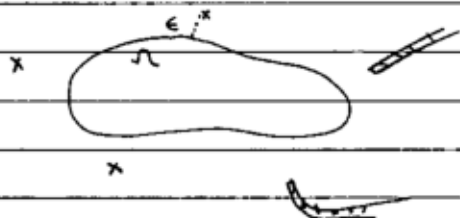
proof:



To say that $f(z)$ is analytic at a means there is no singularity at a . Any singularity must be some distance ϵ outside Ω .

Of course ϵ can be as small as you like + $f(z)$ still be analytic at a . We cannot have $\epsilon = 0$, however. Thus, $\lim_{\epsilon \rightarrow 0} (\text{cut}(\epsilon))$ is not allowed either.

So, let ϵ be the smallest distance from the boundary of Ω to any of the singularities of $f(z)$. Then, let



Ω be the domain obtained by extending Ω by $\epsilon/2$ in all directions. Then $f(z)$ is analytic in closed Ω !

Thus, we can keep extending Ω in this way. First by $\epsilon/2$. Then by $\epsilon/4$, then by $\epsilon/8$. In all these extensions, $\Omega^{(n)}$, $f(z)$ will be analytic.

If Ω were open, you could have a sing. right on the boundary + then extensions are not possible.

✓ G. Corollary: Let $f(x)$ be analytic in closed domain Ω . Let a be pt. on the boundary. Then claim that $f(z)$ is analytic in a finite closed disc around a .

proof: just choose disc radius to be $\epsilon/2$. Then, disc will lie entirely inside the first extension, $\Omega^{(1)}$, where $f(z)$ is known analytic.

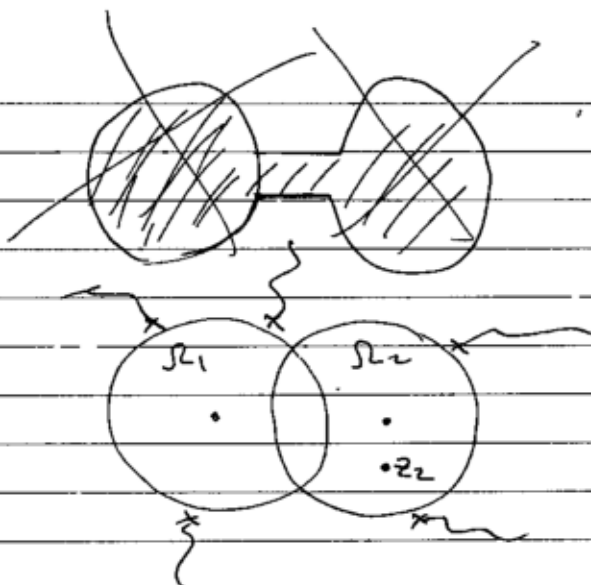
✓ H. Theorem A': If $f(z)$ is analytic in closed Ω , then $f(z)$ can be uniquely continued off a piece of Ω 's boundary.

proof: Let $C = AB = \text{piece of boundary}$. Then C is "finite inside" the first extension $\Omega^{(1)}$. Thus, Theorem A applies as does Theorem D on uniqueness.

better example

In Ω_1 imagine $f_1(z) =$
power series convergent
only in disc Ω_1 ,

In Ω_2 another power
series, $f_2(z)$.

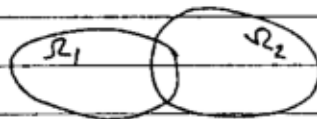


Thus, in this example
 $f_1(z)$ is not defined at point z_2 because power series
 $f_1(z)$ does not converge.

But, $f \equiv f_1 \oplus f_2$ is analytic in $\Omega_1 \cup \Omega_2$.

f_1 is called "the continuation of f_2 into Ω_1 "

Definition: if $f_1 \oplus f_2 = f$ is analytic ~~on Ω~~ [i.e., if f defined as f_1 over Ω_1 and f_2 over Ω_2 is analytic over $\Omega_1 \cup \Omega_2$], then one refers to f_1 as analytic continuation of f_2 , and vice versa.

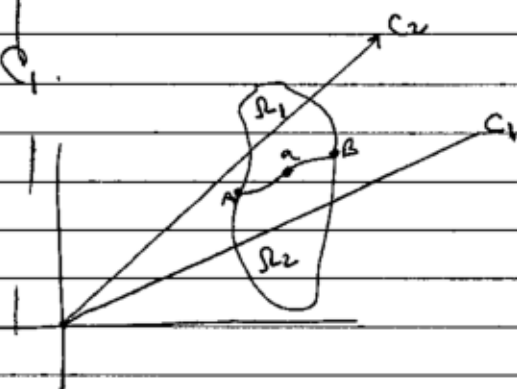


for better example. see ←

Note: This does not imply that $f_1 = f_2$ in all of Ω . Eg, f_1 may have cuts in Ω_2 . f_1 is restricted to Ω_1 by definition, so $f_1 = f_2$ rigorously is nonsense. In some cases, however, f_1 itself may happen to be analytic over all of Ω , not just Ω_1 . In such cases, $f_1 = f$ by uniqueness of f , in addition, f_2 is analytic over all Ω , then $f_2 = f$. In this case only, one may conclude that $f_1 = f_2$ on Ω .

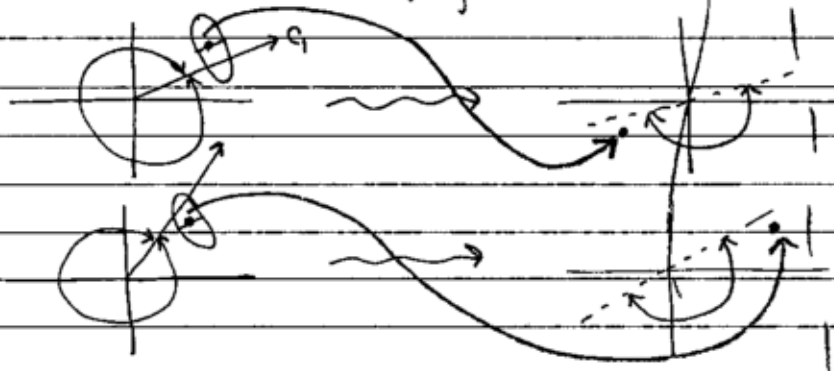
Example: let $f_1(z) = +\sqrt{z}$ with cut C_1 .

Obviously, f_1 is not analytic in all of Ω . "+" implies the branch of $\sqrt{z}$ which includes the + real axis in $\mathbb{C}$ plane. (i.e., $\sqrt{4} = +2$.)



Now, let $f_2(z) = +\sqrt{z}$ with cut C_2 . f_2 is analytic in Ω_2 , but not in Ω_1 (again, + branch).

Question: do f_1 and f_2 have the same values along AB? The answer is: NO, if f_1 and f_2 are both the + branch.



In fact, $f_1(a) = -f_2(a)$.

Thus, in this example, f_2 is not the continuation of f_1 , because ~~they are~~ f_2 doesn't even agree with f_1 on AB!

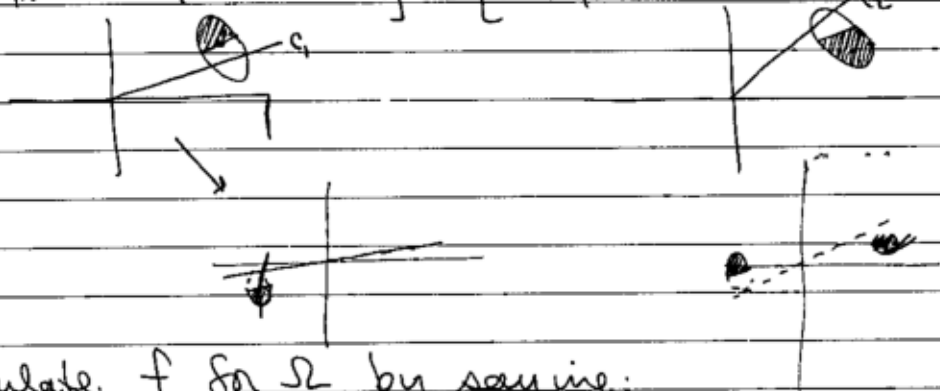
Example 2: Let $f_1 = +$ branch of $\sqrt{z}$ with C_1 on Ω_1 .
 Let $f_2 = -$ branch of $\sqrt{z}$ with C_2 on Ω_2 .

Now $f_1(a) = f_2(a) \forall a$ on AB . So now we have

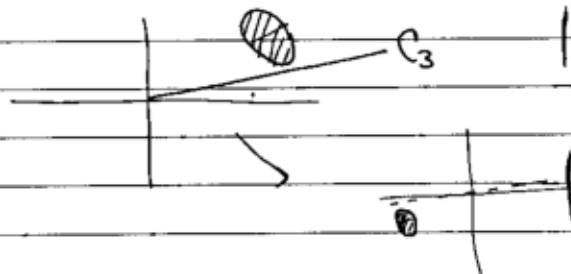
- (a) f_1 analytic in Ω_1
- (b) f_2 analytic in Ω_2
- (c) $f_1 = f_2$ along AB .

Thus, $f_1 \oplus f_2 = f$ is analytic in all of Ω . Clearly $f_1 \neq f_2$.
 And f ~~can be constructed as follows~~ is:

$$f = [+ \text{br of } \sqrt{z} \text{ in } \Omega_1 \text{ with cut } C_1] \oplus [- \text{br of } \sqrt{z} \text{ in } \Omega_2 \text{ with cut } C_2]$$



One could simulate f for Ω by saying:

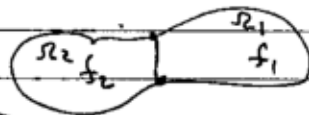


$$f = [+ \text{br of } \sqrt{z} \text{ in } \Omega \text{ with } C_3]$$

I. Theorem: if f_2 is an analytic continuation of f_1 from Ω_1 into Ω_2 , then f_2 is unique. It is the continuation.

proof: we know that $f_1 \oplus f_2 \equiv f$ is analytic over all Ω (by hypotheses on f_1, f_2). Moreover we know that f must take certain values on $\Omega_1 \cap \Omega_2$ specified by f_1 . Thus, there can be no other fctn f' that is anal. in Ω + takes these same values on $\Omega_1 \cap \Omega_2$, so $f_1 \Rightarrow f$ uniquely. (See Theorem D). Since f is unique, + since $f = f_2$ in Ω_2 , we conclude that f_2 is unique.

proof 2: We know from Theorem H that f_2 is uniquely determined by its values on $\Omega_1 \cap \Omega_2$, even if $\Omega_1 \cap \Omega_2$ is just a curve. But these values are preset by f_1 . So $f_1 \Rightarrow f_2$ uniquely.



Example: ~~consider the Euler beta function:~~

~~$B(s,t) = \int_0^1 dx x^{s-1} (1-x)^{t-1}$ as function of s .~~

~~If $\text{Re } s > 0, x$. Do gamma fctn instead:~~

Example: $\Gamma(s) = \int_0^\infty dx x^{s-1} e^{-x}$ gamma function.

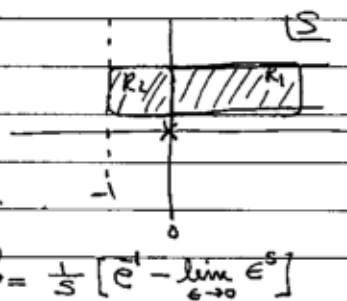
For $s > 0$ this is fine. Even if $\text{Re } s > 0$, still analytic because no problem at the 0 endpoint from $\int_0^\infty dx x^{s-1}$.

(Parts says: $x^{s-1} = \frac{1}{s} \frac{d}{dx}(x^s)$ so:

$$\Gamma(s) = \frac{1}{s} x^s e^{-x} \Big|_0^\infty + \frac{1}{s} \int_0^\infty dx x^s e^{-x}$$

$$= \frac{1}{s} \left(\lim_{x \rightarrow \infty} x^s e^{-x} - \lim_{x \rightarrow 0} x^s e^{-x} \right) + \frac{1}{s} \int_0^\infty dx x^s e^{-x} = \frac{1}{s} (0 - 0) + \frac{1}{s} \int_0^\infty dx x^s e^{-x}$$

"parts" blows for $s < 0$



For $s \leq 0$, $\Gamma(s)$ given by this integral diverges. [Let $s \geq 0$ be Ω_1 and $\Gamma \equiv f_1$ there].

But parts integration provides a continuation into Ω_2 for $\Gamma(s)$, + reveals a pole at $s=0$. These agree for $\text{Re } s > 0$ [i.e., $\Omega_1 \cap \Omega_2$]. Thus, $\Omega_2 \cap \Omega_1$, we have found the unique continuation.

Discuss Γ continuation ✓

① We start with $\Gamma(s) = \int_0^1 dx x^{s-1} e^{-x}$, good for $\text{Re } s > 0$.
Rewrite as:

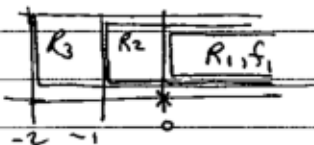
$$\Gamma(s) = \frac{1}{s} (\bar{e}^1 - \lim_{\epsilon \rightarrow 0} \bar{e}^\epsilon) + \frac{1}{s} \int_0^1 dx x^s e^{-x}$$

Now if we go to, say, $s = -1/2$, we know that $\Gamma(s)$ as written blows. In the second expr, we see exactly where it blows.

So, we restrict the first integral to $\text{Re } s > 0$.

Let: $f_1(s) = \int_0^1 dx x^{s-1} e^{-x}$

$$f_2(s) = \frac{\bar{e}^1}{s} + \frac{1}{s} \int_0^1 dx x^s e^{-x}$$



② Notice that, in $s \in R_1$, $f_1(s) = f_2(s)$, because then $\lim_{\epsilon \rightarrow 0} \bar{e}^\epsilon = 0$. Also, we see that, aside from a pole at $s=0$ (which I exclude in picture), f_1 is analytic in R_1 , f_2 is analytic in $R_2 \cup R_1$.

Therefore: $f_2(s)$ is the unique analytic cont of $f_1(s)$ to the R_2 region.

③ We can of course keep going:

$$\int_0^1 dx x^s e^{-x} = \frac{1}{s+1} \int_0^1 dx \left(\frac{d}{dx} x^{s+1} \right) e^{-x}$$

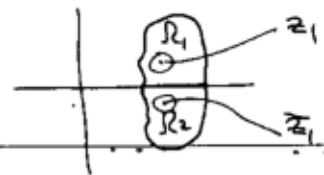
So use: $= \frac{1}{s+1} [\bar{e}^1 - \lim_{\epsilon \rightarrow 0} (\bar{e}^\epsilon)^{s+1}] + \frac{1}{s+1} \int_0^1 dx x^{s+1} e^{-x}$

$$f_3(s) = \frac{\bar{e}^1}{s} + \frac{\bar{e}^1}{s(s+1)} + \frac{1}{s(s+1)} \int_0^1 dx x^{s+1} e^{-x}$$

In region R_2 [ie, in $(-1 < s < 0)$], $\lim_{\epsilon \rightarrow 0} (\bar{e}^\epsilon)^{s+1} \rightarrow 0$, so $f_2(s) = f_3(s)$. But note that $f_3(s) \neq f_1(s)$ in R_2 . But f_3 is anal in $R_3 \cup R_2 \cup R_1$, f_2 is anal in $R_2 \cup R_1$. Thus: f_3 is the unique contin to R_2 .

④ In general: $f_n(s) = \bar{e}^1 \left[\frac{1}{s} + \frac{1}{s(s+1)} + \dots + \frac{1}{s(s+1)(s+2)\dots(s+n-2)} \right] + \frac{1}{s(s+1)\dots(s+n-2)} \int_0^1 dx x^{s+n-2} e^{-x}$

So, this is good for $s > -(n-1)$. Either $f \equiv (f_1, R_1) \cup (f_2, R_2) \cup \dots \cup (f_n, R_n)$ or f_n are anal. over $R_n \cup R_{n-1} \cup \dots \cup R_2 \cup R_1$, so use f_n .
Note how this "bares" all the poles of $\Gamma(s)$.



Schwarz Reflection

J. Theorem: if $f_1(z)$ is analytic in Ω_1 as shown, then $\overline{f_1(\bar{z})}$ is analytic in Ω_2 . ($\Omega_2 = \Omega_1^*$)

proof: let z_1 be in Ω_1 . $f_1(z)$ anal. in $\Omega_1 \Rightarrow$

$$\lim_{\eta \rightarrow 0} \left[\frac{f_1(z_1 + \eta) - f_1(z_1)}{\eta} \right] = \text{indep of phase of } \eta.$$

Expand f_1 at z near z_1 : $f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^n$

Then:

$$f_1(\bar{z}) = \sum a_n (\bar{z} - \bar{z}_1)^n \quad \text{for } \bar{z} \text{ in lower disc, } \Rightarrow \bar{z} \text{ is in upper disc.}$$

Then: $\overline{f_1(\bar{z})} = \sum \bar{a}_n (z - \bar{z}_1)^n$, for z in lower disc:

Thus, $f_2(z) \equiv \overline{f_1(\bar{z})} = \sum \bar{a}_n (z - \bar{z}_1)^n$ is a convergent power series defined over lower disc.

Conclusion: $f_2(z) \equiv \overline{f_1(\bar{z})}$ is analytic in Ω_2 if $f_1(z)$ is in Ω_1 .

K. Theorem: if $f_1(z)$ is analytic in Ω_1 and real on real axis, then $f_2(z) \equiv \overline{f_1(\bar{z})}$ is the unique anal. continuation of f_1 into Ω_2 (Sch. reflection principle)

proof: We know $f_1(z)$ anal. in Ω_1 and $f_2(z)$ anal. in Ω_2 . If $f_1(z)$ is real on AB, then:

$$f_1(z)|_{AB} = \overline{f_1(z)|_{AB}} = \overline{f_1(\bar{z})|_{AB}} = f_2(z)|_{AB}$$

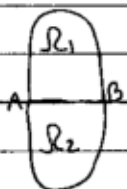
$$f_1(z)|_{AB} = \overline{f_1(z)|_{AB}} = \overline{f_1(\bar{z})|_{AB}} = f_2(z)|_{AB}$$

$\uparrow f_1 \text{ real.} \quad \quad \quad \uparrow z \text{ real.}$

Thus, since $f_1 = f_2$ on AB, we see that by definition, $\overline{f_1(\bar{z})}$ is a cont. of f_1 , and by Theorem I, it is the unique continuation.

Note: we do not conclude that $f_1 = f_2 = \text{analytic over } \Omega$!

L. Theorem: if $f_1(z)$ happens to be analytic over all of $\Omega_1 + \Omega_2 = \Omega$, then $f_2(z) \equiv f_1(\bar{z})$ is also analytic over all Ω .



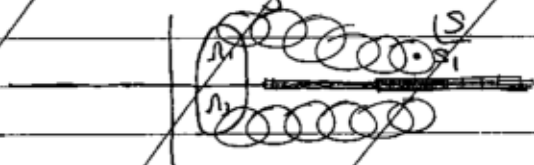
proof: apply Theorem 5 for f_1 in Ω_2 . Obvious.

M. Corollary: If $f_1(z)$ analytic over $\Omega_1 + \Omega_2 = \Omega$, and if $f_1(z)$ real on real axis.

Then 1. $f_2(z) \equiv f_1(\bar{z})$ is the unique anal. contin of f_1 in Ω_2
AND 2. $f_1(z) = f_2(z)$ in all Ω . ("real analyticity")

proof: we know (1.) from Theorem K. Thus, $f = (f_1, \Omega_1) \oplus (f_2, \Omega_2)$ is analytic over all Ω . But, so is f_1 . Moreover, f and f_1 agree on AB ; Thus, by Theorem D, $f = f_1$. Similarly, $f = f_2$, using Thm L. Since $f_1 = f = f_2$, conclude that $f_1 = f_2$ throughout Ω .

Application: in particle scattering theory, $T(s, t)$ usually has cuts in the s -plane, with real branch points. If we draw all cuts along the $+$ real axis, we have:



Next, if we assume a priori that $T(s)$ is real on the real axis, then we can

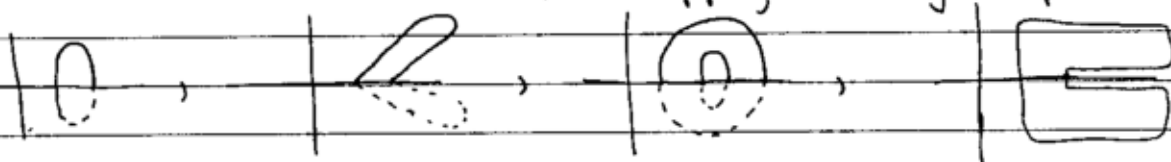
apply Corollary M above to region shown to conclude that:

$$T(s) = T^*(s^*) \quad \text{in } \Omega = \Omega_1 \cup \Omega_2.$$

Next, we continue $T(s)$ beyond Ω_1 to point s_1 .

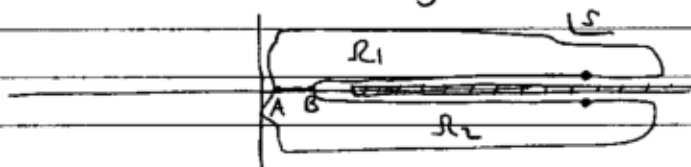
Schwartz application:

- ① Note that the theorems J, K, L, M apply to any shape Ω , eg:



- ② Keep in mind that $f_1(z)$ is supposed to be analytic in Ω which is closed; ie, $f_1(z)$ anal. on that real axis segment. Thus, cannot have a cut on special segment.

- ③ Consider $T(s, t)$ as $T(s)$, the scattering amplitude. If you draw all cuts along the $+ \text{real}$, you get:



Now, a priori, assume that $T(s)$ is real on segment AB. Also, assume $T(s)$ is analytic in $\Omega_1 + \Omega_2$. Then by corollary M, we know:

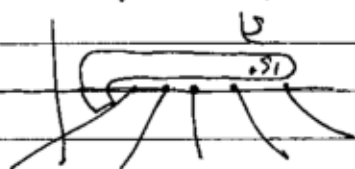
$$T(s) = T^*(s^*) \quad \forall s \in \Omega = \Omega_1 + \Omega_2$$

In particular: $T(|s| + i\epsilon) = T^*(|s| - i\epsilon)$

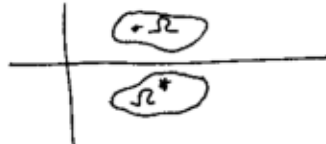
$$\begin{aligned} \text{Then: } T(s + i\epsilon) - T(s - i\epsilon) &= \text{Disc } T \\ &= T(s + i\epsilon) - T^*(s + i\epsilon) \\ &= 2i \text{Im } T(s + i\epsilon) \end{aligned}$$

- ④ I know of no reason ~~why~~ to justify the above a priori assumption. (except TP invariance for elastic, maybe).

- ⑤ If you draw the cuts some other way: Then $T(s) = f_1(z)$ is not analytic in Ω and you cannot conclude that $T(s) = T^*(s^*)$ at s_1



Complex Variables



Theorem: if $f(z)$ is analytic in R , then $g(z) \equiv f(z^*)$ and $h(z) \equiv f^*(z)$ are both non-analytic. But $k(z) \equiv f^*(z^*)$ is analytic in R^* .
[g in R^* , h in R]

Proof: $f(z)$ analytic $\Rightarrow f(z) = \sum a_n (z - z_1)^n$ for disc in R .
The derivative of $f(z)$ is of course the same from all directions ~ that's what analyticity means:

$$\begin{aligned} f'(z) &= \frac{f(z+\eta) - f(z)}{\eta} = \frac{\sum a_n (z+\eta - z_1)^n - \sum a_n (z - z_1)^n}{\eta} \\ &= \sum a_n [z^n + n \cdot \eta \cdot z^{n-1} + o(\eta^2) - z^n] / \eta \quad (\text{repl. } z \rightarrow (z + z_1)) \\ &= \sum a_n n (z - z_1)^{n-1} \text{ independent of } \eta! \end{aligned}$$

(a) now look at:

$$\begin{aligned} g'(z) &= [g(z+\eta) - g(z)] / \eta = \left[\frac{f}{\eta} (z^* + \eta^*) - \frac{f}{\eta} (z^*) \right] / \eta \\ &= \sum a_n \left[\left(\frac{z^* + \eta^*}{-z_1} \right)^n - \left(\frac{z^*}{-z_1} \right)^n \right] / \eta = \sum a_n n \left(\frac{z^* - z_1}{-z_1} \right)^{n-1} \times \left(\frac{\eta^*}{\eta} \right) \end{aligned}$$

$\searrow z \rightarrow (z - z_1)$

$g'(z)$ depends on which η you choose:

$$\eta = |\eta| e^{i\phi} \Rightarrow \eta^* / \eta = e^{-2i\phi}$$

Thus $g(z)$ is not analytic.

$$\begin{aligned} \text{(b) Similarly, } h'(z) &= \sum a_n^* [(z+\eta)^n - z^n] / \eta \\ &= \sum a_n^* n (z^*)^{n-1} \cdot (\eta^* / \eta) \Rightarrow h(z) \text{ not analytic.} \end{aligned}$$

$$\begin{aligned} \text{(c) Finally, } k'(z) &= \sum a_n^* [(z^* + \eta^*)^n - (z^*)^n] / \eta \\ &= \sum a_n^* z^{n-1} \cdot n \left(\frac{\eta}{\eta^*} \right) \Rightarrow k(z) \text{ is analytic.} \end{aligned}$$

$$\begin{aligned} \text{(d) By the way, } f(z) &= \sum a_n z^n \\ k(z) &= \sum a_n^* z^n \end{aligned}$$

Only if $f(z)$ is "real analytic" (the a_n are real #'s) can you say:

$$f^*(z^*) = f(z)$$

Theorem of Analyticity

- ① $f(z)$ is analytic means that the derivative $f'(z)$ as defined by

$$f'(z) = \lim_{\Delta z} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

is the same no matter how $\Delta z \rightarrow 0$.

- ② For example, if $f(z) = z^2$, then $f'(z) = \frac{2z \cancel{\Delta z}}{\cancel{\Delta z}} = 2z$.

- ③ However, consider $g(z) = f(z^*) = (z^*)^2 = (z^c)^* = f^*(z)$.

In this case we have

$$g'(z) = \frac{[(z + \Delta z)^*]^2 - [z^*]^2}{\Delta z}$$

$$\approx \frac{2z^* (\Delta z)^*}{(\Delta z)}$$

$$\begin{aligned} \text{If } \Delta z = \epsilon, \text{ then get } &= 2z^* \\ \text{If } \Delta z = i\epsilon, \text{ then get } &= -2z^* \end{aligned}$$

Thus, this $g(z)$ is not analytic.

- ④ In general, if $f(z)$ is analytic, then $f(z^*)$ is not analytic. Power series proof:

$$f(z) = \sum_n a_n z^n \text{ since analytic}$$

$$\begin{aligned} \Rightarrow f'(z) &= \left[\sum_n a_n (z^n + n z^{n-1} \Delta z) - \sum_n a_n z^n \right] / \Delta z \\ &= \sum_n a_n n z^{n-1} \end{aligned}$$

$$\text{But } g(z) = f(z^*) = \sum_n a_n (z^*)^n \Rightarrow$$

$$\begin{aligned} g'(z) &= \left(\sum_n a_n (z^{*n} + n z^{*n-1} \Delta z^*) - \sum_n a_n (z^*)^n \right) / \Delta z \\ &= \sum_n a_n n (z^*)^{n-1} \frac{\Delta z^*}{\Delta z} \text{ does not exist.} \end{aligned}$$

Schwarz reflection in 2 variables

$$f(z, w)$$

$$z = x + iy$$

$$w = u + iv$$

Assume that $f(x, u)$ is real — you are away from cuts in both variables. Then, if analytic in both variables, can say:

$$f(z, w) = a + bz + cw + dz^2 + ew^2 + fzw + \text{etc.}$$

where all $a, b, c, \dots$ are real. i.e.,

$$f(z, w) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{nm} (z)^n (w)^m \quad \{a_{nm}\} \text{ real.}$$

Now, we know that $(AB)^* = A^* B^*$, not too difficult, so also

$$(z^*)^n = (z^n)^*$$

Thus,

$$\begin{aligned} f(z^*, u) &= f(x + iy, u) = \sum_{nm} a_{nm} (z^*)^n (u)^m \\ &= \left(\sum_{nm} a_{nm} (z)^n (u)^m \right)^* \\ &= f^*(z, u) \end{aligned}$$

Notice that this works when the other variable is real.

More generally:

$$\boxed{f(z^*, w^*) = f^*(z, w)}$$

if $f(z, w)$ is real analytic.

The Logical Structure of Carlson's Theorem.

1. First, define a set of functions of a single complex variable such that if you go off on a ray in the RHP (including the pure imaginary rays), the function obeys a certain condition (the Carlson Condition). Any function which obeys this condition is called a Carlson Function. Or, you might say: this function is "Carlson."
2. The Carlson Condition is this: $f(z)$ must be analytic in the closed Right Half Plane, and as you go off on some ray in this RHP, $|f(z)|$ must be less than a constant times $e^{\pi|z|}$.
3. Examples: $f(z) = e^{i\pi z}$ is not Carlson because if you go down the imaginary axis, $z = -i/z$ and $f(z)$ goes as $e^{\pi/z}$ which violates the Condition. Thus, $\sin(\pi z)$ is also not Carlson. Similarly, $e^{\pi z}$ is not Carlson on the plus real axis ray. So, $\sinh(\pi z)$ is not Carlson. On the other hand, e^{az} for $|a| < \pi$ is Carlson. Any polynomial in z is Carlson. More special examples are discussed below.
4. Question: Suppose you have a certain function defined only on the positive integers. Can you define a unique analytic continuation off the integers? In general, the answer is NO. For example, if your function was $f(z) = 0$, then here are two different analytic continuations off the integers: $f_1(z) = 0$, $f_2(z) = 0 + \sin(\pi z)$.
HOWEVER, if you have a function defined on the integers which is Carlson, then within the class of Carlson functions there does exist a unique analytic continuation off the integers! This will be a direct consequence of Carlson's Theorem below.
5. Carlson's Theorem: if $f(z)$ is Carlson and vanishes on the positive integers, then the only Carlson continuation of $f(z)$ is $f_1(z) = 0$. (Consider the example above. $f(z) = 0$ is Carlson. So, $f_1(z) = 0$ is a Carlson continuation. But $f_2(z) = 0 + \sin(\pi z)$, albeit a continuation, is not a Carlson continuation.)
6. Corollary: a Carlson function defined on the integers has a unique Carlson continuation off the integers. (If not true, you could find two different Carlson continuations. The difference of these two would also be Carlson, and would vanish on the integers. But by the theorem, such a Carlson function must be 0 everywhere.)

Conclusions about Carlson's Theorem and Related Theorems.

1. First, I will quote Carlson's Theorem directly and accurately from T: p186

[5.81. If $f(z)$ is regular and of the form $O(e^{k|z|})$, where $k < \pi$, for $R(z) \geq 0$, and $f(z) = 0$ for $z = 0, 1, 2, 3, \dots$, then $f(z) = 0$ identically.]

2. There are several questions here. First, the conclusion is really only $f(z) = 0$ identically for $\operatorname{Re}(z) > 0$ because $f(z)$ has no definition elsewhere. ~~Re~~ of course the only analytic continuation of $f(z) = 0$ to the left half-plane is $f(z) = 0$. Note the equality part of $\operatorname{Re}(z) \geq 0$. This has the important implication that you must include the imaginary rays.

3. The real question is $\forall$ what is the meaning of $O(\dots)$. According to the first page of his book, this must mean as $|z| \rightarrow \infty$, $|f(z)| < A e^{k|z|}$ where A is an "absolute positive constant". This inequality should hold for $|z|$ larger than some value. Seems fairly unambiguous. Here is an almost good proof which shows an alternative way to write this condition:

$$\lim_{|z| \rightarrow \infty} \frac{\ln |f(z)|}{|z|} < k \Rightarrow \frac{\ln |f|}{|z|} < k \quad \forall |z| > R.$$

$$\Rightarrow \ln |f| < k|z| \Rightarrow |f| < e^{k|z|}$$

4. This statement of the theorem gives with a reduction of one of Hilles theorems which I have done in some notes there, except there you need $\operatorname{Re}(z) > \frac{1}{2}$. I accept T's statement of the theorem.

5. HOWEVER, none of these guys Hille or Titch. gives a proof I like. Titch. is simply wrong, I think. And Hille does something more general than I want. I really should look for another book to reference on this.

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pp 176-188 are of interest.

5.6 The Theorems of Poincaré.

First a quick reminder of the maximum-modulus theorem: if $f(z)$ is analytic inside some finite closed contour, then it is of course bounded on the domain D which includes the boundary. The big fact is that the maximum of $|f(z)|$ actually must occur somewhere on the boundary curve itself.

So, for openers Titch in section 5.6 generalizes just slightly the max-mod theorem by extending it to a case where $f(z)$ fails to be analytic at one point on the boundary. Not too significant? Ah, but when that one point becomes infinity, this theorem becomes very basic! Better work it through.



$f(z)$ = analytic in/on C .

$|f(z)| \leq M$ on C , except at P

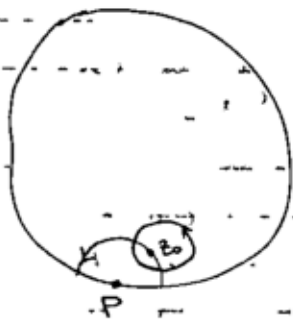
Also, $\exists w(z)$ analytic $\neq 0$, inside C with $|w(z)| \leq 1$ inside C .

assume: we can bypass P as shown such that, for z on the bypass curve,

$|[w(z)]^\epsilon \cdot f(z)| \leq M$ for any $\epsilon > 0$, no matter how close bypass comes to P .

Then: $|f(z)| \leq M$ at all points inside C .

proof: $F(z) \equiv w(z)^\epsilon f(z)$ is analytic inside C . So, for some $z_0 \in C$, $|F(z)| \leq M$ on curve around z_0 . Then $|F(z_0)| \leq M$ too.



$\Rightarrow |f(z)| < M \cdot |w(z)|^{-\epsilon}$ but $|w| \leq 1$, $(\frac{1}{2})^{-4} = 2^4 \dots$

Even if $\epsilon \rightarrow 0$, $|f(z)| \leq M$.

As I read this it seems simply to say: if $f(z)$ is bounded in the neighborhood of P , then P doesn't really matter anyway and the max-mod theorem works. I.e., just remove that damn point.

Well now for the future theorems he says he won't use this result.

5.61 Theorem:

Assume (a) $f(z)$ analytic in wedges D (P , closed)

(b) $|f(z)| \leq M$ on the two ~~ray~~ lines

(c) $f(z) \leq \exp[r^\beta]$, where $\beta < \alpha$, uniformly in angle of wedges.



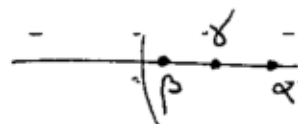
$$z^\delta = e^{\delta \ln z} = e^{\delta \ln |z|} e^{i\delta \theta}$$

$$\operatorname{Re}(z^\delta) = e^{\delta \ln |z|} \cos \delta \theta.$$

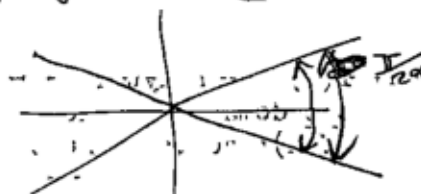
Then: $|f(z)| \leq M$ in all D .

proof:

$$F(z) \equiv e^{-\epsilon z^\gamma} f(z) \text{ where}$$



$\epsilon > 0$.



$$|F(z)| = |f(z)| \times \exp[-\epsilon |z|^\gamma \cos(\delta \theta)]$$

decay

On line $\cos(\delta \theta) = ?$

$\theta = \arg(z)$

$\cos(\delta \theta) > 0$ if $|\delta \theta| < \frac{\pi}{2}$. But $\delta < \alpha$

I will assume $\delta > 0$. Then $\delta < \alpha \Rightarrow |\delta| < |\alpha|$ so

$$\cos(\delta \theta) = \cos|\delta \theta| > 0 \text{ because } |\delta \theta| < \frac{\pi}{2}$$

because $|\delta \theta| \leq \frac{\pi}{2}$

So, on the line, $\leftarrow$ assumed

$$|F(z)| < |f(z)| \leq M \quad \text{new fact:}$$

Next, he shows that on an arc (red), $|F(z)| \rightarrow 0$ as arc moves out. So move arc out until $|F(z)| \leq M$. Then

$|F(z)| \leq M$ on closed curve $\Rightarrow |F(z)| \leq M$ inside.

Ergo, $|f(z)| \leq M e^{\epsilon |z|^\gamma}$

The proof given here is simply no good, though I assume the theorem is correct. The theorem uses heavily the fact that $\epsilon > 0$, and gets the conclusion,

$$|f(z)| \leq M \cdot \exp[+\epsilon |z|^\gamma] \text{ in wedge.}$$

No matter how small ϵ is, this is not a bound at all on $|f(z)|$. You cannot set $\epsilon = 0$ because then proof does not go through. SLOP!!!

Nevertheless, I will quote the next important theorem before I goes into details with the indicator function:

Theorem: if $f(z) \rightarrow a$ on two straight lines separated by any angle, and if $f(z)$ is analytic and bounded in the region between the lines, then $f(z) \rightarrow a$ uniformly over the entire region.

5.7 The P-L function $h(\theta)$ (6 of 10 pages now gone).

define: " $f(z)$ is of order ρ " means

$$\limsup \left[\frac{\ln |f(z)|}{|z|^{\rho+\epsilon}} \right] = 0 \text{ for all } \epsilon > 0.$$

example: $f(z) = e^{(a+ib)z^\rho} = e^{(a+ib)|z|^\rho e^{i\theta}}$

$$z^\rho = e^{\rho \ln |z|} = e^{i\rho\theta} |z|^\rho$$

$$= a \cos(\rho\theta) |z|^\rho - b \sin(\rho\theta) |z|^\rho$$

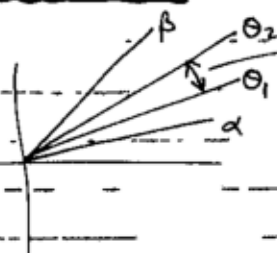
$f(z) = e^{c z^\rho}$ is of order ρ .

$$|f(z)| = e$$

$$\frac{\ln |f(z)|}{|z|^{\rho+\epsilon}} = [a \cos(\rho\theta) - b \sin(\rho\theta)] \cdot |z|^{-\epsilon} \rightarrow 0. \text{ So}$$

$$h(\theta) \equiv \limsup \frac{\ln |f(z)|}{|z|^\rho}$$

5.7.1 Theorem



$$\theta_2 - \theta_1 < \frac{\pi}{\rho}$$

Assume $f(z)$ is of order ρ such that:

$$h(\theta_1) \leq h_1, \quad h(\theta_2) \leq h_2.$$

$$\text{Let } H(\theta) = a \cos(\rho\theta) + b \sin(\rho\theta) \text{ s.t. } H(\theta_1) = h_1, \quad H(\theta_2) = h_2.$$

Claim: $h(\theta) \leq H(\theta)$ for θ in (θ_1, θ_2) .

STOP

old but OK

[Titchmarsh
"Theory of Functions"
1932, 1939]

Trying to clarify Carlsom's Theorem. (is this Donezied.?)

① First some notation of his book:

a) A means a real, positive constant, so

$$\begin{aligned} x &< A \\ \Rightarrow 2x &< A \end{aligned}$$

b) $f(x) = O\{\phi(x)\}$ means $|f(x)| < A\phi(x)$ if x is near some limit.

② p185 [5.8] Let $f(z) = O(e^{k|z|})$ for $\operatorname{Re}(z) \geq 0$.

i.e., strip

[5.6] Phragmén-Lindelöf.

If: Let $f(z)$ be an. in. wedge, closed:

Suppose $|f(z)| \leq M$ on the rays.

Also, assume that

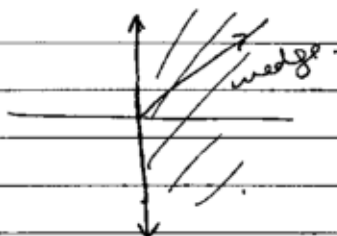
$$\lim_{r \rightarrow \infty} |f(z)| < e^{r^\beta}$$

over entire wedge (on any ray therein)

Then: $|f(z)| \leq M$ in entire wedge.

(with $\beta < \alpha$!)

Example:



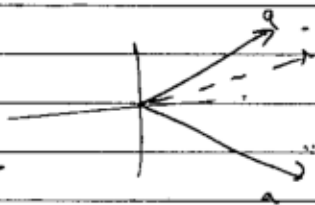
$$\theta = \frac{\pi}{2} \text{ so } \alpha = 1.$$

If $|f(z)| < e^{r^\beta}$ with $\beta < 1$ along any ray,

Then ...

fixed by
Eden.

[5.63] If $f(z) \rightarrow a$ on two rays, and



if $f(z)$ is bounded in wedge,
then $f(z) \rightarrow a$ on any ray in wedge.

[5.64] If $f(z) \rightarrow a$ on 1 and $f(z) \rightarrow b$ on another,
and if $f(z)$ is regular & bounded in wedge, then
 $a=b$ and $f(z) \rightarrow a$ on all rays in wedge.

[5.8] Carlson: If $f(z)$ is regular and

if " $f(z)$ is of the form $O(e^{k|z|})$ for $\operatorname{Re}(z) \geq 0$ "

and if $f(z) = O(e^{-a|z|})$ $a > 0$ on imaginary axis,

then $f(z) \equiv 0$ everywhere.

I would take this to mean: if along any ray in $\operatorname{Re}(z) \geq 0$
(and that includes the im. axis itself) - if along
any ray you have

$$|f(z)| \sim e^{k_0 |z|}$$

And if in addition, along the real ray you have

$$|f(z)| \rightarrow e^{k_0 |z|} \quad \text{where } k_0 < 0$$

Then $f(z) \equiv 0$.

old but OK

Carlson's Theorems

As stated in Titchmarsh 1939, the theorem is not quite clear, so I will try to state it more clearly.

CT#1: Assume you have a function $f(z)$ that is analytic for all $\operatorname{Re}(z) \geq 0$. Suppose that this function goes to 0 along both imaginary rays at least as fast as an expo, so that $\lim_{n \rightarrow \infty} |f(\pm in)| < e^{-an}$ for some $a > 0$.

Suppose also that along any ray in the RHP, $f(z)$ does not $kz^{(2)}$ blow up faster than an exponential, i.e. $\lim_{n \rightarrow \infty} |f(re^{i\theta})| < e$.

Then Carlson claims that $f(z) \equiv 0$ in RHP.

Example 1: $f(z) = e^{iz}$. Along $+i$ ray, you have $|f(in)| \rightarrow |e^{-n}| \rightarrow 0$.
Thus, not a candidate.

Example 2: $f(z) = e^{iz^2}$. OK along $+i$ ray, but blows up on $-i$ ray, so no good.

Example 3: $f(z) = e^{z^2}$. Along both rays, this goes as $e^{-y^2} \rightarrow 0$, so OK on the im axis. However, along ray ϕ we have $\lim |f(re^{i\phi})| = e^{r^2 \cos(2\phi)}$. This blows up too fast to satisfy theorem.

(1) Note that must be analytic at $z=0$. This rules out $|z|$, $\sqrt{z^2}$.

(2) k can be as large as you like, but expo must be 1, not 1^3 or some other power.

CT#2: If $f(z) = 0$ on all integers $0, 1, 2, \dots$, and if $|f(z)|$ is bounded by $O(k^{|z|})$ on all rays, incl. imag axis, and where $k < \pi$, and if $f(z)$ analytic for all $\operatorname{Re}(z) \geq 0$,
then $f(z) \equiv 0$.

"condition 2" for class 2" fctns.

Corollary: If $f_1(z)$ and $f_2(z)$ are two functions satis. condition 2, and if $f_1(z) = f_2(z)$ on the integers, then $f(z) = f_1(z) - f_2(z)$ satis. all of CT#2 and $f(z) \equiv 0$ so $f_1(z) \equiv f_2(z)$.

Corollary: If $f_1(z)$ satis. cond 2, then its analytic continuation of the integers is unique. There is no other function $f_2(z)$ satisfying cond 2 + agreeing on the integers.

So Recall of Carlson:

Suppose f_z is given on the integers, and we are thinking of continuing into complex z . Then we know that:

"If f_z is class 2, then there is no other class 2 function g_z that coincides with f_z on the integers."

[Thus, within Class 2, f_z has a unique continuation.]

Outside class 2 there are many "other" functions such as $f_z + 4z \sin(\pi z)$.

Conversely, we can say

"If f_z is not class 2, then there are many continuations and (probably) none of these is class 2."

In regge theory, we use f_z^+ and restrict ourselves to class II so the continuation is unique.

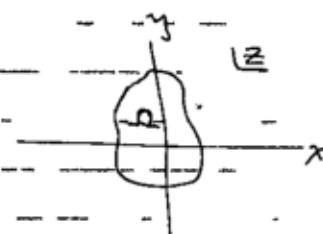
Sept 4, 1976

2.

Hille's book: some notes pursuant of Carlsom's Theorem.

- ①. Meaning of: The function of support of a closed, convex set. (Minkowski).
(Section 2.3 p 31)

- (a) Choose a closed, bounded, convex region, Ω , in z -plane.
Construct a real-valued function of another complex variable w in this way:



$$F(w) = \max_{\substack{\text{as } z \text{ runs} \\ \text{over } \Omega}} \operatorname{Re} [\bar{z} \cdot w]$$

$$w = u + iv$$

$$z = x + iy$$

Thus, once you construct $F(w)$, you know that $F(w) \geq ux + vy$.

Now, $ux + vy \leq F(w)$ defines, for each w , a half-plane in the z -plane. You will. For every $z \in \Omega$, $ux + vy \leq F(w)$, [w -fixed]. Therefore, all of Ω lies to one side of $ux + vy = F(w)$.

But also, for some z , call it z_0 , in Ω , we must have $F(w) = ux + vy$.

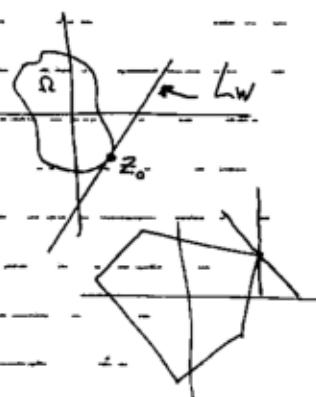
Therefore, z_0 must lie on the boundary of Ω , and on the line. Thus, for each w we are constructing a line which is tangent to Ω somewhere.

L_w is called "the line of support of Ω at z_0 ".

If z_0 = vertex, there may be ∞ # of

support lines, but generally only one.

don't know yet.



What, then, is $F(w)$? Is its significance? You can compute it as follows: pick a w ; this sets slope of L_w . Then position L_w to be tangent. This sets $F(w)$.

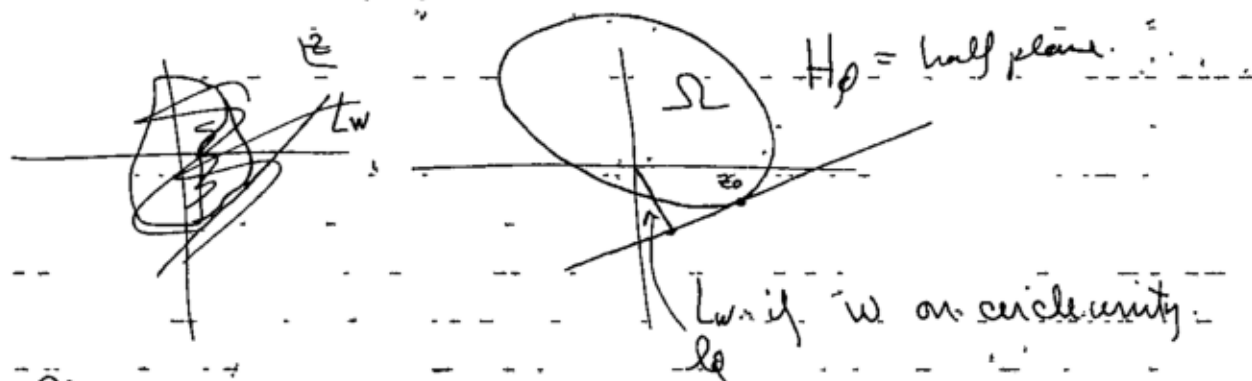
$F(w)$ is called: "The function of support of Ω "; obviously it is a set of axis-intercepts determined by the geometry of Ω .

- (b) So far, w can be anywhere in w -plane. Seems redundant, since $\arg(w)$ is what sets slope of L_w . Same effect of "tracing" Ω is had by letting $|w|=1$ and varying only $(\arg w)$.

In this case, $w = \cos \phi + i \sin \phi = u + iv$, and

$$L(w) = x \cos \phi + y \sin \phi.$$

Note that (Specially) $L(w)$ is in this case the distance from:



New names:

$$L_w \rightarrow \ell_\phi$$

$$f(w) \rightarrow \sigma(\phi) = \text{new support fct.}$$

(c) significance: as we let ϕ go from 0 to 2π , the series ℓ_ϕ "trace out" boundary of Ω . This lets you define the set Ω as:

$$\Omega = \bigcap_\phi H_\phi$$

Yes, Ω is "represented" as the intersection of all those half-planes! I can feel how this concept will be useful.

② meaning of growth indicator: (p54, vol II).

(a) First define: "exponential type function":

~~$g(z)$ is analytic for $\operatorname{Re}(z) \geq 0$~~

$f(z)$ analytic for $\operatorname{Re}(z) \geq 0$ and $\exists$ constant a such that

$$\text{for } |z| > R, \quad |f(z)| \leq e^{a|z|}$$

On my language, $f(z)$ is not hyper-expo, eg, e^{az^2} is no go.
 e^{5z} is expo, $\geq e^{45z}$ is expo because $\leq e^{6z}$ for $|z|$ large enough.

(b) Now measure the growth of $f(z)$ like so: ("growth indicator function")

$$h(\arg(z)) = \lim_{|z| \rightarrow \infty} \frac{1}{|z|} \ln |f(z)| \text{ along say } \arg(z).$$

Examples:

(b1) $f(z) = z^n$, $\ln |f(z)| = n \ln |z|$, so $h(\phi) \equiv 0 \forall \phi$.

(b2) $f(z) = e^{z^n}$, $\ln |f| = |z|^n \Rightarrow h = +\infty$, indicating hyper-expo presence $n > 1$.

(b3) $f(z) = z^n e^{az}$, $\ln |f| = \ln(|z|^n e^{a|z|})$
 $= n \ln |z| + a|z|$

So $h(\phi) = \frac{n \ln |z| + a|z|}{|z|} = a$

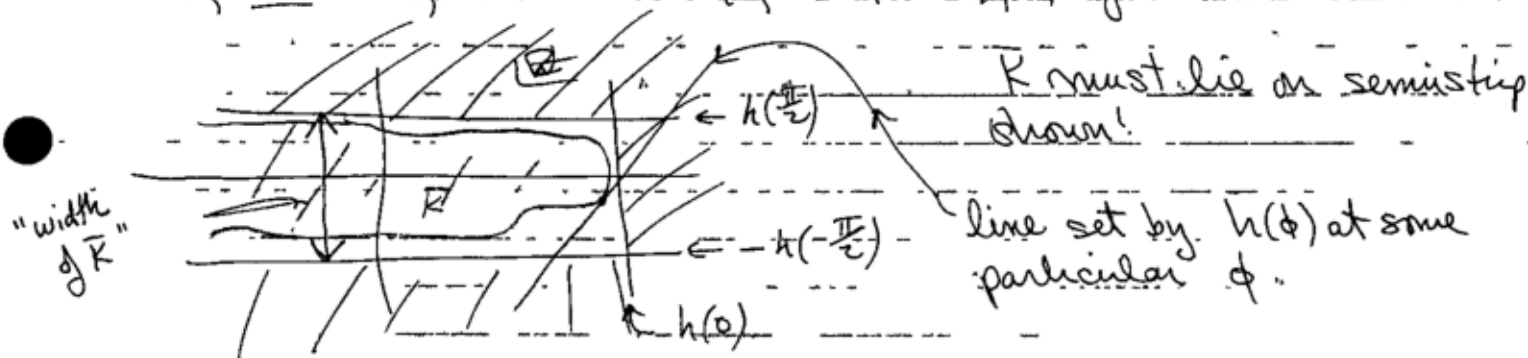
Only "senses" the expo, not the power!

③ The sets K and $\bar{K}$: Take a function $f(z)$ and compute from

it $h(\phi) = \lim_{|z| \rightarrow \infty} \frac{\ln |f(z)|}{|z|}$, for $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$. Now,

any function of ϕ may be regarded as the support function of some convex set in the ϕ -plane. ~~Call this now the w -plane, reversing earlier notation.~~

Where is $\bar{K}$ in ϕ -plane? Depends, of course, on what $f(z)$ is, but they claim that if $f(z)$ is expo, you know that



Somehow the shape of K contains info. on the nature of the function $f(z)$.

K = "the indicator diagram" of $f(z)$

(b) Suppose $f_m(z) \equiv e^{-2\pi i m z} f(z)$. Then

$$h_m = \frac{1}{|z|} (\cancel{2\pi i m} \ln |e^{2\pi i m z}| + h)$$

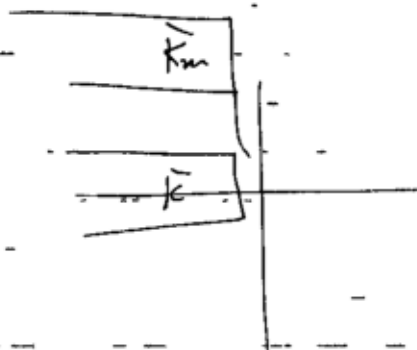
$$= 2\pi m \frac{\sin \phi}{|z|} + h = 2\pi m \sin \phi + h(\phi)$$

So $h_m(\phi) = h(\phi) + 2\pi m \sin \phi$ $\phi = \arg(z)$.

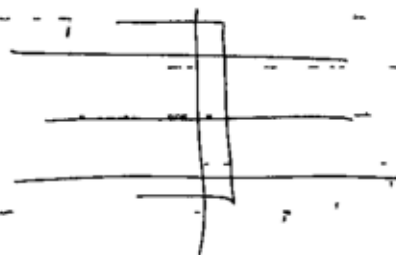
Where is K_m relative to K ?

$$h_m\left(\frac{\pi}{2}\right) = h\left(\frac{\pi}{2}\right) + 2\pi m$$

K_m is shifted up by $2\pi m$.



Note that for $z = 0, 1, 2, \dots$ $f_m = f$



④ Super-Theorem 11.3.1 (change now to variable w):

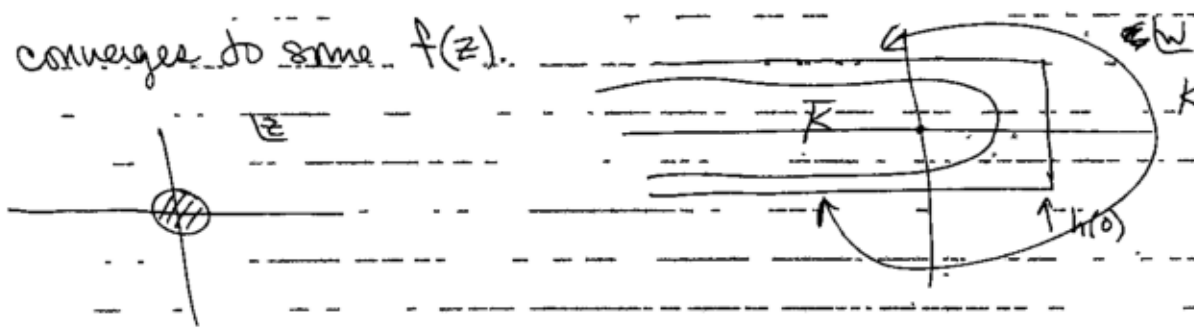
Conditions of theorem:

- (a) $g(w)$ analytic in w .
- (b) $g(w)$ of expo. type for $\operatorname{Re}(w) \geq \beta < 0$.
- (c) the indicator $h(\phi)$ is bounded; so $h(0)$ is bounded.
- (d) K is the indicator diagram of $g(w)$, but shifted so as to straddle or be in $-\pi$ to π strip.

Results of theorem:

(A) from the values of $g(w)$ at $w = 0, 1, 2, \dots$ construct a power series $f(z) = \sum_{n=0}^{\infty} g(n) z^n$

Now, for $|z| < e^{-h(0)}$, ie, not too large, a circle, this series converges to some $f(z)$.



Now pause: we could consider set K as if it were in the z -plane. Instead consider:

set in z -plane $\equiv e^{-K}$

Hold it: where does semi-strip go in mapping $z = e^{-w}$.



$\Rightarrow |z|$ indep of w'' , so $\uparrow$ is a circle eqn
 $\arg(z) = -w''$, so $\arg(z)$ restricted

Now you see why wanted K to shadde! $\uparrow$ at $w=0$ is int

(B), not only does the series $f(z)$ converge in disc $|z| < e^{-h(0)}$, but $f(z)$ has analytic continuation throughout region called e^{-K} , see picture. The complement of e^{-K} .

pause: confusion on K versus K . K is as shown $\frac{\pi i}{h(0)}$.

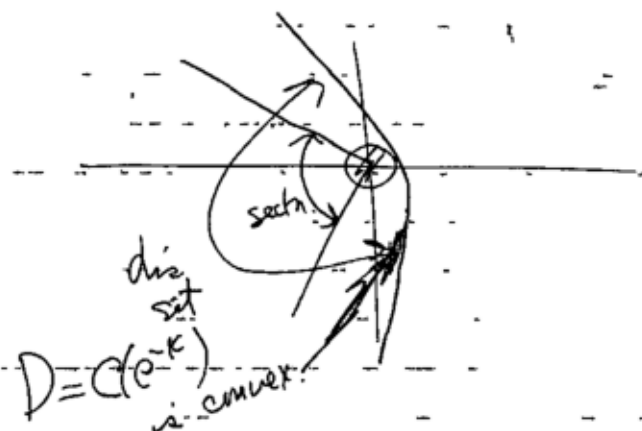
K is not the complement, but the reflection in real axis of K ! So similar to K .

So, e^K, e^{-K} have relation



detail: starlike means $\sim$ convex, see p. ____ vol 1.

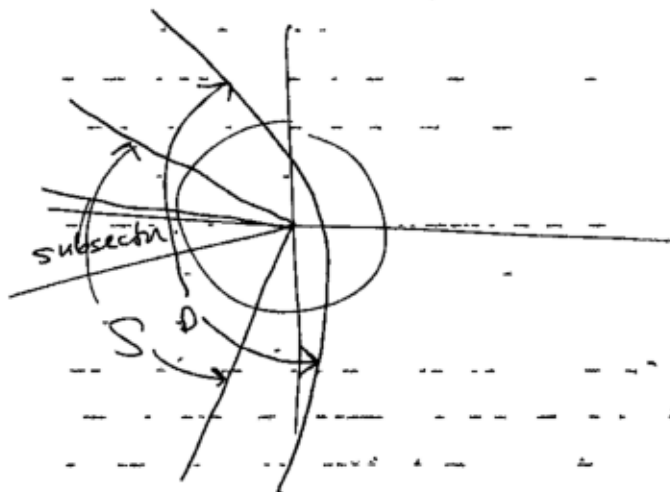
(C) The set $C(e^{-k})$ is starlike, so my picture is wrong



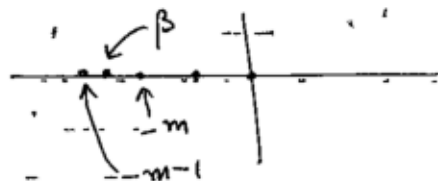
(D) If $\text{width } k < 2\pi$ (as I have been assuming) then $\underline{D} = C(e^{-k})$ contains the sector shown, called S.

(E) Define $\delta = 1 - \beta$, whichever is smaller. (recall $\beta < 0$).

Then $z^\delta f(z)$ is bounded in every subsector of S with $|z| > 1$.



(F) Suppose β lies here:



Then inside S we have, this asymptotic expansion for $f(z)$ when $|z| > 1$, inside sector S:

$$f(z) = -\frac{g(-1)}{z} - \frac{g(-2)}{z^2} - \dots + O(|z|^\beta)$$

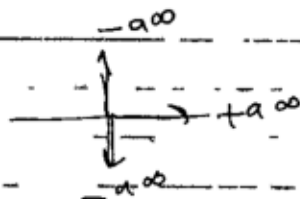
Hey: by sheer accident, this theorem probably has some bearing on the fact that $F(a, b; c; z)$ fails to converge in a certain sector. ^{as $|z| \rightarrow \infty$} Come back to this at some future time. By the way, Erdelyi's disappointing letter came today on this subject.

At this point comes a 7 page proof of all these results, with good example picture. Then Theorem 11.3.2 does the special case that $h(\phi) = -\infty$.

detail: consider e^{az^2} $a = \text{real}$. Then $|e^{az^2}| = e^{a(x^2 - y^2)}$
and $\lim_{|z| \rightarrow \infty} \frac{|e^{az^2}|}{|z|} = \frac{a(x^2 - y^2)}{\sqrt{x^2 + y^2}}$

For $\phi = 0$, get $\frac{ax^2}{x} = ax$

For $\phi = \pm i\frac{\pi}{2}$ get $\frac{-ay^2}{y} = -ay$



So here, $h(\phi)$ ranges from $-\infty$ to $+\infty$ if $a > 0$.

example: $g(w) = \frac{1}{\Gamma(1+\alpha w)}$ $\alpha > 0$. The gamma function

$\ln |g(w)| = \text{such that } h(\frac{\pi}{2}) = h(-\frac{\pi}{2}) = \alpha \frac{\pi}{2}$ (yes!)

and $h(\phi) = -\infty$ elsewhere i.e. $|\phi| < \frac{\pi}{2}$ of course!

$$\frac{1}{\Gamma(w)} \sim w^{-w} \sim |w|^{-|w|} \text{ so}$$

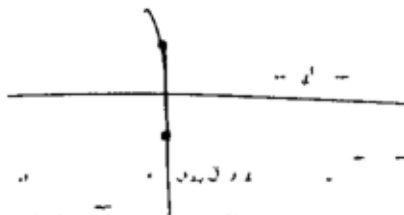
$$\frac{\ln \left(\frac{1}{|\Gamma(w)|} \right)}{|w|} = \frac{-|w|(\ln |w|)}{|w|} = -\ln |w| \rightarrow -\infty$$

example: $g(w) = \sin \pi w$. $\ln |g(w)| = \ln |\sin \pi w|$

$= \ln |e^{i\pi w} - e^{-i\pi w}|$ well as $|w| \rightarrow \infty = e^{\pm i\pi w}$ so $| | = e^{\pm \pi |w| \sin \phi}$

$\Rightarrow \ln = \pi |w| \sin \phi \Rightarrow h(\phi) = \pi \sin \phi$ for $|\phi| < \frac{\pi}{2}$

Claim is that $K =$



Theorem 11.3.3 of Carleson:

Conditions: (a) $g(w)$ = analytic in $\text{Re}(w) > -\frac{1}{2}$, expo type

(b) $h(\pm \frac{\pi}{2})$ both $\leq \pi$ (not $< \pi$).

(c) K lies inside $\frac{+\pi}{- \pi}$

(d) $\text{sech} \pi n \times g(-\frac{1}{2} + i n) \in L_2(-\infty, \infty)$

(e) $g(n) = 0 \quad n = 0, 1, 2$

Conclusion: $g(w) \equiv 0$.

OK, but not really the way I want it. Can't I replace (c)

with $|h(\phi)| < \pi$ a smolting.

Maybe my Carleson Theorem is a weaker version of the above. Eg,

suppose $g(z) < e^{+\pi z}$. Then

$$\int_{-\infty}^{\infty} \text{sech} \pi n \cdot g(-\frac{1}{2} + i n) dn$$

looks at the ends like, say $n \rightarrow +\infty$

$$\text{ch} \pi n = e^{\frac{\pi n}{2}} + e^{-\frac{\pi n}{2}} \sim e^{\frac{\pi n}{2}}$$

$$\text{sech} \pi n \rightarrow e^{-\frac{\pi n}{2}}$$

Then $\sum_{n=0, \infty} e^{-\frac{\pi n}{2}} e^{+\pi n}$ would converge.

So, $\left[g(w) < e^{+\pi |w|} \text{ if } \phi = \pm \frac{\pi}{2} \text{ will cause (d) to be satisfied.} \right.$
 $\left. (\phi = \arg(w)). \right]$

Also, $g(w) < e^{+\pi|w|}$ for $\phi = \pm i\frac{\pi}{2} \Rightarrow$ ~~for~~

Tells you nothing about the width. But suppose $g(w) = e^{-2\alpha w}$
 i.e., $|g(w)| < e^{+\pi|w|}$

$$\Rightarrow \frac{\ln |g(w)|}{|w|} < \pi \rightarrow h(\phi) < \pi$$

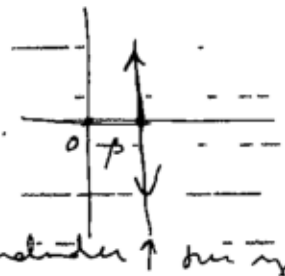
But we could have $h(\frac{\pi}{2}) + h(-\frac{\pi}{2}) = w(\tilde{k}) = \frac{\pi}{2} + -3\alpha \dots$

Okay, but p55 says cannot be negative! So $w(\tilde{k}) < 2\pi$ since $h(\phi) < \pi$. Thus, it will fit into (c)

So $g(w) < e^{+\pi|w|}$ for $\phi = \pm \frac{\pi}{2}$ causes (b) and (c) to be satisfied

Next, $g(w)$ must be $\leq e^{\alpha|w|}$ under to be "expo type". But
 this means $\text{Re}(w) \geq \text{some } \beta$.

def expo type: $|g(w)| \leq e^{\alpha|w|}$ for $|w| \geq \text{some } R$
 $\text{Re}(w) \geq \text{some } \beta$.



β unimportant.

So, if $\alpha = 2\pi$, then $\alpha = 2\pi$ for $\phi = \pm \frac{\pi}{2}$, violates (b), (c), (d).

Need $\alpha < \pi$!

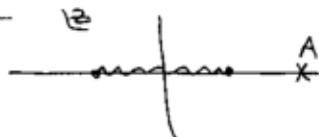
So $g(w) < e^{+\pi|w|}$ for $|\phi| \leq \frac{\pi}{2}$ and $g(w)$ analytic $\text{Re}(w) \geq -\frac{1}{2}$
 cause (a) to be satisfied

The $-\frac{1}{2}$ is a new element here, can probably be relaxed via a weaker theorem.

Note: Calculation given as problem on p259.

Complex Variable: a study of $f(z) = \sqrt{z^2 - 1}$. The phase question.

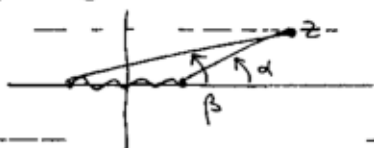
① The cut structure is



② Choose positive square root at A. This specifies one of the sheets.

③ When $z = -50$, what is the phase of $f(z)$? Naively you'd guess $f(-50) \cong +50 = \text{real}$, I think this is wrong.

④ The only way I know to do this is setup two angles.



$$z-1 = |z-1| e^{i\alpha}$$

$$z+1 = |z+1| e^{i\beta}$$

$$\alpha = \arg(z-1)$$

$$\beta = \arg(z+1)$$

$$\text{Then } \sqrt{z^2 - 1} = \sqrt{(z+1)(z-1)} = \sqrt{|z+1||z-1|} e^{i\alpha/2} e^{i\beta/2}$$

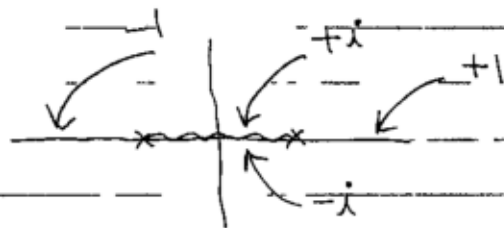
$$= |\sqrt{z^2 - 1}| e^{i(\alpha+\beta)/2} \quad \checkmark$$

When $\alpha = \beta = 0$, $z > 1$ you get $\pm \text{real}$, as desired.

On top of the cuts $\beta = 0$, $\alpha = \pi \Rightarrow \text{phase} = e^{i\pi/2} = +i$

On left side, $\alpha = \pi$, $\beta = \pi$, $\text{phase} = e^{i\pi} = (-1)$

⑤ Picture shows phases when $z = \text{real}$:



⑥ Therefore, if you stay on the sheet specified, then

$$f(-50) \approx -50$$

⑦ This agrees with

$$\lim_{|z| \rightarrow \infty} \sqrt{z^2 - 1} = z \quad \checkmark$$

A converse of Carlson's Theorem.

1. First we restate the theorem:

(a) $f(z)$ is analytic in $\text{Re}(z) > 0$.

(b) there exists an R such that, for all $|z| > R$ with $\arg(z)$ in RHP range,
 $|f(z)| < \exp(k|z|)$ where $k < \pi$.

(c) $f(z)$ is Carlson.

Theorem says: (a) and (b) $\Rightarrow$ (c).

2. What is the converse? $\text{not. (c)} \Rightarrow \text{not. (a and b)}$
 $\Rightarrow (\text{not. a}) \text{ and/or } (\text{not. b})$

Suppose (a) is still true. Then $\text{not. c} \Rightarrow \text{not. b}$.

The idea is that if you have a set of conditions $a, b, \dots$ which imply x , then $\text{not. } x$ says that at least one of those conditions is false. If you assume that all the conditions except one are true, then $\text{not. } x$ implies that that one unassumed condition is false.

3. So if (a) is true, then $f(z)$ is not Carlson means there exists no ~~any~~ R such that for $|z| > R$ $|f(z)| < \exp(k|z|)$ where $k < \pi$. Therefore no matter what R you choose, this must be false:

$$|f(z)| < \exp(k|z|) \text{ where } k < \pi \text{ for all } |z| > R$$

In other words, this must be false:

$$|z| > R \Rightarrow |f| < \exp(k|z|) \text{ with } k < \pi$$

I am getting confused. Here is theorem I want to prove:

~~There exists an R such that for $|z| > R$ $|f(z)| < \exp(k|z|)$ for any $k < \pi$.~~ Or maybe: for

any R , there exists a point z_1 such that $|f(z_1)| > e^{k|z_1|}$ for all $k < \pi$.

If there were no such point, then $|f(z)|$ would be less than $e^{k|z|}$ for all $|z| > R$ for all $k < \pi$, and then f would be Carlson. This is OK, forget the logic.

Functions of Exponential Form.

① definition: $f(z)$ is of "expo form" if $|f(z)| < e^{k|z|}$ for $|z| > \text{some } R$, for some real k , for some range of $\arg(z)$.

② Theorem: if $|f(z)| < e^{k|z|} \forall |z| > R$, then $\exists k'$ very close to k such that $|f(z)| < A \cdot e^{k'|z|}$ for $A = \text{any positive constant}$.

proof: if $A \geq 1$, choose $k' = k$, QED. For $A < 1$, choose $k' = k + \frac{1}{R} \ln\left(\frac{1}{A}\right)$.

Then $|f| < e^{k|z|} \not\leq e^{k'|z|}$ (since $k' > k$) = $e^{k|z|} \cdot e^{\frac{|z|}{R} \ln\left(\frac{1}{A}\right)}$
 Then $|f| < e^{k|z|} = e^{k'|z|} \cdot e^{-\frac{|z|}{R} \ln\left(\frac{1}{A}\right)} = A e^{k'|z|}$ QED.

③ Theorem: from the above, $|f| < e^{k|z|} \not\leq \frac{1}{n} e^{k|z|}$ for $|z| > R$, where $n > 1$. I.e., you have to increase k a little.

④ Theorem: if $|f| < |z|^a e^{k|z|}$, then $\exists k'$ near $k \Rightarrow |f| < e^{k'|z|}$, all for $|z| > R$.

proof: have $\ln|f| < a \ln|z| + k|z|$
 want $\ln|f| < k'|z|$

So need: $a \ln|z| + k|z| < k'|z| \Rightarrow \text{need } (k' - k) > a \frac{\ln|z|}{|z|}$

But $\frac{\ln R}{R} > \frac{\ln|z|}{|z|}$ so if $(k' - k) = a \frac{\ln R}{R}$, then $(k' - k) > a \frac{\ln|z|}{|z|}$

Ergo: choose $k' = k + a \frac{\ln R}{R}$. (If $a < 0$, choose $k' = k$).

⑤ Theorem: (version of ②): if $|f| < C e^{k|z|}$, then $\exists k' \Rightarrow |f| < e^{k'|z|}$.

Namely, $k' = k + \frac{1}{R} \ln C$. If $C < 1$, choose $k' = k$.

⑥ definition: $f(z)$ is "Carlson" if $|f(z)| < e^{k|z|}$ for some $k < \pi$, $|z| > R$, $k < \pi$, plus other conditions. Here we emphasize: "There exists an R such that $\forall |z| > R, |f| < e^{k|z|}$ for some $k < \pi$."

⑦ Theorems: if $|f(z)| \leq C e^{k|z|}$, or $|f| \leq |z|^a e^{-k|z|}$ for $k < \pi$, then (+ other conditions) $f(z)$ is still Carlson!

proof: in either case we can find $k' \geq |f(z)| < e^{k'|z|}$ if $\pi - k = \epsilon$, then we can find huge R such that $(k' - k) = \epsilon/2$ so that $k' < \pi$ also. Thus, $\exists R$ such that $|f| < e^{k'|z|}$, $k' < \pi$, so $f(z)$ is Carlson.

Theorem: Let $F(z, n)$ be defined for $n = 0, 1, 2, \dots, \infty$. Assume that $|F(z, n)| < C_n e^{+k|z|}$ for $|z| > R$, $k = \text{real constant}$, $C_n = \text{set of positive real constants such that } \sum_{n=0}^{\infty} C_n \text{ converges to some } C$. Define $f(z) = \sum_{n=0}^{\infty} F(z, n)$ and assume this converges absolutely (+ uniformly) for $|z| > R$. Then: $|f(z)| < C e^{+k|z|}$

proof: $f(z) = \sum_{n=0}^{\infty} F(z, n)$ abs conv.

$$\Rightarrow |f(z)| \leq \sum_{n=0}^{\infty} |F(z, n)|$$

For $|z| > R$, $|F(z, n)| < C_n e^{k|z|}$ so

$$|f(z)| \leq \sum_{n=0}^{\infty} C_n e^{k|z|} \quad \text{by "series comparison test"}$$

$$= e^{k|z|} \cdot C \quad \blacksquare$$

comments: if $|F(z, n)| < C_n e^{+k|z|}$ where $\sum_{n=0}^{\infty} C_n \rightarrow C$, then

$$\sum |F| \leq C \quad \forall |z| > R \text{ so series must converge}$$

uniformly for $|z| > R$; uniformity is redundant assumption,

reversal: if $|F(z, n)| < D_n e^{+k|z|}$ and $\sum F$ converges abs and uniformly $|z| > R$, then $|f| \leq \sum |F| < \sum D_n e^{+k|z|} < e^{k|z|} (\sum D_n)$,

you may not conclude that $\sum D_n$ converges! (Eg,

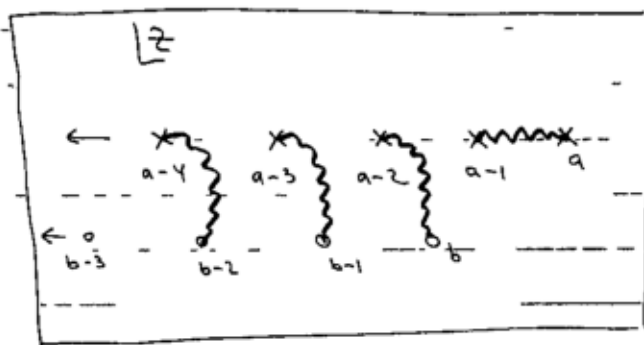
$$|F| \leq \frac{10}{n^3} e^{k|z|} \Rightarrow |f| \leq \sum \frac{10}{n^3} e^{k|z|} \text{ but } \sum \frac{10}{n^3} \rightarrow \infty \text{). So } \sum C_n = C$$

is essential.

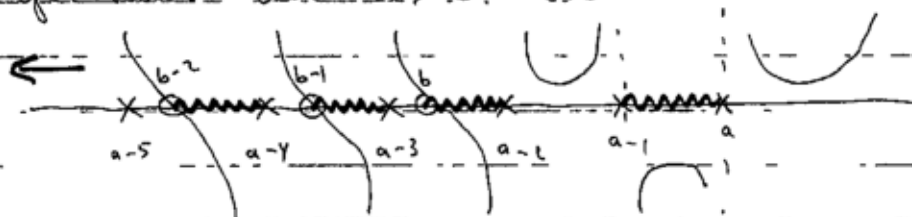
$$\boxed{\left(\sum C_n = C \right) \Rightarrow \left(\text{uniformity} \right) \text{ on } |z| > R}$$

Case study: $f(z) = \sqrt{\frac{\Gamma(z-a)}{\Gamma(z-b)}}$

- ① Assume general a, b then:
How do you draw
the cuts? Anyway you like

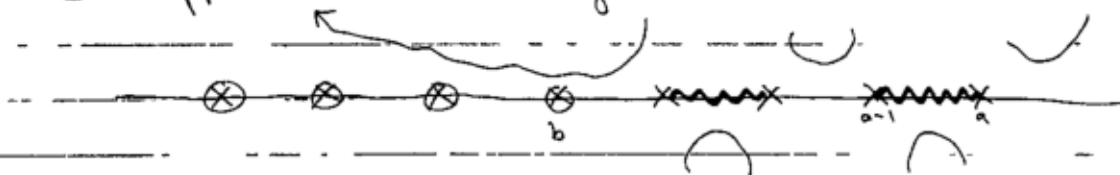


- ② Standard way: use fact that $\Gamma(z)$ is real analytic. Let a, b be real for now. Let $a > b$. Then

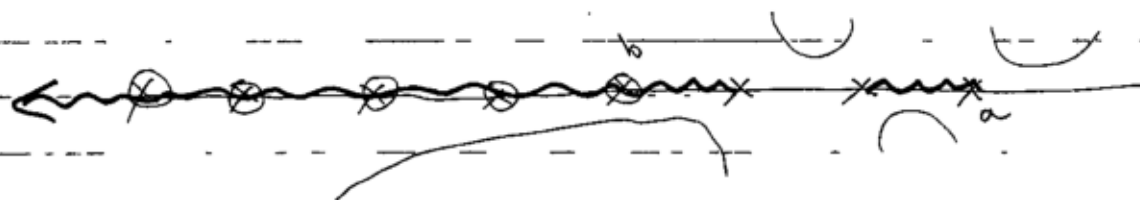


We start with a, b real + set up all cuts on real axis. Then
move a, b wherever you like + cuts move accordingly.
They keep going off to the left.

- ③ Suppose $a-b = \text{integer}$. Then



$a-b = \text{even}$



$a-b = \text{odd}$

This corresponds to:



- ④ The idea is: given a, b
choose cuts in some definite way.

Cuts: $f(z) = \left[\frac{\Gamma(z-a)}{\Gamma(z-b)} \right]^{\frac{1}{2}}$ (more)

① One approach is to use $\Gamma(x+1) = x\Gamma(x)$ so that

$$\Gamma(z-a) = \frac{1}{(z-a)} \Gamma(z-a+1) = \frac{1}{(z-a)(z-a+1)} \Gamma(z-a+2) \text{ etc.}$$

Then, eg, we have

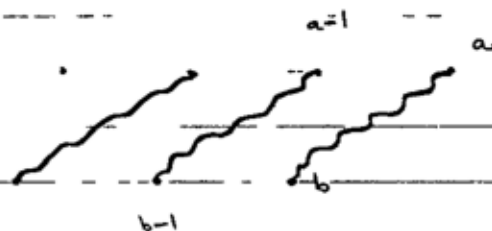
$$f(z) = \left[\frac{(z-b)(z-b+1)\Gamma(z-a+2)}{(z-a)(z-a+1)\Gamma(z-b+2)} \right]^{\frac{1}{2}}$$

This reveals 4 of the singularities. If we group them as

$$= \left[\frac{(z-b)}{(z-a)} \right]^{\frac{1}{2}} \left[\frac{z-b+1}{z-a+1} \right]^{\frac{1}{2}} \left[\frac{\Gamma(z-a+2)}{\Gamma(z-b+2)} \right]^{\frac{1}{2}}$$

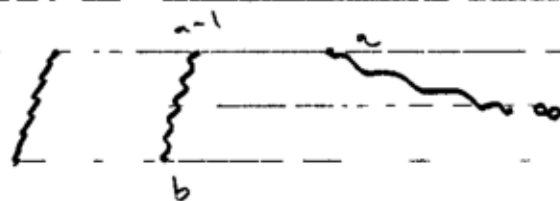
Then you would draw:

Clearly, you could keep on going in this way.



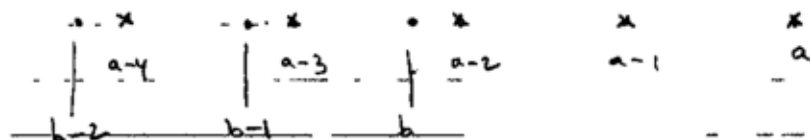
Many other choices. You could have done:

$$f(z) = \left[\frac{1}{(z-a)} \right]^{\frac{1}{2}} \left[\frac{(z-b)}{(z-a+1)} \right]^{\frac{1}{2}} \left[\frac{\Gamma(z-a+2)}{\Gamma(z-b+1)} \right]^{\frac{1}{2}}$$



You really don't know how cuts go until you have the elementary factors.

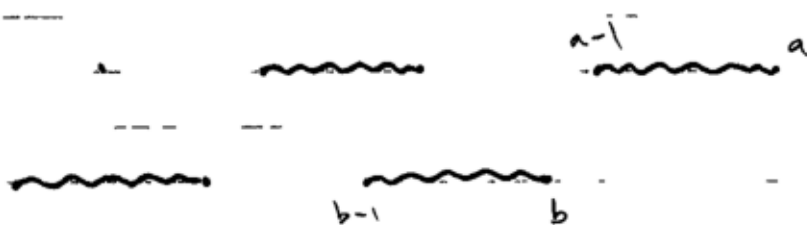
② Suppose a and b are both real. HOLD.



③ Here is still another choice:

$$f(z) = \frac{\sqrt{(z-b)(z-b+1)}}{\sqrt{(z-a)(z-a+1)}} \cdot \sqrt{\frac{P(z-a+1)}{P(z-b+2)}}$$

Notice that $\sqrt{(z-a)(z-a+1)}$ is ~~not~~ has no branching at ∞ .



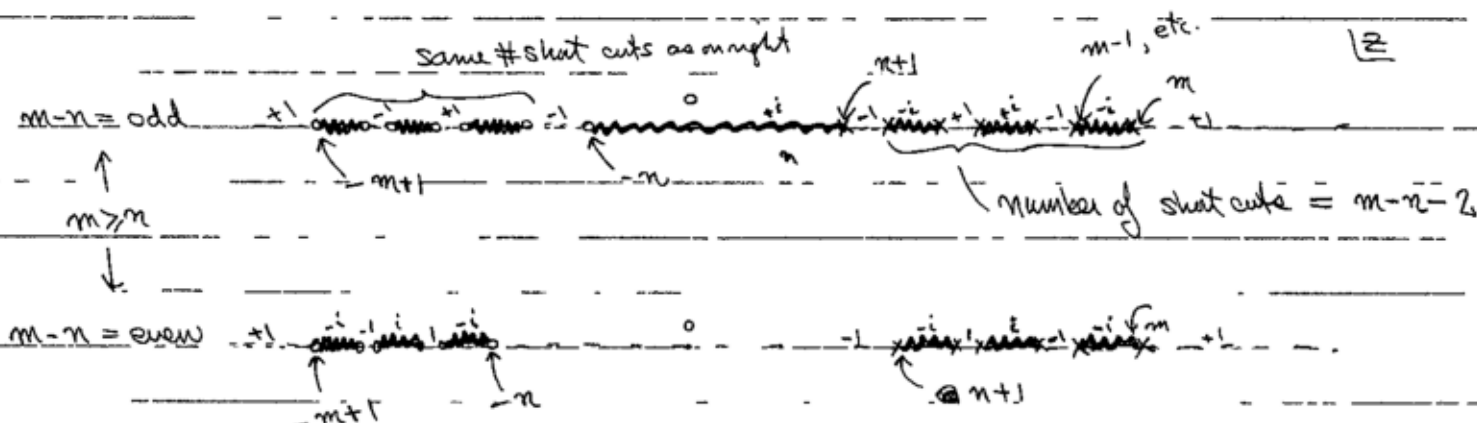
So here is another reasonable way to do your cuts.

The cut structure of the function:

$$f(z) = + \left[\frac{\Gamma(z+m)\Gamma(z-n)}{\Gamma(z+n)\Gamma(z-n)} \right]^{\frac{1}{2}}$$

1. I do not have the patience to draw all the special cases I have worked out. Instead, let us consider just two cases. In both these cases, we shall assume that both m and n are integers. However, the result is probably just the same for m and n both half integers.

2. In the case $m > n$ we get two kinds of pictures depending on whether the difference is even or odd. Namely,



3. Suppose n is larger than m . Then the picture looks just the same except everywhere you see a label m replace it with n , and vice versa. I have worked out an example of this but you can do it anyway by simply inverting the inside of the bracket. This inversion does not change the position of the branch points. The x marks poles, the o mark zeroes.

4. Phasing. From experience with simpler functions, imagine starting on the far right where we choose the phase at zero. Then move left. As you cross over each zero, you gain 90° , so I imagine that with each pole you lose 90° . See labels on the pictures. It would appear that if you start with $+1$ on the far right, you also have $+1$ on the far left, unlike the simpler function (z^2-1) .

These phase claims are born out if you just take the large z limit of the function. You get simply 1. This is the same in all directions. Thus, you must have plus phase in the negative direction.

5. Compare with earlier example: $f(z) = + \sqrt{z^2-1}$ $z > 0 \Rightarrow f(-z) = - \sqrt{(-z)^2-1}$ $z > 0$
Here we picked up a $(-)$ phase going from far right to far left. But in the present example this is not so: $= - f(z)$

$$f(-z) = + \left[\frac{\Gamma(-z+m)\Gamma(-z-n)}{\Gamma(-z+n)\Gamma(-z-n)} \right]^{\frac{1}{2}} = \left[\frac{(z-n)!}{(z-m)!} (-1)^{m-n} \frac{(z+n)!}{(z+m)!} (-1)^{m-n} \right]^{\frac{1}{2}}$$

$$= + \left[\frac{(z+n)!(z-n)!}{(z+m)!(z-m)!} \right]^{\frac{1}{2}} = + \left[\frac{\Gamma(z+n)\Gamma(z-n)}{\Gamma(z+m)\Gamma(z-m)} \right]^{\frac{1}{2}}$$

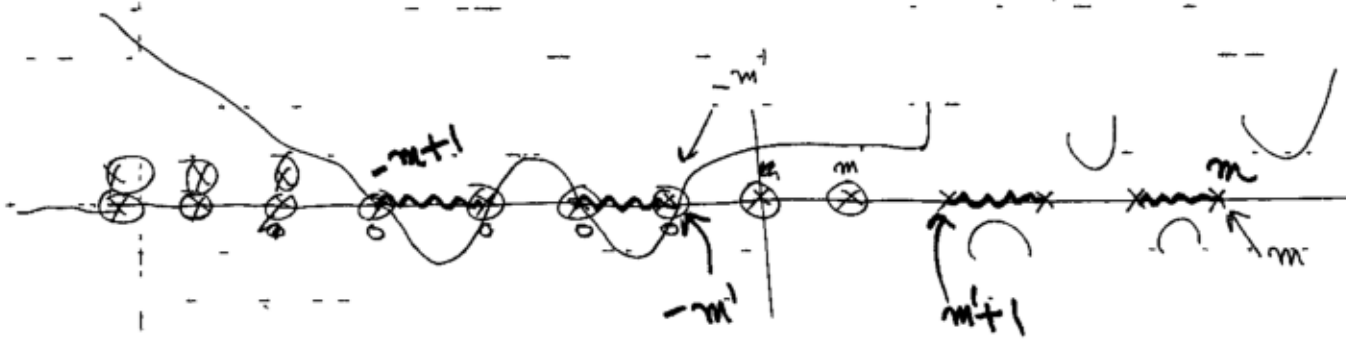
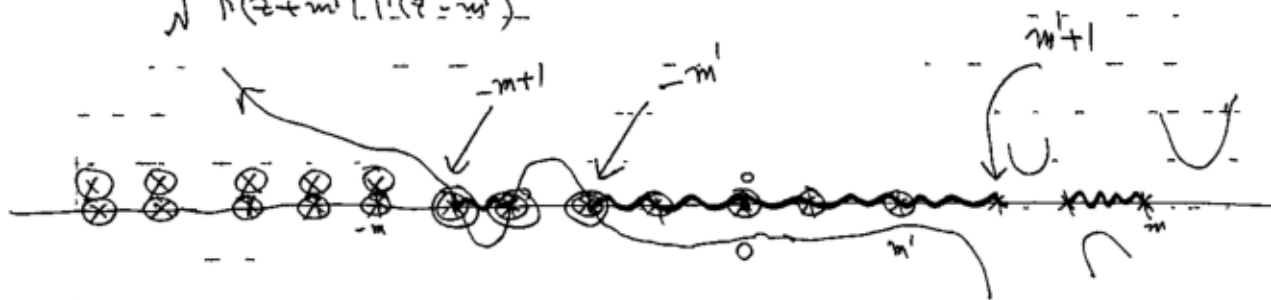
6. Application:

$$\left[\frac{\Gamma(z+m)\Gamma(z-n)}{\Gamma(z+n)\Gamma(z-m)} \right]^{\frac{1}{2}} \Rightarrow {}_m g(-z-1) = + \left[\frac{(z+n)!(z-n)!}{(z+m)!(z-m)!} \right]^{\frac{1}{2}} = {}_m g(z)$$

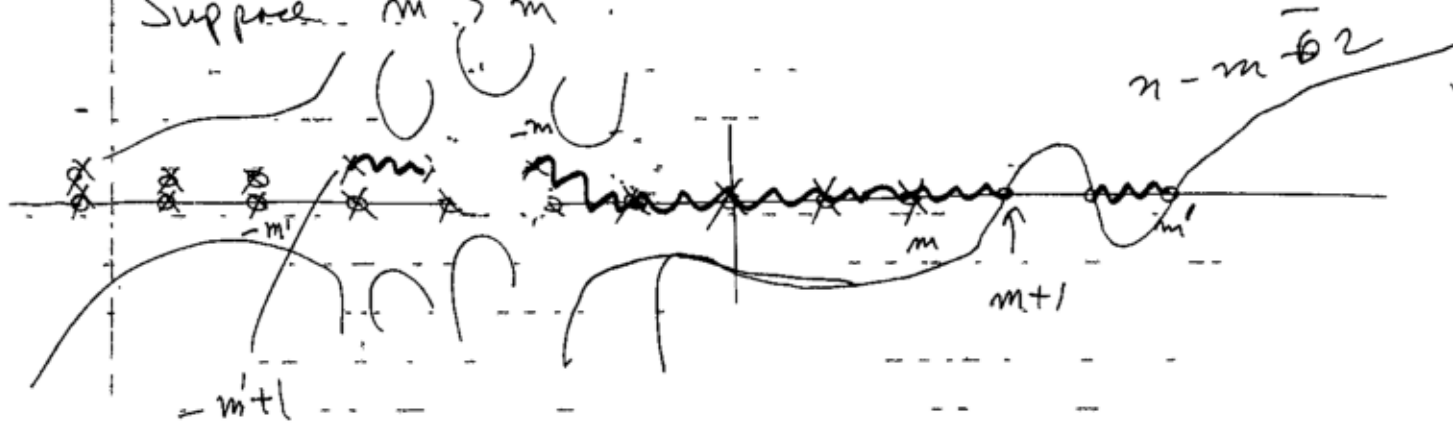
7. I suspect you can make this argument: if this little relation holds for large z , and if both sides are analytic in z aside from cuts, then relation must hold for all z . Hence:

$$\left[{}_m g(-z-1) = {}_m g(z) \right]$$

$$\sqrt{\frac{\Gamma(z+m)\Gamma(z-m)}{\Gamma(z+m')\Gamma(z-m')}} \quad (m' > m)$$



Suppose $m' > m$?



Large z limit

$$\sqrt{\dots} = \left[(z)^{-m-m'} (z)^{-m+m'} \right]^{\frac{1}{2}} = \left[(z)^0 \right]^{\frac{1}{2}} = 1$$

Analytic probe of $\sqrt{G_{mn}^j}$ where $m-n = \text{integer}$. (D_k^+ in mind)

①
$$\sqrt{G_{mn}^j} = \left[\frac{\Gamma(j+1+m)}{\Gamma(j+1-m)} \frac{\Gamma(j+1-n)}{\Gamma(j+1+n)} \right]^{\frac{1}{2}}$$

If we follow the previous sheet, then when $m-n = \text{odd}$, the phase of the inside of $[\]^{\frac{1}{2}}$ is not 0° but is probably $\pm i$ for j above cut. [rule: you gain 90° going over a pole]

So first surprise is: $\sqrt{G_{mn}^j}$ is not defined for $j = \text{real}$, in D_k^+ region, unless you say $j \pm i\epsilon$! (When $m-n = \text{odd}$).

Also, when $m-n = \text{even}$, phase of ~~inside of~~ $[\]^{\frac{1}{2}}$ is $\textcircled{-1}$.

② Can this be clarified via:

$$\frac{\Gamma(j+1-n)}{\Gamma(j+1-m)} = \frac{\Gamma(-j+m)}{\Gamma(-j+n)} \frac{\sin \pi(-j+m)}{\sin \pi(-j+n)}$$

$$\left[\frac{\Gamma(j+1+m)\Gamma(-j+m)}{\Gamma(j+1+n)\Gamma(-j+n)} \frac{\sin \pi(-j+m)}{\sin \pi(-j+n)} \right]^{\frac{1}{2}} \quad \text{so what!}$$

③ Should we "cut" $\sqrt{G_{mn}^j}$ differently for D_k^+ ? No!

④ Conclusion so far:

$$\sqrt{G_{mn}^{j \pm i\epsilon}} = \begin{cases} (\pm i) \sqrt{G_{mn}^j} & m-n = \text{odd} \\ -\sqrt{G_{mn}^j} & m-n = \text{even} \end{cases}$$

The j -cut structure of $\sqrt{G_{\mu\nu}^j}$

$$\textcircled{1} \quad \sqrt{G_{\mu\nu}^j} = \sqrt{\frac{\Gamma(j+\mu+1)\Gamma(j-\nu+1)}{\Gamma(j-\mu+1)\Gamma(j+\nu+1)}}$$

$$\sqrt{G_{\mu\nu}^{-j-1}} = \sqrt{\frac{\Gamma(-j+\mu)\Gamma(-j-\nu)}{\Gamma(-j-\mu)\Gamma(-j+\nu)}} =$$

$$= \sqrt{\frac{\Gamma(j+\mu+1)\Gamma(j-\nu+1)}{\Gamma(j-\mu+1)\Gamma(j+\nu+1)} \times \frac{\sin\pi(-j-\mu)\sin\pi(-j-\nu)}{\sin\pi(-j+\mu)\sin\pi(-j-\nu)}}$$

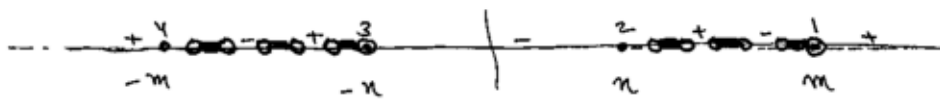
$$\text{I.e.,} \quad \sqrt{G_{\mu\nu}^{-j-1}} = \sqrt{\cancel{G_{\mu\nu}^j}} = \sqrt{(-1)^{\mu+\nu}} \sqrt{G_{\mu\nu}^j}$$

$\textcircled{2}$ OK, it is very hard to work with $\sqrt{G_{\mu\nu}^j}$ unless μ, ν are lattice. Then we get the analysis given earlier, namely:

$$\boxed{\sqrt{G_{\mu\nu}^{-j-1}} = (-1)^{\mu+\nu} \sqrt{G_{\mu\nu}^j} \quad \mu, \nu \in \text{lattice.}}$$

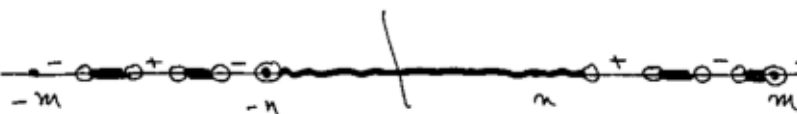
Analysis of $h(z) = + \left[\frac{\Gamma(z+m) \Gamma(z-n)}{\Gamma(z-m) \Gamma(z+n)} \right]^{\frac{1}{2}}$ fast method

① Nice to take $m > 0$ and $m > |n|$. In fact I'll take $n > 0$ also here. These orders is:



② Numbers show the order of activation. At $z=m$ we have 0. Then zeroes until cancellation at $z=n$. Then $m, \frac{1}{2}$ always cancel so forget them. At $z=-n$ start new string of zeroes. Repeat of R.H.S. Finally, draw in same rule. Case shown has $m-n = \text{even}$.

③ For $m-n = \text{odd}$, get



④ Phases: gain 90° over each 0. Thus, you get $-$ in case $(m-n) = \text{odd}$.

⑤ Therefore, if

$$k(z) = + \left[\frac{(z+m)! (z-n)!}{(z-m)! (z+n)!} \right]^{\frac{1}{2}}$$

$z > 0$

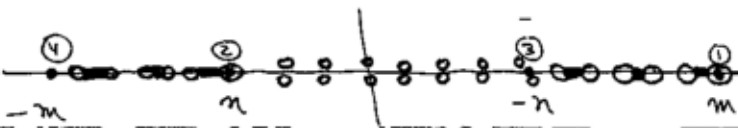
Then

$$k^{mn}(-z-1) = (-1)^{m-n} \left[\frac{(z+m)! (z-n)!}{(z-m)! (z+n)!} \right]^{\frac{1}{2}} = (-1)^{m-n} k^{mn}(z), \quad z > 0.$$

⑥ This is the same as in Andrews & Gunson:

$$k^{mn}(-z-1) = (-1)^{m-n} k^{mn}(z)$$

⑦ Suppose $m > 0, m > |n|$, but $n < 0$. Then

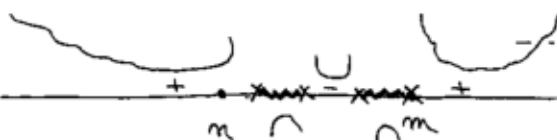


$A(z)$ is a simple zero of $h(z)$, hence analytic. Same phase analysis as before.

⑧ I assume result is same $\forall m, n$.

Another function: $h(z) = \left[\frac{\Gamma(m-z)}{\Gamma(n-z)} \right]^{\frac{1}{2}}$

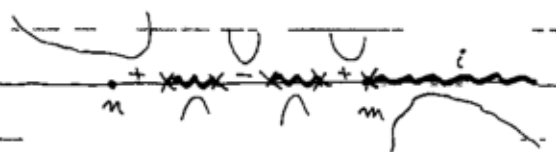
① Let $m \geq n$ and analyze.



$$\lim_{z \rightarrow -\infty} \left[\frac{\Gamma(m+z)}{\Gamma(n+z)} \right]^{\frac{1}{2}} = \left[\frac{1}{z} \right]^{\frac{m-n}{2}} = \left[(-z) \right]^{\frac{m-n}{2}}$$

always + on left, but alt sign on right

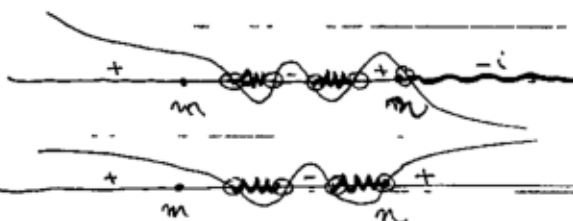
Thus, the phase of this function on the positive real z axis is either 1 or i .



② Alternate point of view: $h(z) = \left[\frac{\Gamma(z+1-n)}{\Gamma(z+1-m)} \right]^{\frac{1}{2}} \cdot (-1)^{\frac{m-n}{2}}$. As this shows, the ratio $\frac{\Gamma}{\Gamma}$ is + at large real z ; but $(-1)^{m-n}$ modulates sign of $[\]$. Perhaps $(-1)^{m-n} = e^{+i\pi(m-n)}$. These example gives $e^{+i\pi \cdot 5/2} = +i$. ✓ Does this same form work for $m < n$?

③ Case $n \geq m$.

$$e^{+i\pi(m-n)/2} = e^{+i(-5)/2} = -i \quad \checkmark$$



④ So, we tentatively conclude that our choice of phasing is consistent with

$$h(z) = e^{\pm i\pi(m-n)/2} \left[\frac{\Gamma(z+1-n)}{\Gamma(z+1-m)} \right]^{\frac{1}{2}}$$

where this is + when z is +. This I have already been analyzing.

On the left, it should have phase $\Rightarrow h(z) = \text{real}$.

Simple discontinuities,

① Let $f(z) = (z-a)^\alpha$ and cut as

Then:

$$\text{disc}_{z < a} (z-a)^\alpha = |z-a|^\alpha \left(e^{i\theta} \right) \Big|_{-\pi}^{\pi} = \underline{2i \sin \pi \alpha (a-z)^\alpha}$$

Once you say $(z-a)^\alpha = |z-a|^\alpha e^{i\theta\alpha}$

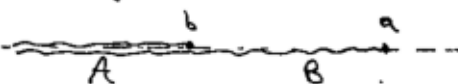
You have: $\text{disc} = 2i \sin \pi \alpha$

I.e., $z-a = \text{real} \Rightarrow f(z)$ real + when $z-a > 0$.

This is real analytic.

② Let $f(z) = (z-a)^\alpha (z-b)^\beta$ where $\text{Re } a = \text{Re } b$ and $\text{Re } a > \text{Re } b$.

Then picture is



Clearly:

$$\text{disc}_B f(z) = (z-b)^\beta \cdot 2i \sin \pi \alpha (a-z)^\alpha$$

But look at region A.

$$z-a = |z-a| e^{i\theta_1}$$

$$z-b = |z-b| e^{i\theta_2}$$

$$\text{Then } \text{disc}_A f(z) = |z-a|^\alpha |z-b|^\beta e^{i\theta_1\alpha + i\theta_2\beta} \Big|_{\theta_1=\theta_2=\pi}^{\theta_1=\theta_2=-\pi}$$

$$= \underline{2i \sin[\pi(\alpha+\beta)] (a-z)^\alpha (b-z)^\beta}$$

③ Repeat ② by adding discs.

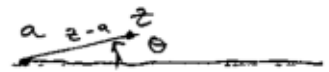
$$\text{disc}_B f(z) = (z-b)^\beta 2i \sin \pi \alpha (a-z)^\alpha + (z-a)^\alpha 2i \sin \pi \beta (b-z)^\beta$$

$$\text{But } (z-b)^\beta = (b-z)^\beta e^{-i\pi\beta} \quad (z-a)^\alpha = (a-z)^\alpha e^{+i\pi\alpha}$$

$$S_0 = (a-z)^\alpha (b-z)^\beta \cdot 2i \cdot \left[\underbrace{\sin \pi \alpha e^{-i\pi\beta}}_{\sin \pi(\alpha+\beta)} + \underbrace{\sin \pi \beta e^{+i\pi\alpha}}_{\sin \pi(\alpha+\beta)} \right]$$

So either way gives same.

① Let $f(z) = (a-z)^\alpha$. Then



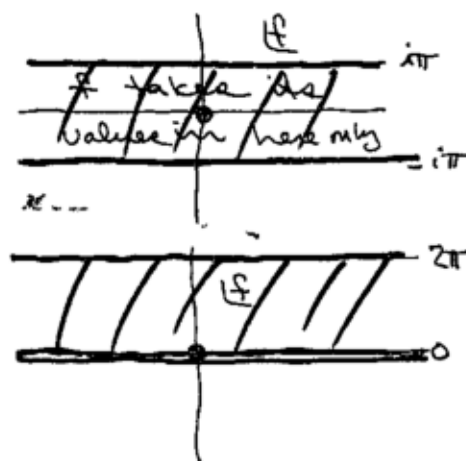
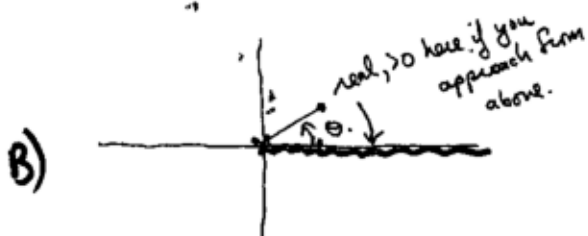
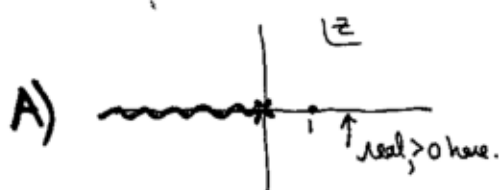
$$\text{disc } f(z) = |a-z|^\alpha e^{i\alpha(\theta-\pi)} \Big|_{2\pi-}^0 = |a-z|^\alpha (e^{-i\pi\alpha} - e^{+i\pi\alpha})$$

since
$$\begin{aligned} z-a &= |z-a| e^{i\theta} \\ a-z &= |z-a| e^{i(\theta-\pi)} \end{aligned} \quad = -2i \sin(\pi\alpha) |a-z|^\alpha$$

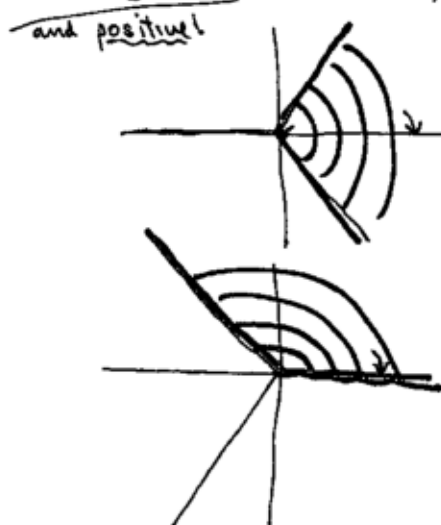
Thus: $\text{disc. } (a-z)^\alpha = -2i \sin(\pi\alpha) (z-a)^\alpha$

The meaning of principle value.

1. A reasonable definition of principle value might be this: choose your cut in such a way that the "principle value function" takes real, positive values somewhere on your selected sheet. The notion of principle value does not force you to draw your cut in any special way.
2. Example: for $f(z) = \ln(z)$, we can draw cut in lots of ways. Here are two standard ways:



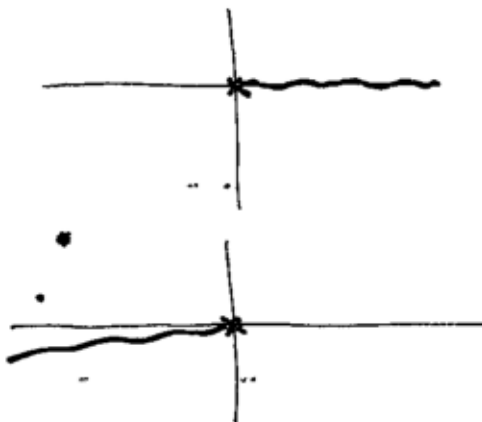
- 3/. Any other position of the cut is also acceptable. The important fact is that you want your "image"-region-to-include the real axis. Otherwise you are not on the "principle branch".
4. Here is another example: $f(z) = z^a$. Consider z -planes shown in A) and B) above. Again, the point here is to define the principle branch so that when z takes real positive values, the image is real. For $a=1/3$:



5. Another example:

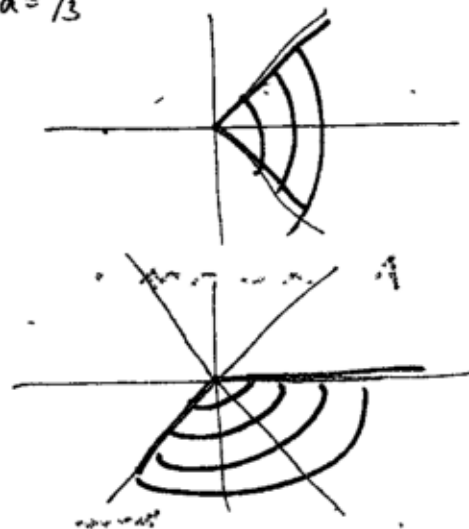
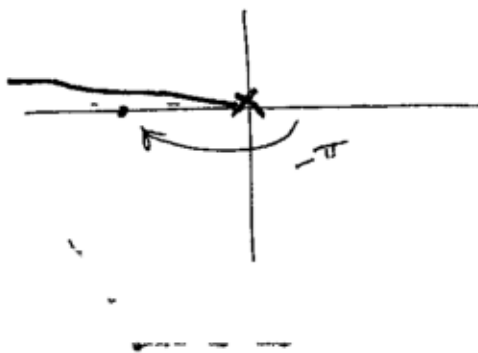
$$f(z) = (-z)^a$$

Be careful. In this case you must define principle branch by saying: when z lies on the negative real axis, the function is real positive. Again, you can draw the cut as you like. $a = 1/3$



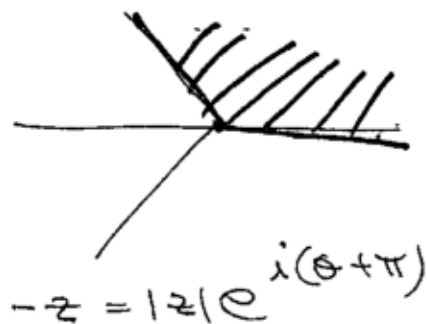
$$z = |z| e^{i175^\circ}$$

$$\Rightarrow -z = |z| e^{i(5^\circ)}$$



$$z = |z| e^{i\theta}$$

$$-z = |z| e^{i(\theta - \pi)}$$



$$-z = |z| e^{i(\theta + \pi)}$$

-i su

More cut analysis. how to connect things up

- ① Consider our usual $f(z) = \sqrt{(z+1)(z-1)} = \sqrt{z^2-1}$.

Is there a singularity at $z = \infty$? Naively, $f(z) \rightarrow z$, so not?
Actual calc:

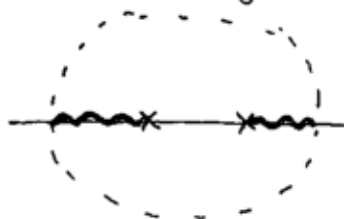
$$f\left(\frac{1}{t}\right) = g(t) = \sqrt{\left(\frac{1}{t}+1\right)\left(\frac{1}{t}-1\right)} = \sqrt{\frac{(1+t)(1-t)}{t}}$$

$\Rightarrow f(1/t) = t^{-1/2} \sqrt{1-t^2}$. This has a simple pole at $t=0$,
so $f(z)$ has simple pole at ∞ . Yes.

- ② There is not a branch point at ∞ ! Thus, you can draw:



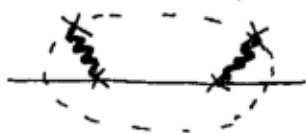
But you can deform this cut. Recall that "point ∞ " is all of "large circle". Thus you can say:



- ③ The usual way for doing this is $\uparrow$
Note that cut $\rightarrow \infty \not\Rightarrow$ branch points at ∞ . Here, the cut "passes through" $z = \infty$ like it would pass through any other point.

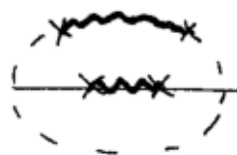
- ④ Now consider $f(z) = (z+1)^a \cdot (z-1)^b$

Here, you must draw bp's internally for each of the product functions:



Each one has, in general, two bp's.

If $a \rightarrow$ integer, then one entire cut vanishes.
It is illogical to draw this as:
because as $a \rightarrow$ integer, you get nonsense!



Just for fun, we could say

$$f(z) = \left(\frac{z+1}{z-1} \right)^{a-b}$$

In this case, $f(z) \rightarrow 1$ so no bp at ∞
and only way to draw it is:

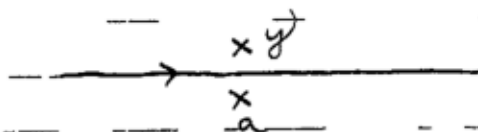


Now as $a-b \rightarrow \text{integer}$, both bp's vanish together
which is OK.

Example of contour pinching.

① How about: $f(y) = \int_{-\infty}^{\infty} \frac{dx}{(x-a)(x-y)}$ $a = \text{fixed}$

Assume looks like this:



Since $f(x) \rightarrow x^{-2}$, we can close either way. Close up to get:

$$f(y) = \oint_{\gamma} \frac{dx}{(x-a)(x-y)} = 2\pi i \times \frac{1}{(y-a)}$$

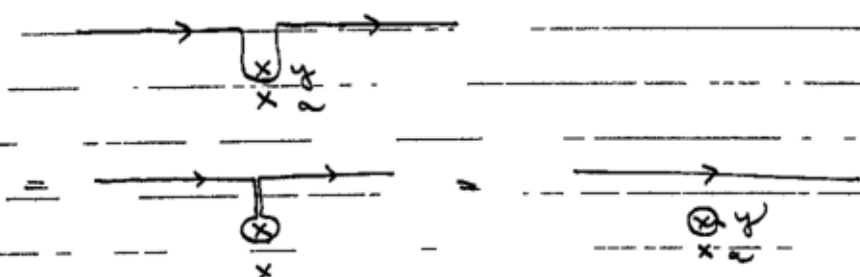
Close down to get

$$f(y) = \oint_{\gamma} \frac{dx}{(x-a)(x-y)} = -2\pi i \times \frac{1}{(a-y)}$$

} same either way.

② Now let $y \rightarrow a$. What happens? Solution shows clearly a pole, but we want to see this without "doing" the integral.

Just prior to contact,



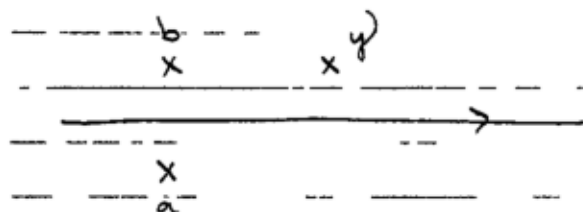
$$\text{So: } f(y) = \underbrace{\int_{-\infty}^{\infty} \frac{dx}{(x-a)(x-y)}}_{\text{background}} + \underbrace{\oint_{\gamma} \frac{dx}{(x-a)(x-y)}}_{\text{pinched contour}}$$

In my example this happens to vanish, but in general this is not so. This integral becomes background.

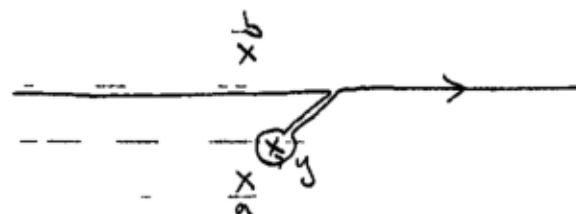
This then gives $2\pi i \times \frac{1}{(y-a)}$ and reveals the pole

Better example:

$$f(y) = \int_{-\infty}^{\infty} \frac{dx}{(x-a)(x-b)(x-y)}$$



Now let $y \rightarrow a$ to get



Thus, we find that as $y \rightarrow a$,

$$f(y) = \int_{-\infty}^{\infty} \frac{dx}{(x-a)^2(x-b)} + \underbrace{\oint_{y \rightarrow a} \frac{dx}{(x-a)(x-b)(x-y)}}_{= 2\pi i \times \frac{-1}{(y-a)(y-b)}}$$

So, we can now probe the singularity:

$$f(y, y \text{ near } a) = \underbrace{\int_{-\infty}^{\infty} \frac{dx}{(x-a)^2(x-b)}}_{f(a,b)} + 2\pi i \frac{1}{(y-a)(a-b)}$$

shows that this integral has a pole!

- ④ To summarize: we consider an contour integral which is a function of some parameter -- which parameter controls the position of a pole in the integrand. We have other known poles in the integrand. We want to know if there are any singularities in this external parameter. Even if we adjust the parameter so that its pole lies right on the integration contour, we do not get a singularity at this value of the parameter because we can simply deform the contour. But, if the parameter varies in such a way that its pole pinches the contour onto some other pole, then we may compute the singular part of the integral by computing the residue of the integrand at the pinch point.

2/97 7

The problem of double-contour integrals and pinching.

1. This subject comes up on page 3-14 of BDW review. I don't really think it has much to do with pinches.

2. First, I claim there are certain instances where a double integral does not have any meaning. For example, what is this integral:

$$\int_0^x dy \int_0^y dx \cdot 1 = ?$$

This simply does not mean anything! You cannot have the endpoint of the outer integration depending on the dummy variable of an inside integration. This is just plain nonsense.

3. Now consider this double contour integral, with the contours as shown:

$$\int_{C_1} dy \int_{C_2} dx \cdot \frac{1}{(x-a)} \cdot \frac{1}{(x-y)}$$

The solid lines show the integration contours. The dotted lines show the locus of the pole $1/(x-y)$. In one plane, the location of this pole depends on where you are on the other integration contour.

4. Now, we are going to consider closing down one or the other or both of these contours. I claim that in this respect, the contour C_2 depends on the variable y . The reason is this: if you change y as you would change it in order to close down the C_1 -contour, you are going to have to deform the C_2 contour to avoid the pole at $x=y$ which is impinging upon it. You cannot leave C_2 where it was. Thus, we should really write $C_2(y)$.

However, on the other hand, if you were to try to close the C_2 contour, you would not have to deform C_1 at all because the pole at $y=x$ always stays away from the contour C_1 . Thus, C_1 is in this sense independent of x .

Therefore, the above double integration written in the order shown makes complete sense:

$$I = \int_{C_1} dy \int_{C_2(y)} dx \cdot \frac{1}{(x-a)} \cdot \frac{1}{(x-y)} = \int_{C_1} dy \cdot \frac{1}{(a-y)} = 1$$

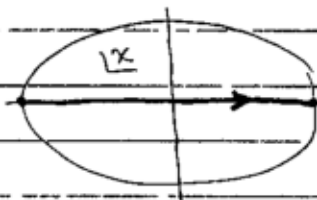
5. The contours taken in the other order simply have no meaning, just as in 2 above.

A pinch is suggested, but really it doesn't mean anything.

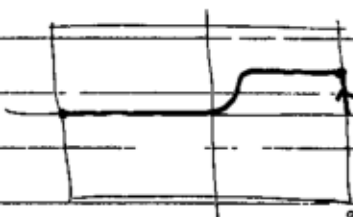
Delta Function: how to shift the contour and still make sense.

1. Consider the standard representation for the delta function. The usual contour is as shown:

$$2\pi\delta(k) = \int_{-\infty}^{\infty} e^{ikx} dx$$



2. Now, how can you deform this contour without changing anything? Here is an example of what you ~~cannot~~ do:



you just have to approach ∞ in the same direction to be OK.

$$\int_a^{a+ib} dx e^{ikx} = \frac{1}{ik} e^{ikx} \Big|_a^{a+ib} = \frac{1}{ik} e^{ika} (e^{-kb} - 1)$$

Here we are looking at the finite little green segment as we move it off to the right. Thus, we seek:

$$\lim_{a \rightarrow +\infty} \frac{e^{ika}}{ik} (e^{-kb} - 1)$$

This "function" of k is actually a generalized function like the delta function, so we treat the limit in the sense of distributions. Here are some facts I know in the distributional sense: recall that

aside:

$$\left. \begin{aligned} \frac{\sin ka}{k} &\approx \pi \delta(k) \\ \text{pf} \left(\frac{\cos ka}{k} \right) &= 0 \end{aligned} \right\} \text{ as } a \rightarrow +\infty.$$

But,

$$\left. \begin{aligned} \frac{\sin ka}{k} (e^{-kb} - 1) &\rightarrow 0 \\ \frac{\cos ka}{k} (e^{-kb} - 1) &\rightarrow 0 \end{aligned} \right\} \text{ as } a \rightarrow \infty.$$

proof:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos ka}{k} (e^{-kb} - 1) g(k) dk &= \int_{-\infty}^{\infty} \frac{\cos x}{x} (e^{-x/a} - 1) g\left(\frac{x}{a}\right) dx \\ &= \int_{-\infty}^{\infty} \frac{\cos x}{x} \left[-x \frac{b}{a} + O\left(\frac{x^2}{a^2}\right) \right] g\left(\frac{x}{a}\right) dx \\ &= -\frac{b}{a} g(0) \cdot \int_{-\infty}^{\infty} \cos x dx = 0 \end{aligned}$$

Be more careful:

$$\left\langle \lim_{a \rightarrow \infty} \left[\frac{\cos ak}{-k} (e^{-bk} - 1) \right] f \right\rangle$$

$$= \int_{-\infty}^{\infty} dk \lim_{a \rightarrow \infty} \left[\frac{\cos ak}{k} (e^{-bk} - 1) \right] f(k)$$

$$= \int_{-\infty}^{\infty} dx \lim_{a \rightarrow \infty} \frac{\cos x}{x} (e^{-bx/a} - 1) f(x/a)$$

The tick is needed here, for a while. ↑

$$= f(0) \cdot \int_{-\infty}^{\infty} dx \cdot \frac{\cos x}{x} = 0$$

So it seems that this limit only works in the limit of the pseudofunction distribution.

3. Therefore, we are able to show that the little green segment vanishes in the pseudodistribution sense. I don't think there is any real problem with the "pseudo" here.

4. To avoid this possible confusion with the pseudo-business, let's consider instead the effect of simply raising the entire contour up to higher level. Then we have two green segments and:

$$\left(\int_a^{a+ib} - \int_{-a}^{-a+ib} \right) e^{ikx} dx = \frac{e^{ika}}{ik} (e^{-ikb} - 1) - \frac{e^{-ika}}{ik} (e^{-ikb} - 1)$$

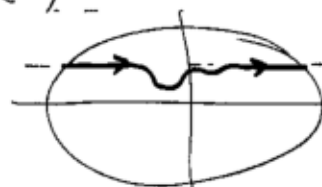
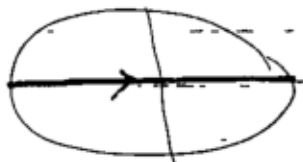
$$= (e^{-kb} - 1) \cdot \frac{1}{ik} (e^{ika} - e^{-ika}) = \frac{2}{ik} (e^{-kb} - 1) \cdot \frac{\sin ka}{k}$$

But $\frac{\sin ka}{k} \rightarrow \pi \delta(k)$ for sure, so

$$= 2\pi \delta(k) \cdot (e^{-kb} - 1) = 2\pi \delta(k) (1 - 1) = 0$$

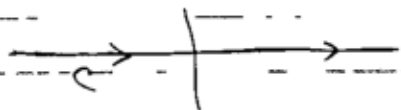
5. Therefore, there is absolutely no question but that you can raise or lower the contour without changing anything. Easy to show this directly:

$$\int_{-ia+a}^{ia+a} e^{ikx} dx = e^{ika} \cdot 2\pi \delta(k) = 2\pi \delta(k)$$



Contours and change of variable : over for complex conjugation →

① Consider: $I = \int_{-\infty}^{\infty} dx f(x) = \int_C dx f(x)$



② Change variables to $y = g(x)$ $x = x(y)$ $y = y(x)$.
assume clear 1-to-1 map. Then:

$$dx = \frac{dx(y)}{dy} dy \quad f(x) = f(x(y))$$

$$C \rightarrow C' = y(C)$$

So

$$I = \int_{C'=y(C)} \left[\frac{dx(y)}{dy} \right] dy f(x(y)) \quad \text{Let } y(x) = -x$$

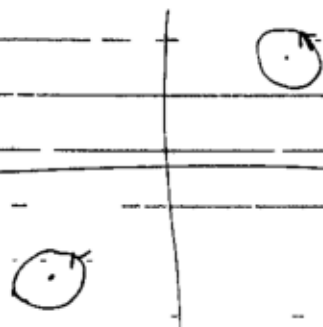
In this case $C = \rightarrow \Rightarrow C' = \leftarrow$

So:

$$I = \int_{-\infty}^{\infty} (-1) dy f(-y) = \int_{-\infty}^{\infty} dy f(-y) \quad \checkmark$$

③ So: $I = \int_C dz f(z)$

$$\Rightarrow I = \int_{g(C)} \left[\frac{dz}{dy} \right]^{-1} dy f[z(y)]$$



④ Better: let $y = y(z)$ and $z = z(y)$ be the map.

$$I = \int_C dz f(z) = \int_{y(C)} dz(y) f(z(y))$$

⑤ Example: $I = \oint_{|z|=1} dz f(z)$ $y = -z$ $C = \bigcirc$ $y(C) = \bigcirc$

$$I = - \oint_{|y|=1} dy f(-y)$$

complex conjugation:

$$\text{Let } a = \int_C dz f(z) \approx \sum_i \Delta z_i \cdot f(z_i)$$

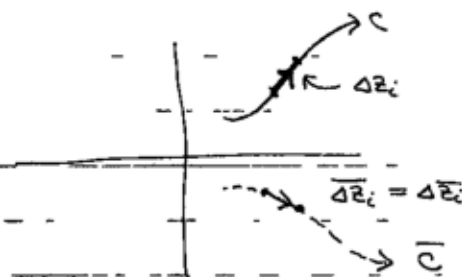
$$\text{Then } \bar{a} = \sum_i \overline{\Delta z_i} \cdot \overline{f(z_i)}$$

$$\text{But } \Delta z_i = z_{i+1} - z_i \quad \text{so } \overline{\Delta z_i} = \overline{z_{i+1} - z_i} = \Delta \bar{z}_i$$

$$\text{So: } \bar{a} = \sum_i \Delta \bar{z}_i \cdot \overline{f(z_i)}$$

Where is new contour:

$$\text{So } \bar{a} = \int_{\bar{C}} d\bar{z} \overline{f(z)}$$

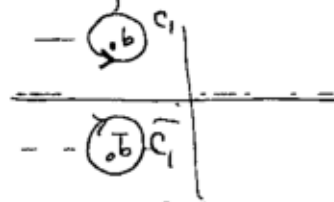


But $\bar{z}$ is only a dummy variable, so rename it z
(like $\int_\gamma dz f(z) = \int_\gamma ds f(s)$)

$$\bar{a} = \int_{\bar{C}} dz \overline{f(z)}$$

Note that $f(z)$ analytic $\Rightarrow \overline{f(z)}$ = analytic also.

$$\text{Test: } 1 = \frac{1}{2\pi i} \int_{C_1} dz \frac{1}{z-b}$$



$$\Rightarrow 1 = -\frac{1}{2\pi i} \int_{\bar{C}_1} dz \frac{1}{z-b} \quad \checkmark$$

Conclusion:

$$a = \int_C dz f(z) \Rightarrow \bar{a} = \int_{\bar{C}} dz \overline{f(z)}$$

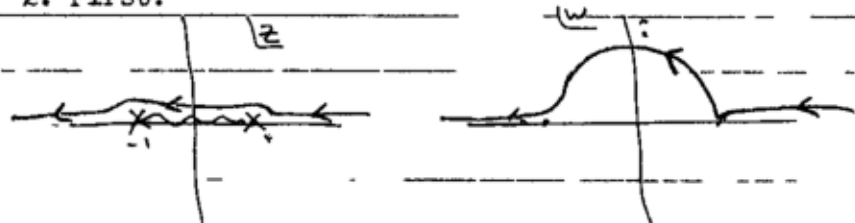
where $\bar{C}$ = reflected contour in real axis.

Analytic structure of $z = \operatorname{ch} q$ and $q = \operatorname{ch}^{-1} z$.

1. Comment: this is one of the more complicated situations because you get a "reflected cut" in your original plane, which cut wanders through another cut.
Consider in two stages:

$$q = \operatorname{ch}^{-1}(z) = \ln[z + \sqrt{z^2 - 1}] = \ln w \quad w = z + \sqrt{z^2 - 1}$$

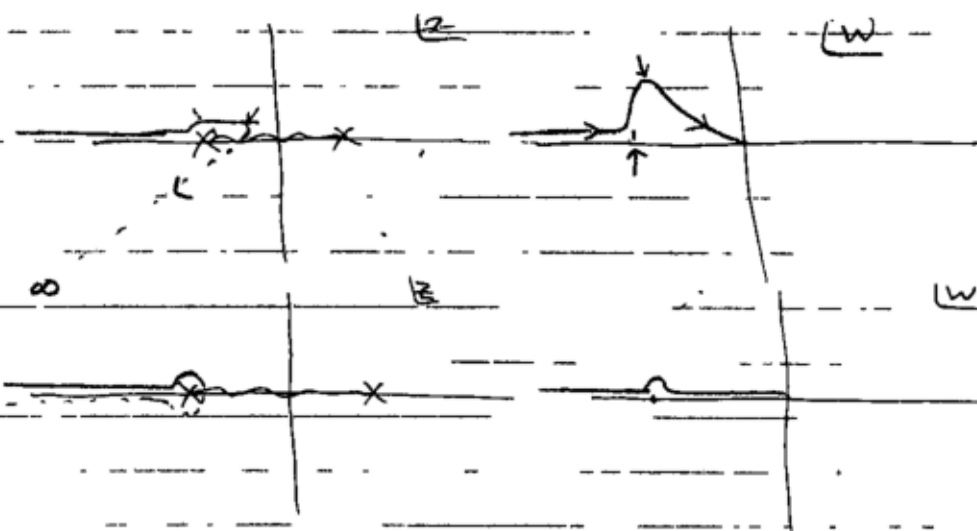
2. First:



$$4 - \sqrt{15} = u$$

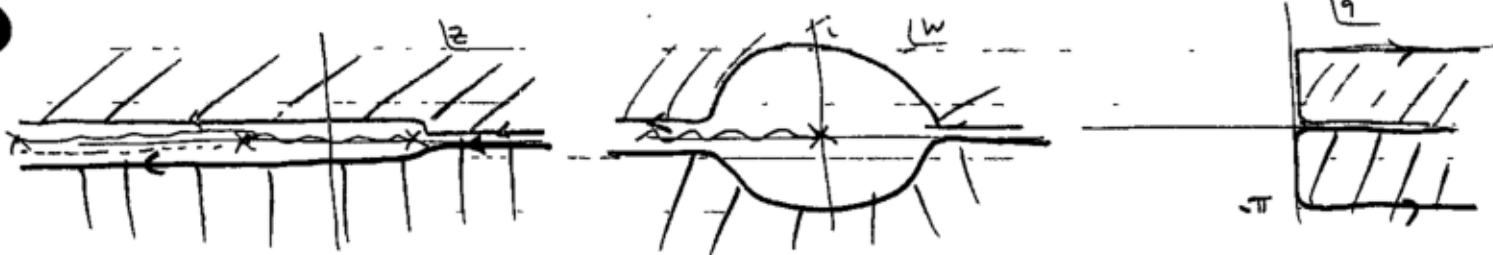
$$-4 +$$

Here we show the z -plane for the function w . All points on this sheet of the z -plane map into points which lie outside the unit circle in the w -plane. Any line which crosses the unit circle in the w -plane, must cross through the cut in the z -plane. As an example, we consider this line:



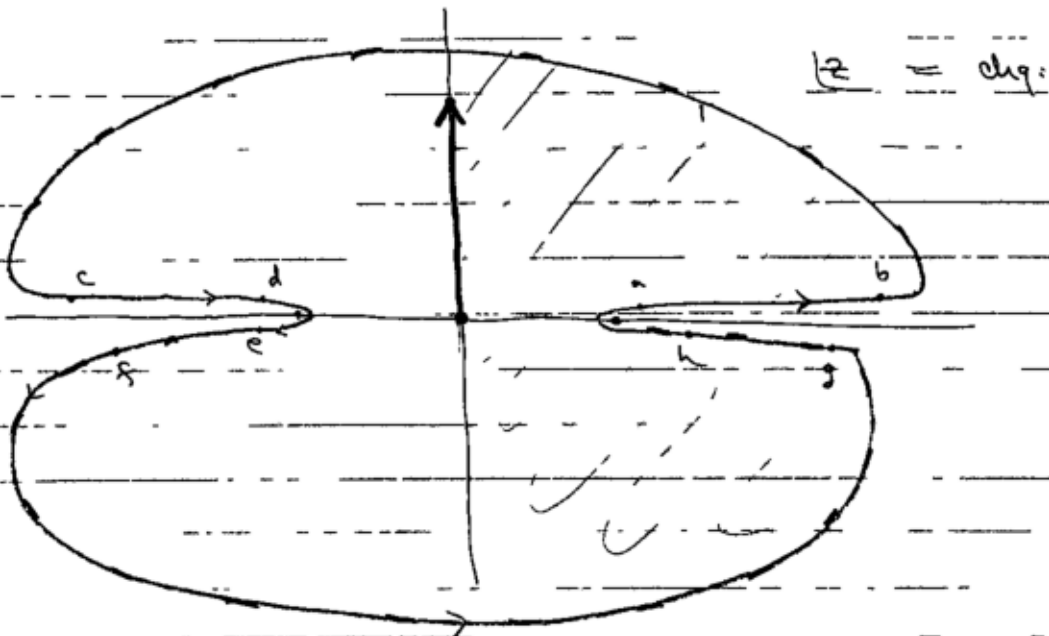
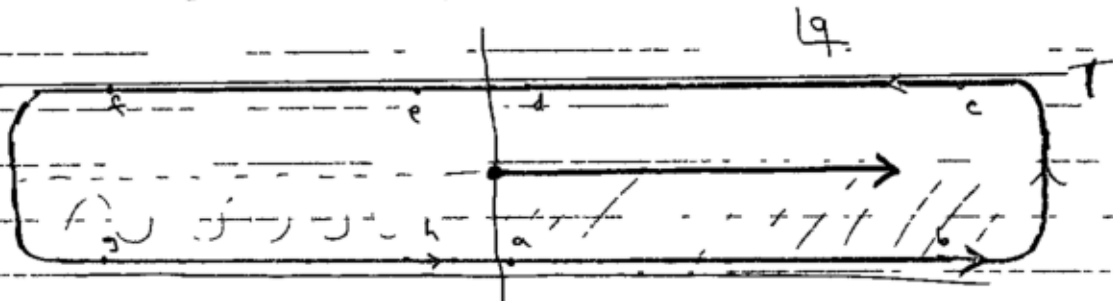
You have to work on the details to really see this, but the top picture shows the actual penetration of the unit circle. As we try to draw the line straight off to the left in the w -plane, we have to make a detour at $w = -1$ in order to know what happens in the z -plane.

3. If we now consider $q = \ln w$, we see that there will be a cut in the w -plane just where we have been drawing the above lines just for fun. Thus, we may consider the picture above as the w -plane for the function $q = \ln(w)$. This cut at $w = 0$ to $-\infty$ is then reflected into the z -plane as the above pictures show. So:



Notice that the chosen sheet of the z -plane maps into the right half q -plane strip.

4. So far we have considered the function $q = \text{ch}^{-1} z$. This is not an analytic function of z because there are two cuts to worry about. One of the cuts is obvious and comes from the SQRT. The other is more subtle and is the one whose both ends are at infinity but on different sheets. We now wish to consider the function $z(q) = \text{ch} q$. This function is analytic in the finite q -plane. No matter how you wander in q , you will get something sensible in z . Here are some pictures:

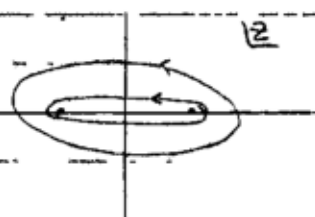
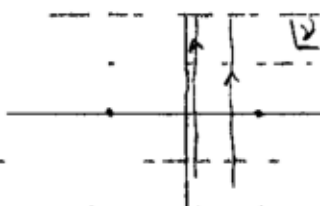


As q wanders through the 0 to π box shown, $z = \text{ch} q$ wanders through the entire z -plane without obstruction. The phase of z can be defined, eg, as $-\pi < \arg z < \pi$

~~More on the map $z = \cosh w$~~

More on the map $z = \cosh w$

- ① Goal is to learn about $|z + \sqrt{z^2 - 1}| = \text{constant}$. At once, change variables to $z = \cosh w$. Analyze this map.



In more detail, set $z = z' + iz''$ (1)
 $w = v' + iv''$

Then find: $\left. \begin{aligned} z' &= \cosh v' \cos v'' \\ z'' &= \sinh v' \sin v'' \end{aligned} \right\}$ (2)

This implies: $\frac{(z')^2}{(\cosh v')^2} + \frac{(z'')^2}{(\sinh v')^2} = 1$ (3)

$$\frac{(z')^2}{(\cos v'')^2} - \frac{(z'')^2}{(\sin v'')^2} = 1 \quad (4)$$

- ② This says: (3) $\therefore$ lines of constant v' are ellipses in z 
(4) $\therefore$ lines of constant v'' are hyperbolas in z 

Map is analytic $\Rightarrow$ conformal $\Rightarrow$ orthogonal level curves $\checkmark$.

- ③ Ellipses: $a = \cosh v'$, $b = \sinh v'$ so longer along x axis.
As $v' \rightarrow 0$, ellipse folds down around $(-1, 1)$.

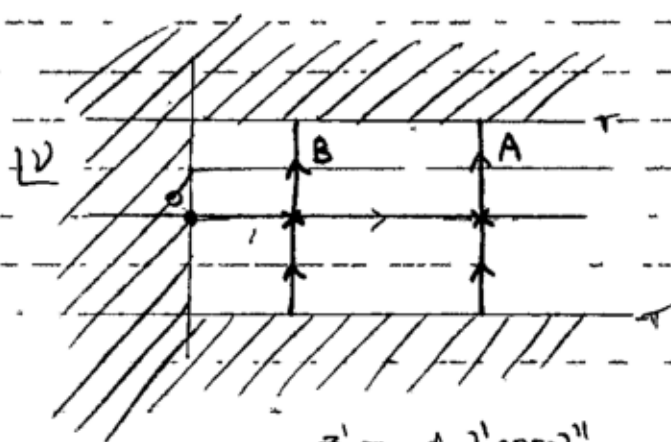
- ① Sheets: if you choose some $v' + i$ vary v'' , z moves along ellipse. But clearly z is periodic 2π in v'' . Thus we can show cut & found earlier:



As v'' rises more, you wind onto ∞ # of sheets in z .
But recall there is another cut $(-1, 1)$, so to stay on this sheet of z we must keep $\text{Re}(w) > 0$.

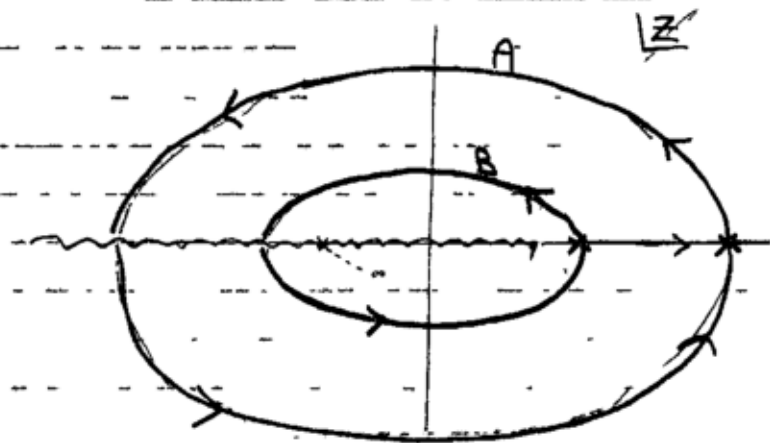
So entire z -plane corresponds to a semiship

⑤ Make sketches of ellipses + hyperbolas:

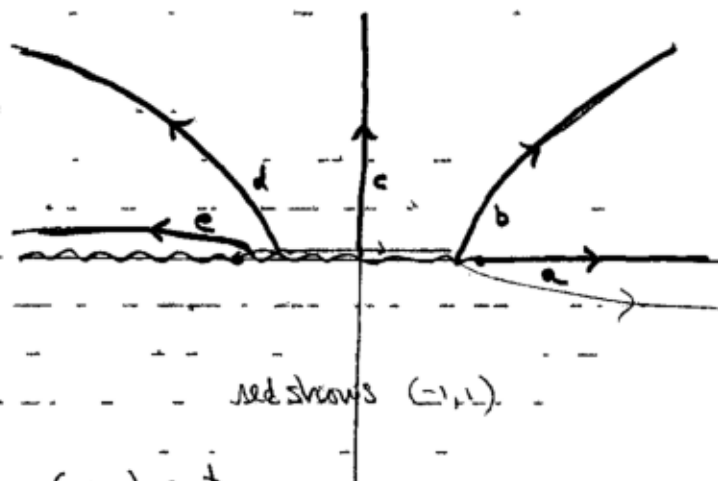
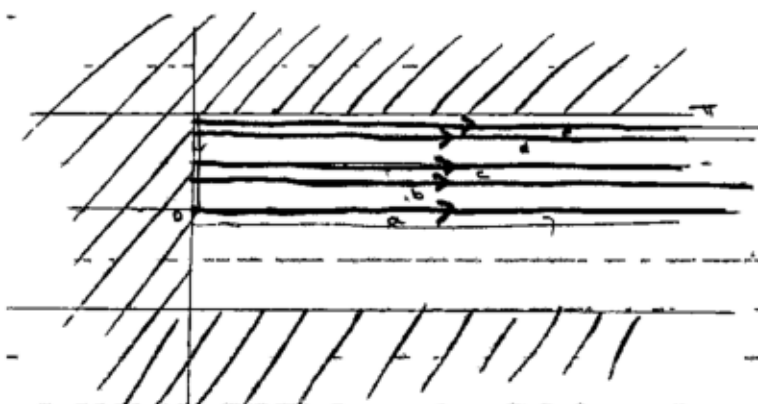


$$z' = dw' \cos v''$$

$$z'' = dw' \sin v''$$



⑥ The hyperbolic branches obvious from



In general the curves sprout out of the $(-1, 1)$ cut.

⑦ How would you define this sheet without pictures?

$$z = dw$$

$$\begin{aligned} w: & \operatorname{Re} w > 0 \\ & |dw| < \pi \\ z: & |\arg(z+1)| < \pi \end{aligned}$$

$$\begin{aligned} v' & > 0 \\ |v''| & < \pi \end{aligned}$$

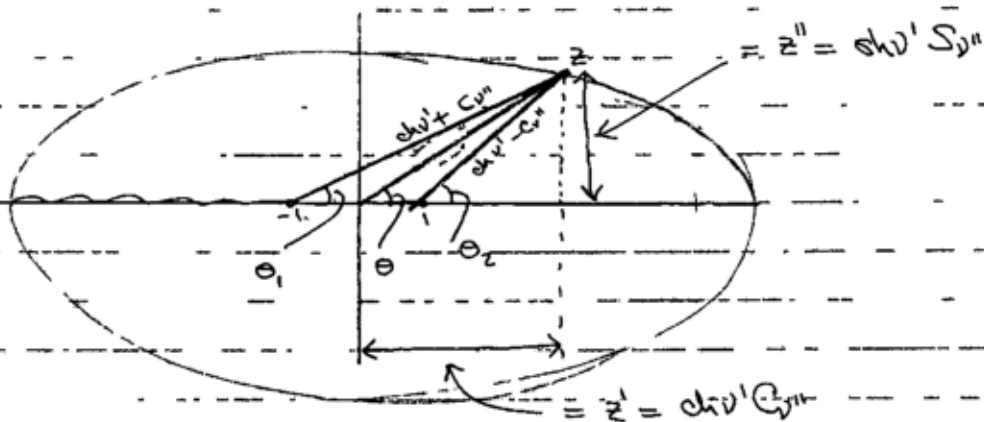
⑧ Facts: from $z = dw$, $|z + \sqrt{z^2 - 1}| = |e^v| = e^v$

So $|z + \sqrt{z^2 - 1}| = \text{const} \Rightarrow e^v = \text{const}$. Note that const must be > 1 , except when $-1 < z < 1 \Rightarrow \text{const} = 1$.

$$\arg(z + \sqrt{z^2 - 1}) = v'' \quad \text{so} \quad |\arg(z + \sqrt{z^2 - 1})| < \pi \Rightarrow |v''| < \pi \Rightarrow |\arg(z+1)| < \pi$$

details on angles in z -plane related to V', V'' of $z = \cosh \zeta$...

① The picture



② The facts:

$$\theta_1 = \arg(z+1)$$

$$\Theta_2 = \arg(z-1).$$

$$\theta = \arg(z)$$

$$\cot \theta_1 = \frac{1+z'}{z''}$$

$$\cot \theta_2 = \frac{-1 + z'}{z''}$$

$$\cot \theta = \frac{z'}{z''}$$

③

$$\cot(a+b) = \frac{\cot a \cot b - 1}{\cot a + \cot b}$$

$$\cot(a-b) = \frac{\cot a \cot b + 1}{\cot a - \cot b}$$

④ Examples:

$$\cot(\theta - \theta_2) = \frac{\cot \theta \cot \theta_2 + 1}{\cot \theta - \cot \theta_2} = \frac{z_1^2 + z_2^2 - z_1}{z_2}$$

$$\cos(\theta_1 - \theta_2) = \frac{z_1^2 + z_2^2 - 1}{2 z_1 z_2} = \frac{\sinh^2 u - \sinh^2 v}{2 \sinh u \sinh v}$$

⑤ More facts:

$$\cos \theta_1 = \frac{z+1}{dv'+Cv''}$$

$$\sin \theta_1 = \frac{z''}{C_V' + C_V''}$$

$$\cos \theta_2 = \frac{z' - 1}{c h \nu' - c \nu''}$$

$$\sin \theta_2 = \frac{z''}{c_{v'} - c_v''}$$

$$\cos \theta = \frac{z'}{|z|}$$

$$\sin \theta = \frac{z''}{|z|}$$

On dealing with $f(\cos \theta)$ or $f(\sin \theta)$.

- ① Consider $z = \cos \theta = \cos(\pm i \theta)$. Want to map θ to z . So the idea will be to cover the z -plane once. Now:

$$\cos(\theta' + i \theta'') = \cos \theta' \cos(i \theta'') - \sin \theta' \sin(i \theta'')$$

$$= \cos \theta' \cosh \theta'' - i \sin \theta' \sinh \theta''$$

$$= z' + i z''$$

$$z' = \cos \theta' \cosh \theta''$$

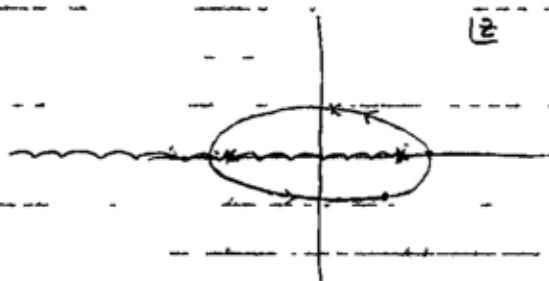
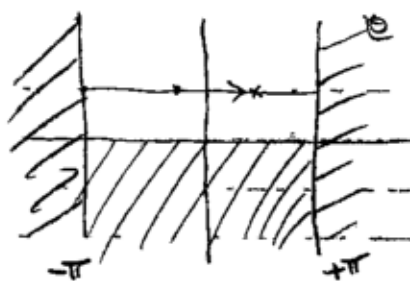
$$z'' = -\sin \theta' \sinh \theta''$$

$\Rightarrow$

- ② Furthermore, we should restrict to $-\pi < \theta' < \pi$ since periodic. Next, easy to show that:

$$\frac{(z')^2}{(\cosh \theta'')^2} + \frac{(z'')^2}{(\sinh \theta'')^2} = 1$$

this is curve of constant $\theta'' = \text{Re}(\theta)$. Vary θ' , hold θ'' same, you must be on ellipse in z -plane:



So $\theta'' \geq 0$ will cover entire z -plane. But structure must be: as shown.

- ③ With this choice, we have

$$z = \cosh \theta$$

$$z = \cos \theta$$

$$\text{Re} z \geq 0$$

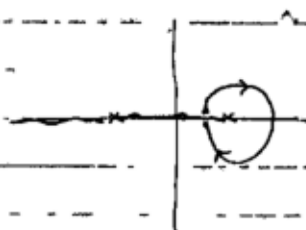
$$\text{Im} z \geq 0$$

$$\text{Thus: } \theta = +i \theta''$$

$$\Rightarrow \text{Im} \theta = \text{Re} z$$

So $\theta = +i \theta''$ is the only correct relation in this convention

- ④ Now let $\theta = \text{pos. real \#}$. To go from θ to $-\theta$ is unambiguous: keep path in allowed region:

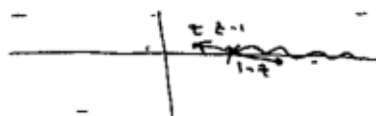


z rotates around the branch at $z = 1$. Of course $\cos \theta$ ends up at the same value.

But, consider $f(z) = \sqrt{z-1}$. This starts out $= 1 \cdot e^{+i\frac{\pi}{2}}$, ends up at $1 \cdot e^{-i\frac{\pi}{2}}$ so

$\sqrt{\cos\theta - 1}$ changes sign as $\theta \rightarrow -\theta$!

Consider: $g(z) = \sqrt{1-z}$. Starts at $\sqrt{1-1} = 0$, but ends up at $\arg(1-z) = 2\pi$.



So $g(0) = 1$, but $g(-0) = -1$.

Thus, $\sqrt{1-\cos\theta}$ changes sign.

Well $\sqrt{1-\cos\theta} = \sqrt{2} \sin \frac{\theta}{2}$, so that was expected.

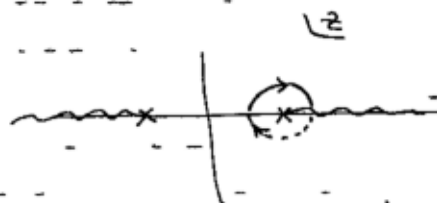
$\sin\theta = \sqrt{1-\cos^2\theta}$ also changes sign for same reason.

⑤ General idea: consider some $f(z)$. As $\theta \rightarrow -\theta$, in unambiguous way shown, then z wraps once around ~~branch at~~ point $z=1$, as shown. Now, if $f(z)$ has a branch point at $z=1$, then $\theta \rightarrow -\theta$ will cause a change in $f(z)$. Namely you say $\theta \rightarrow -\theta$ is a symmetry of $f(\cos\theta)$, but obviously that is wrong; as in case:

$$\sin\theta = \sqrt{1-(\cos\theta)^2} = f(\cos\theta).$$

⑥ Application:

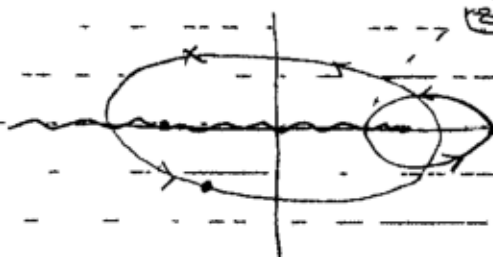
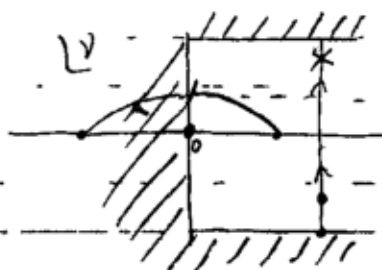
$$\langle m | e^{-i\theta J_z} | m' \rangle = d_{mm'}(\cos\theta) \quad \theta > 0$$



If $\theta \rightarrow -\theta$, z goes as shown. Cut caused by $(1-\frac{z}{2})^{\frac{m-m'}{2}}$, so you pick up a phase $(-1)^{m-m'}$.

Thus, it does not matter which way you convention θ .

① Now consider some $f(\text{ch } v)$. Then we have:



Now what happens if $v \rightarrow -v$? First of all, z goes off the principle sheet of $v = v(z)$, but that's OK

What happens? as $v \rightarrow -v$, z wraps around $z=1$ in a direction which depends on which way you bypass the origin in the v -plane. If $f(z) = f(\text{ch } v)$ has a branch at $z=1$, then $f(z)$ will change accordingly.

② Example:

$$\langle m | e^{-ivK_z} | m' \rangle = d_{mm'}^j(\text{ch } v) \quad v > 0$$

$$\Rightarrow \langle m | e^{+ivK_z} | m' \rangle = (-1)^{m-m'} d_{mm'}^j(\text{ch } v) \quad v > 0$$

Because as $v \rightarrow -v$, $d(z)$ has z wrap me

consistency check:

$$\langle m | e^{-ivK_z} | m' \rangle = d_{mm'}^j(\text{ch } v)$$

$$\Rightarrow \langle m' | e^{+ivK_z} | m \rangle = d_{mm'}^j(\text{ch } v) = (-1)^{m-m'} d_{m'm}^j(\text{ch } v) \quad \checkmark$$

③ Example: suppose $f(v) = \mathcal{D}_{\mu\mu'}^j(\text{ch } v)$ $j = \text{real}$
 $v > 0$ $\mu, \mu' = \text{both imag}$

Then $f(-v) = ?$ Well $\mathcal{D}_{\mu\mu'}^j(z) \sim$ (and!) $\sim$

Has three branch points at $z=1$! Consider: Let

$$\mathcal{D}_{\mu\mu'}^j(\text{ch } v) \rightarrow e^{-i\pi(\mu-\mu')} \mathcal{D}_{\mu\mu'}^j \text{ if you go a certain way. The}$$

$$\mathcal{D}_{\mu\mu'}^j(\text{ch } v) \stackrel{?}{=} \frac{1}{\pi} \sin \pi(\mu-\mu') = \mathcal{D}_{\mu\mu'}^j(\text{ch } v) - \mathcal{D}_{\mu\mu'}^j(\text{ch } v)$$

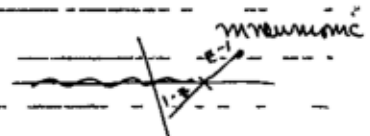
$$\rightarrow e^{-i\pi(\mu-\mu')} \mathcal{D}_{\mu\mu'}^j = \mathcal{D}_{\mu\mu'}^j e^{+i\pi(\mu-\mu')} \rightarrow \text{multiple of } Q!$$

Solution to the cut problem.

- ① If we insist that $|\arg_s(z-1)| < \pi$ and $|\arg_s(1-z)| < \pi$,
then the only possible relation is

$$\arg_s(z-1) \equiv \arg_s(1-z) \pm \pi$$

$$\text{i.e., } {}_s(z-1) \equiv {}_s(1-z) e^{\pm i\pi}$$



The reason is: when $\text{Im} z > 0$, $(1-z)$ lies in lower half and then $\arg(1-z) < 0$, so we must add π to get something in range of $\arg_s(z-1)$; and vice versa.

- ② Thus, at once we get ${}_s(z-1)^\alpha = {}_s(1-z)^\alpha e^{\pm i\pi\alpha}$

- ③ Or go through logs

$$\left[\begin{array}{l} {}_s(z-1)^\alpha \equiv e^{\alpha \ln_s(z-1)} \\ {}_s(1-z)^\alpha \equiv e^{\alpha \ln_s(1-z)} \end{array} \right]$$

$$\text{But } {}_s(z-1) \equiv {}_s(1-z) e^{\pm i\pi} \Rightarrow \ln_s(z-1) = \ln_s(1-z) \pm i\pi$$

$$\begin{aligned} \text{Thus: } {}_s(z-1)^\alpha &= \exp[\alpha \ln_s(z-1)] \\ &= \exp[\alpha \ln_s(1-z) \pm i\pi\alpha] \\ &= {}_s(1-z)^\alpha e^{\pm i\pi\alpha} \end{aligned}$$

This is valid for general complex α :

$$\boxed{{}_s(z-1)^\alpha = {}_s(1-z)^\alpha e^{\pm i\pi\alpha}}$$

or

$$(z-1) = (1-z) e^{\pm i\pi}$$

special case: $\alpha = \frac{1}{2}$ (done first)

① definition of $\sqrt{z-1}$: positive when $z > 1$ $|\arg(z-1)| < \pi$ $\sqrt{z-1} \equiv +1 \cdot e^{\frac{i}{2}\arg(z-1)}$

② definition of $\sqrt{1-z}$: positive when $z < 1$ $|\arg(1-z)| < \pi$ $\sqrt{1-z} \equiv +1 \cdot e^{\frac{i}{2}\arg(1-z)}$

$\sqrt{}$ is for "standard"

③ Now define some new functions: using the usual cut-to-the-left mnemonic:

$$" \sqrt{1-z} " \equiv \sqrt{z-1} e^{-\frac{i}{2}\pi} \quad \text{Im } z \geq 0 \quad 0 < |\arg(z-1)| \leq \pi$$

This new function is not strictly defined anywhere on the real axis. Each of the factors rules out one half of the real axis. However, if you approach the real axis on the left from either direction, you easily get the same result, so this function is uncut on the left. Moreover, it is positive and real:

$$\text{for } z = -5 \pm i\epsilon \quad " \sqrt{1-z} " = 1 \cdot e^{-\frac{i}{2}(\pm\pi)} = 1 > 0$$

4. The functions $\sqrt{1-z}$ and $" \sqrt{1-z} "$ certainly look the same. For complex z , both are unambiguous through their definitions. You must keep in mind that the standard function has its z characterized by $\arg(1-z)$ which must be less than π in magnitude, whereas the $" "$ function is characterized by $\arg(z-1)$ which must be less than π in magnitude. In fact, both these functions agree on the negative real axis in sign!

Investigation in detail shows that they are the same function:

$$" \sqrt{1-z} " = -\sqrt{1-z} = -\sqrt{z-1} e^{-\frac{i}{2}\pi} \quad \text{Im}(z) \geq 0$$

This is the one to remember

5. Now define the other $" "$ function and see what happens:

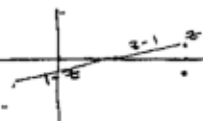
$$" \sqrt{z-1} " \equiv \sqrt{1-z} e^{-\frac{i}{2}\pi}$$

using cut-to-the-right mnemonic (note how opposite to the above mnemonic for left going cut)

Again, this function is defined for all complex z unambiguously. Look at positive real z :

$$\text{for } z = 2 \pm i\epsilon \quad " \sqrt{z-1} " = 1 \cdot e^{\frac{i}{2}(\mp\pi)} = 1 \cdot e^{-i\pi}$$

Notice the minus sign!



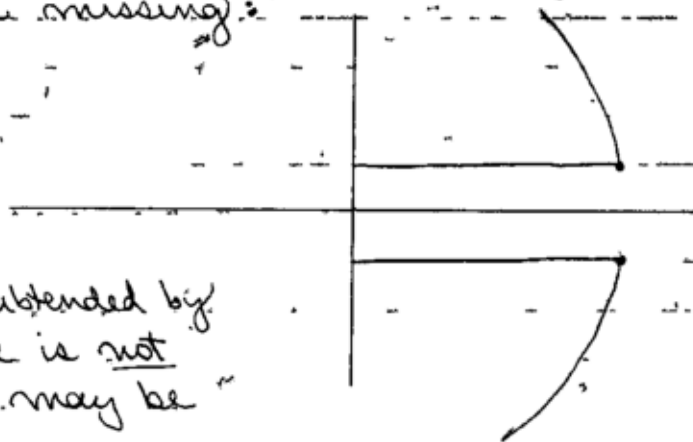
Investigation in detail shows that $" \sqrt{z-1} "$ and $\sqrt{z-1}$ are related like so:

$$" \sqrt{z-1} " = e^{-i\pi} \sqrt{z-1}$$

A theorem about great circles

① Suppose you know that an integrand decays expo in any direction in the RHP except parallel to real axis. Can you conclude that "great circle" vanishes? (see ⑥ ⑦ for answer)

② What is meant by great circle? Usually we mean it with a finite piece missing:



However, the angle subtended by the missing piece is not finite, so there may be some question.

③ Let's jump right to an example:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} = (2^{1-p} - 1) \zeta(p) \quad p? \quad GR \quad [Re p > 0]$$

$$= -\frac{1}{2\pi i} \oint_{\gamma} \frac{dk \, k^{-p}}{\sin \pi k} \quad \text{by usual SWT.}$$

Now, question is: can we "open" the contour? The integrand is dominated by $\exp(-\pi |\Im(k)|)$ in any tilted direction, but by k^{-p} in the real direction.

④ The resolution of this matter really hinges on the small pieces of contour near the "corners". We are really stuck with going all the way down to the real axis.

Let $k = Re^{i\theta}$. Then: $dk = iR d\theta$ so

$$-\frac{1}{2\pi i} \int_{\theta=0}^{\theta=\epsilon} \frac{iR d\theta \cdot (Re^{i\theta})^{-p+1}}{\sin \pi [Re^{i\theta}]} \quad \text{this 1 caused by "length" of contour}$$

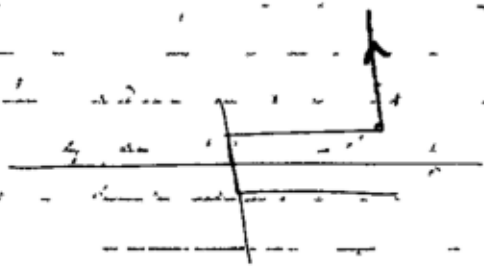
Go very close to $\theta=0$:

$$\sim \int_{\theta=0}^{\epsilon} \frac{d\theta (R)^{-p+1}}{\sin \pi R}$$

② This analysis is not precise enough.

Perhaps the relevant integral is:

$$\lim_{R \rightarrow \infty} \left[\int_{R+i}^{R+i\infty} \frac{dk k^{-p}}{\sin \pi k} \right] \quad \text{IE}$$



Convert this to a real integral:

$$R + ix = k \Rightarrow \lim_{R \rightarrow \infty} \left[\int_1^{\infty} \frac{ix (R+ix)^{-p}}{\sin \pi (R+ix)} \right]$$

$$= \lim_{R \rightarrow \infty} \frac{(i)^{1-p}}{(i)} \left[\int_1^{\infty} \frac{dx (x-iR)^{-p}}{\sin \pi (x-iR)} \right]$$

So not at all obvious to me.

$$(R+ix)^{-p} = e^{-p \ln(R+ix)} \approx e^{-p \ln R (1+i \frac{x}{R})}$$

$|\sin \pi (x-iR)|$ bounded?

$$= |\sin \pi R \cosh \pi x + \cos \pi R i \sinh \pi x| = \sin^2 \pi R \cosh^2 \pi x + \cos^2 \pi R \sinh^2 \pi x$$

= at its smallest, $|\cosh \pi x|$ or $|\sinh \pi x|$.

$$\text{So } |\sin \pi (x-iR)| > \sinh \pi x$$

$$= \sinh^2 \pi x + \sin^2 \pi R$$

$$\text{So } |I| < \int_1^{\infty} \frac{dx |x-iR|^{-p}}{\sinh x} = \int_1^{\infty} \frac{dx (x^2+R^2)^{-p/2}}{\sinh x}$$

$$\approx R^{-p} \left[\int_1^{\infty} \frac{dx (1+x^2/R^2)^{-p/2}}{\sinh x} \right]$$

$$\approx R^{-p} \int_1^{\infty} \frac{dx}{\sinh x}$$

So in this little case we find that as long as $\text{Re } p > 0$.



⑥ Suppose the above vertical integral does converge ^{to 0}. Then it is fairly clear that the upper half vertical integral running up the imaginary axis will converge, since we are assuming our integrand has expo decay in all directions except real. This in turn would imply (assuming no singularities in upper half plane first quadrant) that the horizontal upper contour integral going off to the right must be converging. So what we get is that: you can drop great circle given expo decay in any direction except real provided that the real integral converges.

It need only be power convergent, of course. This explains why you are only allowed to open the contour in our little example here when $\text{Re}(p) > 0$. Actually, what really seems to count is that the integrand must approach zero as you go off into the real direction. Then the little critical integration segment will not cause any trouble.

⑦ Very crudely, consider $\oint dz f(z)$.

The crucial integration piece is $\int_0^{2\pi} d\theta R f(Re^{i\theta})$

$\approx f(R)$. I.e., we look at one unit's worth of GC. If $f(R) \rightarrow 0$,

you are OK.

⑧ Application: let $f(z) = \frac{1}{z^p \sin \pi z}$. If $\text{Re}(p) > 0$, then you can "cut the corner".

$$\text{then: } \sum_{k=1}^{\infty} \frac{(-1)^k e^{-i\phi k}}{k^p} \sim \int \frac{dk e^{-i\phi k}}{k^p \sin \pi k}$$

= convergent if $|\phi| < \pi$. This then serves as a proof that

$$\sum_{k=1}^{\infty} \frac{e^{ik(\phi \pm \pi)}}{k^p} \text{ converges for } \text{Re}(p) > 0, |\phi| \neq \pi.$$

We found the same result by directly examining the series.

Inverse mapping theorem

① Suppose $f(z)$ analytic in some D . Define a new variable $\xi = f(z)$. Can we speak of $z = f^{-1}(\xi) = g(\xi)$? Is $g(\xi)$ analytic in ξ ?

② Let $f(z)$ analytic in disc about $z=0$ wlog. Assume $f'(0) \neq 0$. Then

$$f(z) = f(0) + f'(0)z + (f''(0)/2!)z^2 + \dots$$

$$\text{For } |z| \ll 1, \quad f(z) - f(0) = f'(0)z \Rightarrow z = \left[\frac{f(z) - f(0)}{f'(0)} \right] = \frac{(\xi - f(0))}{f'(0)} \\ = g(\xi) = c \cdot (\xi - \xi_1), \quad g(\xi_1) = 0$$

So if $f'(0) \neq 0$, it seems that $g(\xi)$ is analytic near $\xi_1 = f(0)$

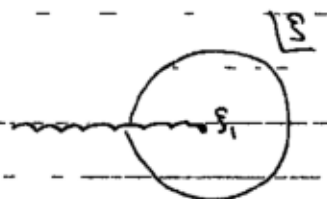
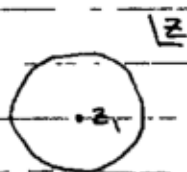
③ What about higher terms? Hopefully, you can neglect these.

① Suppose $f'(0) = 0$. Then $f(z) \approx f(0) + \frac{1}{2} f''(0) z^2$

$$\text{so } (\xi - \xi_1) = \frac{1}{2} f''(0) [g(\xi)]^2 \Rightarrow g(\xi) \approx \sqrt{\frac{2}{f''(0)}} \cdot \sqrt{(\xi - \xi_1)}$$

Now clearly, $g(\xi)$ has a sqrt branch at ξ_1 .

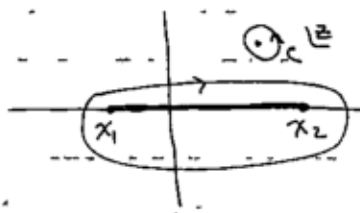
① Theorem: If $f(z)$ is analytic near some z_1 , and $\xi_1 \equiv f(z_1)$, then the inverse function $z = g(\xi) = f^{-1}(\xi)$ will be analytic near ξ_1 except $g(\xi)$ will have a simple $(\xi - \xi_1)^{1/n}$ branch point at ξ_1 , where n is the first non-vanishing derivative $f^{(n)}(z_1)$. So, if $f'(z_1) \neq 0$, there is no branch point.



Question: Suppose someone gives you a real function defined on some portion of the real axis, and zero on some other portion. Is this the discontinuity of some analytic function?

1. Example:
General:

$h(x)$ = the real function



$$f(z) = \frac{1}{2\pi i} \int_C \frac{dz' f(z')}{z' - z}$$

$$= \frac{1}{2\pi i} \int_{x_1}^{x_2} \frac{dx h(x)}{x - z}$$

If you can ignore the great circle and the possible hardness of the endpoints of the integral, then we seem to have a formula telling us the candidate analytic function which corresponds to any $h(x)$, namely:

$$f(z) = \frac{1}{2\pi i} \int_{x_1}^{x_2} \frac{dx h(x)}{x - z}$$

2. Example:

$h(x) = \Theta(x - x_0) x^\alpha \quad \alpha < -1$

$$f(z) = \frac{1}{2\pi i} \int_{x_0}^{\infty} \frac{dx \cdot x^\alpha}{x - z} = \frac{1}{2\pi i} (-z)^{-1} \int_{x_0}^{\infty} \frac{x^\alpha dx}{[1 + (-\frac{1}{z})x]^1}$$

Consult p285 #1 G.R.

$u = x_0, v = 1$

$\mu - 1 = \alpha \quad \beta = (-z)^{-1}$

$\mu - \nu = \alpha$

$$= \frac{(x_0)^{\alpha+1}}{(-z)^{-1} (1-\alpha)} F(1, -\alpha; 1-\alpha; \frac{-1}{(-z)^{-1} x_0})$$

$$\Rightarrow f(z) = \frac{1}{2\pi i} \frac{(x_0)^{\alpha+1}}{(-\alpha)} F(1, -\alpha; 1-\alpha; \frac{-z}{x_0})$$

By the way:

$$P_{1/2, -1/2}(\xi) = \left(\frac{\xi-1}{2}\right)^{1/2} F(1, -\alpha; 1-\alpha; \frac{1-\xi}{2}) / \Gamma(1-\alpha)$$

$$\frac{1-\xi}{2} = \frac{z}{x_0} \Rightarrow \xi = \left(1 - \frac{2z}{x_0}\right)$$

$$P_{1/2, -1/2}\left(1 - \frac{2z}{x_0}\right) = \left(-\frac{z}{x_0}\right)^{-1/2} F(1, -\alpha; 1-\alpha; \frac{z}{x_0}) / \Gamma(1-\alpha)$$

$\underline{S_0}$,

$$f(z) = \frac{1}{2\pi i} \frac{(x_0)^{\alpha+1}}{(-\alpha) \Gamma(1-\alpha)} \left(-\frac{z}{x_0}\right)^{-1/2} P_{1/2, -1/2}\left(1 - \frac{2z}{x_0}\right)$$

not important

(go to next page)

Now, consider

$$F(1, -\alpha, 1-\alpha; \xi)$$

$$a=1$$

$$a+b-c=0$$

$$b=-\alpha$$

$$c=1-\alpha$$

$$b-a=-\alpha-1$$

$$a+1-c=\alpha+1-\alpha=1$$

$$c-a=1-\alpha-\alpha=1-2\alpha$$

$$a+1-b=\alpha+1-\alpha=1$$

$$c-b=1-\alpha-\alpha=1-2\alpha$$

$$b+1-c=-\alpha+1-\alpha=1-2\alpha$$

$$b+1-a=-\alpha+1-1=-\alpha$$

$$u_1 = \frac{\Gamma(1-\alpha)\Gamma(-\alpha-1)}{\Gamma(-\alpha)\Gamma(-\alpha)} u_3 + \frac{\Gamma(1-\alpha)\Gamma(\alpha+1)}{\Gamma(1)\Gamma(1)} u_4$$

$$\Rightarrow F(1, -\alpha; 1-\alpha; \xi) = \frac{\Gamma(-\alpha+1)\Gamma(-\alpha-1)}{\Gamma(-\alpha)^2} (-\xi)^{-1} F(1, \alpha+1; \alpha+2; \xi^{-1}) + \frac{\Gamma(1-\alpha)\Gamma(1+\alpha)}{\Gamma(1)^2} (-\xi)^{\alpha} F(0, -\alpha; -\alpha; \xi^{-1})$$

$$\Gamma(1-\alpha) = (-\alpha)\Gamma(-\alpha)$$

$$\Gamma(1-\alpha)\Gamma(1+\alpha) = (-\alpha)\Gamma(-\alpha)\Gamma(1+\alpha) = \frac{(\alpha)\pi}{\sin\pi\alpha}$$

$$\text{So } F(1, -\alpha; 1-\alpha; \xi) = -\Gamma(-\alpha+1)\Gamma(-\alpha-1) \xi^{-1} F(1, \alpha+1; \alpha+2; \xi^{-1}) + \frac{\pi\alpha}{\sin\pi\alpha} (-\xi)^{\alpha}$$

$$\Rightarrow \text{disc } f(\bar{z}) = \frac{1}{2\pi i} \frac{(x_0)^{\alpha}}{(-\bar{z})} \cdot \frac{\pi\alpha}{\sin\pi\alpha} \cdot \text{disc } \left(-\frac{\bar{z}}{x_0}\right)^{\alpha} = (x_0)^{\alpha} (z)^{\alpha} = x^{\alpha} \checkmark$$

As $x_0 \rightarrow 0$, $\xi \rightarrow 0$ so first term vanishes. Then we get

$$f(z)|_{x_0=0} = \frac{1}{2\pi i} \frac{(x_0)^{\alpha}}{(-\bar{z})} \frac{\pi\alpha}{\sin\pi\alpha} \left(-\frac{\bar{z}}{x_0}\right)^{\alpha} = \frac{(-z)^{\alpha}}{-2i\sin\pi\alpha} = f(z)|_{x_0=0}$$

Summarize this example:

$$h(x) = \Theta(x-x_0) x^{\alpha}$$

$$\Rightarrow f(z) = \frac{1}{2\pi i} (x_0)^{\alpha-1} \Gamma(-\alpha+1)\Gamma(-\alpha-1) z F(1, \alpha+1; \alpha+2; \frac{x_0}{z}) + \frac{(-z)^{\alpha}}{-2i\sin\pi\alpha} = \frac{1}{2\pi i} (x_0)^{\alpha} F(1, -\alpha; 1-\alpha; \frac{z}{x_0})$$

So, this was an example in which I started with a rather arbitrary discontinuity and worked backwards to construct the analytic function whose discontinuity this was! At one time I thought that this example was impossible. I thought there was no analytic function which had this simple discontinuity starting at $x=x_0$. But now I have found such a function. Not an elementary function, of course.

Further exploration of this example: what does the analytic function $f(z)$ do at the branch point?

$$F(1-\alpha, 1-\alpha; \frac{\tau}{x_0}) \quad \text{Re } \alpha < -1$$

$$\text{Re}(a+b-c) = 0$$

The standard test fails.

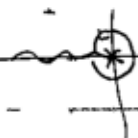
I conjecture the following: as you approach the branch point of $f(z)$, $f(z)$ diverges in some way, log or power. $f(z)$ is not finite. However, the endcap integration around the branchpoint vanishes, I guess. Moreover, as we know, the discontinuity suddenly drops to zero from a finite value as you pass over the branch point.

If $f(z)$ had a finite value at the branch point, then I don't see how the disc could drop suddenly to zero. That would not be possible.

Example of a branchpoint which has these properties:

$$f(z) = \ln z$$

$$(a) \quad f(0) = \infty \text{ diverges}$$



$$(b) \quad \oint f(z) dz = 0 \text{ since } \text{Re } \ln z \rightarrow 0 \text{ "soft"}$$

$$(c) \quad \text{disc } f(z) = 2\pi i, \text{ constant, drops suddenly to zero.}$$

Some facts about branch points.

1. First, consider the logarithmic example. We have:

$$f(z) = \ln z \quad z = Re^{i\theta} \Rightarrow \ln z = \ln R + i\theta$$

$$\text{disc } f(z) = (\ln R + i\frac{\pi}{2}) - (\ln R - i\frac{\pi}{2}) = 2i\pi, \text{ indep of } R.$$

$$\oint_{|R|} dz f(z) = iR \int_{-\pi/2}^{\pi/2} d\theta [\ln R + i\theta] = 2\pi i R \ln R \rightarrow 0$$

- (a) as you approach the branch point from different directions, you get different complex numbers, especially different imaginary part = angle of approach. At the branch point, $f(z)$ is "logarithmically infinite".
- (b) the discontinuity across the cut is a constant. At the end of the cut, it suddenly drops to zero.
- (c) the circle integration around the branchpoint depends on R , but as circle closes down onto the branchpoint, integral vanishes because the contour gets "shorter" faster than the function diverges.

2. Second, consider the simple power example:

$$f(z) = z^a$$

$$z = Re^{i\theta} \Rightarrow f(z) = R^a e^{ia\theta}$$

$$\text{disc } f(z) = R^a [e^{i\pi a} - e^{-i\pi a}] = 2i \sin \pi a R^a$$

$$\oint_{|R|} dz f(z) = iR \int_{-\pi}^{\pi} d\theta R^a e^{ia\theta} = -iR^{a+1} \left[\frac{1}{a+1} \right] \left[\frac{e^{i\pi a}}{2i} - \frac{e^{-i\pi a}}{2i} \right] 2i$$

$$= \frac{2i}{a} R^{a+1} \sin \pi a$$

Case I: $\text{Re } a > 0$

Case II: $\text{Re } a > -1$ but $\text{Re } a < 0$

$$-1 < \text{Re } a < 0$$

Case III: $\text{Re } a < -1$

Case IV: $\text{Re } a = 0$

Case V: $\text{Re } a = -1$

- (a) again, as you approach the branch point; the direction you choose controls the phase of the $f(z)$, so directions do not all give the same thing.
- (b) the cut disc in general depends on R . At the branch point the disc does:
 - 1) vanishes smoothly at branchpoint
 - 2,3,5) disc blows up at branchpoint
- (c) the circle integration depends on R . As closes on branchpoint you find:
 - 1,2) circle integration vanishes
 - 3) circle integration blows up.
- (a) the actual $f(z)$ blows up at branchpoint in case 2,3 but not 1.

	$f(z)$	$\text{disc } f(z)$	
$\text{Re } a > 0$	0	0	1.
$-1 < \text{Re } a < 0$	∞	∞	2.
$\text{Re } a < -1$	∞	∞	3.
$\ln z$	$\ln \infty$	const	4.

Comments:

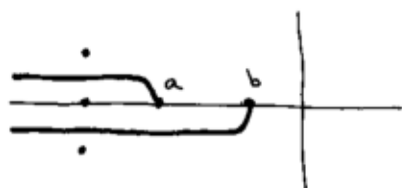
1. Even if the disc blows up, the circle integration may still vanish. (2.i)
2. Even if $f(z)$ blows up, the circle integration may still vanish.
3. The circle integration certainly vanishes if $f(z) = 0$ at branchpoint.
4. If the discontinuity drops from a constant to zero at branchpoint, you should suspect a logarithmic branchpoint.

Comments: Basically, there are not many kind of branchpoints you can make with elementary functions. You can make $\ln z$ and higher chains of same, or you can combine powers with logs.

I suspect, however, that there are much fancier branchpoints possible in functions which are integral representations of elementary functions, ie, special functions and higher functions. For example, $F(a,b;c;z)$ has a branch point at $z=1$ where sometimes the function is a constant at the branch point. None of my examples above could do this.

Comments: In particle physics, the scattering amplitude is supposed to have thresholds which are "soft" but the functions are supposed also to reggeize.

Rule for discontinuity of a product and dividend of two functions with cuts.

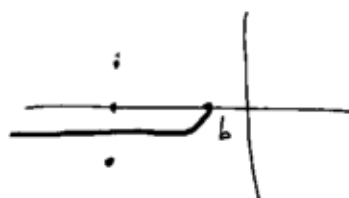
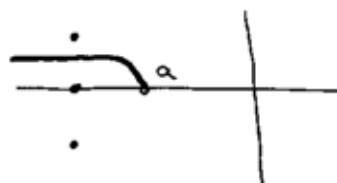


$$f(z) = a(z)b(z)$$

Consider: $a(x) [b(x) - b(x-i\epsilon)] + b(x) [a(x+i\epsilon) - a(x)]$

$$= \{a(x+i\epsilon)b(x+i\epsilon) - a(x-i\epsilon)b(x-i\epsilon)\} = [ab](x+i\epsilon) - [ab](x-i\epsilon)$$

$+ \cancel{a(x)b(x)} - \cancel{b(x)a(x)}$

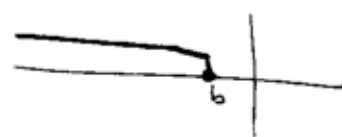
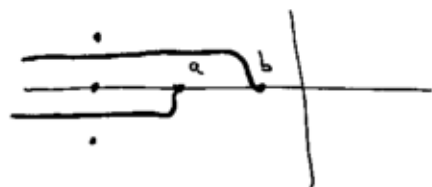


So $\Delta(ab) = a(\text{below its cut}) \Delta b + b(\text{above its cut}) \Delta a$

Let $f(z) = a(z)b(z)$.

Rule: $\Delta f = a(x) \Delta b + b(x) \Delta a$.

Suppose cuts were drawn the other way.



Now $a(x) \Delta b + b(x) \Delta a = a(x) [b(x+i\epsilon) - b(x)] + b(x) [a(x) - a(x-i\epsilon)]$

$$= a(x+i\epsilon)b(x+i\epsilon) - b(x-i\epsilon)a(x-i\epsilon) \quad \text{Same.}$$

Conclusion: Suppose you are given fact that $f(z) = a(z)b(z)$ where a and b have cuts in z . You want to know the disc of the function f . If you agree to draw one cut pulled above and the other cut pulled below, then the resultant formula is unambiguos, it does not matter which one you pull up and which one you pull down.

$$\boxed{f = ab \Rightarrow \Delta f = a \Delta b + b \Delta a}$$

Ans: $b(z) = \frac{1}{c(z)}$. Then: (first cut choice)

$$\Delta b = b(x) - b(x-i\epsilon) = \frac{1}{c(x)} - \frac{1}{c(x-i\epsilon)} = \frac{-\Delta c}{c(x)c(x-i\epsilon)}$$

$$\begin{aligned} \text{So } \Delta f &= a(x) \frac{[-\Delta c]}{c(x)c(x-i\epsilon)} + \frac{1}{c(x)} \cdot \Delta a \times \frac{c(x-i\epsilon)}{c(x-i\epsilon)} \\ &= \frac{1}{c(x)c(x-i\epsilon)} \left[c(x-i\epsilon) \Delta a - a(x) \Delta c \right] \end{aligned}$$

For second cut choice,

$$\Delta b = b(x+i\epsilon) - b(x) = \frac{1}{c(x+i\epsilon)} - \frac{1}{c(x)} = \frac{-\Delta c}{c(x+i\epsilon)c(x)}$$

$$\Rightarrow \Delta f = \frac{1}{c(x+i\epsilon)c(x)} \left[c(x+i\epsilon) \Delta a - a(x) \Delta c \right].$$

So here is result:

$$f(z) = \frac{a(z)}{c(z)}$$

$$\Rightarrow \Delta f = \frac{1}{c(x)c(x\pm i\epsilon)} \left[c(x\pm i\epsilon) \Delta a - a(x) \Delta c \right]$$

where $\pm$ means c cut drawn (above / below).