What is the cut structure of f(z) = F(a,b,c; 1/z²) PhL 2.18.10

Theory

First, here was my approach taken in my Legendre document:

The F function has a cut for $\xi = 1/z^2 \geq 1$, according to our Section 5 above. It is useful to plot this function $\xi = 1/z^2$,

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plot(1/(z^2),z= -4..4, view = 0..4);
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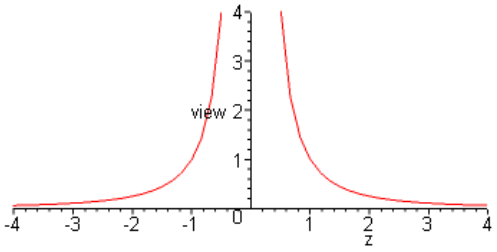
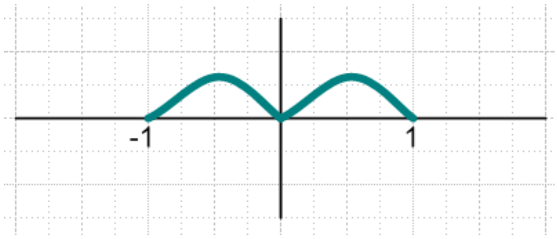


This shows that as z runs (1→0), the argument ξ runs (1→∞). And:
This shows that as z runs (-1→0), the argument ξ runs (1→∞). I like to view this as two cuts in the z plane, the first is (0,1) and the second is (-1,0) [although one can of course draw them as one (-1,1)]. These **two cuts** will appear **green** below. [Note: The hypergeometric *series* F(a,b,c; 1/z²) diverges inside |z| = 1, but the hypergeometric *function* F(a,b,c; 1/z²) which continues this series has these green cuts.]



Now, the presence of a cut suggests that the function has different values on the two sides of the cut. But how would that arise? Consider some z between 0 and 1: R < 1

$\xi = z^{-2}$ $z = Re^{\pm i\varepsilon}$ $\xi = R^{-2}\, e^{\mp\, 2i\varepsilon}$ $= S\, e^{\mp\, i\varepsilon'}$ $S > 1$ $F(a,b,c,\xi)$

So yes, ξ would indeed lie on the cut of F(a,b,c,ξ). For a point z in this range, we really have to use some continuation formula to see what is happening.

Kummer: The formulas starting with p 106 (25) all involve fixed phases and gamma functions, there are no functions of z appearing. We assume "general" values of a,b,c and avoid all gamma function poles. Our function F(a,b,c,ξ) is u_1 on page 105. We would like to relate this to u_3 and u_4 which have 1/ξ. This would be a formula red 134 (I just added these letters in Bateman) which is p 107 (34). So think of this as

$$F(a,b,c,\xi) = A\,u_3 + B\,u_4 = A\,(-\xi)^{-a}F(a,a+1-c;\ a+1-b;\ 1/\xi) + B\,A\,(-\xi)^{-b}F(b,b+1-c;\ b+1-a;\ 1/\xi)$$

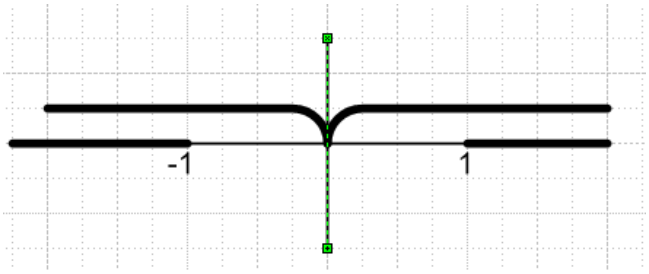
where the second term of course is the first with a↔b, as the symmetry of the LHS requires. Now set $\xi = z^{-2}$ and we get

$$F(a,b,c,\ z^{-2}) = A\,u_3 + B\,u_4 = A\,(-z^{-2})^{-a}F(a,a+1-c;\ a+1-b;\ z^2) + B\,A\,(-z^{-2})^{-b}F(b,b+1-c;\ b+1-a;\ z^2)$$

Consider one of these power factors:

$$(-z^{-2})^{-a} = (-z^2)^a = (0^2-z^2)^a = (0-z)^a\,(0+z)^a$$

We have to say this has a cut to the left and a cut to the right, both branch points at 0. And the residual F functions have branch points for |z| > 1, so here is our picture:



It must be, then, that the outer cuts cancel, and that gives our green-cuts picture! But now we have a way to evaluate this thing say at $z = Re^{\pm i\varepsilon}$ for R = 1/2. The residual F's are happy, and we have

$$(-z^{-2})^{-a} = (0-z)^a\,(0+z)^a = \quad (0-\ Re^{\pm i\varepsilon})^a\,(0+\ Re^{\pm i\varepsilon})^a$$

Now using our theory of power functions "study of (1-z)^a and related v2.doc ", we have

$$(z-0)^{\alpha}{}_{0L} = e^{\pm i\alpha\pi}\,(0-z)^{\alpha}{}_{0R} \qquad\qquad \mathrm{Im}(z) \gtrless 0$$

Then

$$(-z^{-2})^{-a} = (0-z)^a{}_{OR}\,(0+z)^a{}_{OL} = (z-0)^a{}_{0L}\,e^{\mp\, ia\pi}(0+z)^a{}_{OL} = e^{\mp\, ia\pi}\,(z)^{2a}$$

So we can rewrite our equation appropriate for points in the range z = (0,1)

$$F(a,b,c,\ z^{-2}) = A\,e^{\mp\, ia\pi}\,(z)^{2a}\,F(a,a+1-c;\ a+1-b;\ z^2) + B\,A\,e^{\mp\, ib\pi}\,(z)^{2b}\,F(b,b+1-c;\ b+1-a;\ z^2)$$

For example, setting z = 1/2 gives:

$$F(a,b,c,\ 4) = A\,e^{\mp\, ia\pi}\,(1/2)^{2a}\,F(a,a+1-c;\ a+1-b;\ 1/4) + B\,A\,e^{\mp\, ib\pi}\,(1/2)^{2b}\,F(b,b+1-c;\ b+1-a;1/4)$$

and we see that indeed the two numbers will be different. So those green cuts are really there.