An Ahlfors Exercise: PhL 8.18.09

Statement of the Problem. We have three arbitrary points z1,2,3 which define a circle C. We have some other point w at some arbitrary location. We want to show where the "symmetry point" w* is located in a geometric construction. That is to say: given C and w, where is w* located?

The information we do know is this

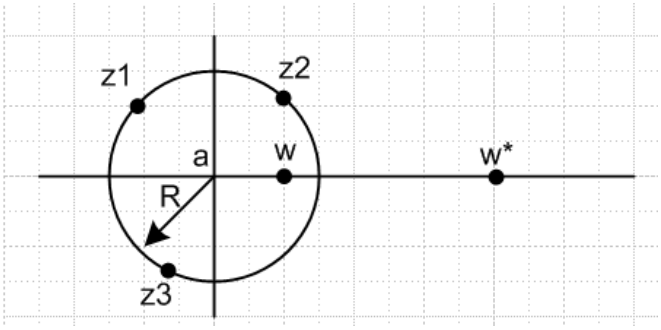
$$(w^*,z1,z2,z3) = \overline{(w,z1,z2,z3)}$$

which when written out says:

$$[(w^*- z_2)/ (w^*- z_3)] / [(z_1- z_2)/ (z_1- z_3)] = [(\bar{w}- \bar{z}_2)/ (\bar{w}- \bar{z}_3)] / [(\bar{z}_1- \bar{z}_2)/ (\bar{z}_1- \bar{z}_3)]$$

I don't want to read Ahlfors construction, I want to "discover it for myself" first, then I will read his thing.

The first order of business is to "guess" where the w* point lies. Here is my guess

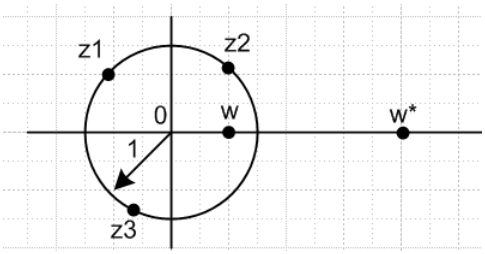


The point a is the center of our circle. We know when the circle is a straight line, w and w* are the obvious reflection points, so we guess that this remains true for the circle but distances are no longer equal. We also know that in that straight line case it does not matter WHERE the three points z_i are located on the line. The relation between w and w* is independent of the location of the z_i as long as they are somewhere on the line. That is to say, given w and the line, we can find w*.

So, I am going to assume that this is true for the circle case as well.

It seems pretty reasonable to choose a = 0 and R = 1 "without loss of generality". We then know that the three z_i satisfy this rule: $x_i^2 + y_i^2 = 1$. We don't care where they are on the circle, it does not matter.

So we now have

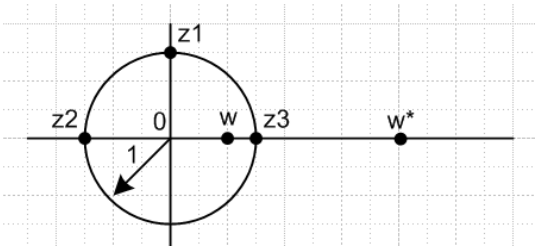


and we might as well put w and w* on the real axis and just think of them as real numbers. Our "assumption" then takes this form:

w>0 w*>0 w* = kw k>1 find k !

(Note: below we shall learn that $k = 1/w^2$ and is not a "constant" as I first thought it might be)

Since things are independent of the three z_i locations (my assumption), let's put these three points at simple locations:



which is to say

$z_1 = i$
$z_2 = -1$
$z_3 = 1$

$$[(w^*- z_2)/ (w^*- z_3)] / [(z_1- z_2)/ (z_1- z_3)] = [(\overline{w}- \overline{z}_2)/ (\overline{w}- \overline{z}_3)] / [(\overline{z}_1- \overline{z}_2)/ (\overline{z}_1- \overline{z}_3)]$$

Installing our various values and putting w and w* as reals, we get

$$[(w^*+1)/ (w^*- 1)] / [(i+1)/ (i- 1)] = [(w+1)/ (w- 1)] / [(-i+1)/ (-i- 1)]$$

$$[(w^*+1)/ (w^*- 1)] / [(i+1)/ (i- 1)] = [(w+1)/ (w- 1)] / [(i-1)/ (i+1)]$$

$$[(w^*+1)/ (w^*- 1)] = [(w+1)/ (w- 1)] / [(i-1)^2/ (i+1)^2]$$

But

$$[(i-1)^2/ (i+1)^2] = (-1 -2i +1)/(-1 +2i + 1) = -2i/2i = -1$$

$$(w^*+1)/(w^*-1) = -(w+1)/(w-1)$$

$$(w^*+1)/(w^*-1) = (w+1)/(1-w) \equiv \alpha$$

In this last equation, all the paren quantities are positive numbers, says the picture. So assume we are given w, then the RHS = $\alpha > 0$, some number. But

$$(w+1) > (1-w) \qquad \text{because} \qquad w > -w$$

Therefore $\alpha > 1$ in fact.

$$(w^*+1)/(w^*-1) = \alpha \qquad (w^*+1) = \alpha\,(w^*-1)$$

$$w^*(1-\alpha) = -\alpha-1 \qquad w^* = (\alpha+1)/(\alpha-1) > 1$$

Now write it all out

$$(\alpha+1)/(\alpha-1) = [\,(w+1)/(1-w) + 1\,] / [\,(w+1)/(1-w) - 1\,]$$

$$= [\,(w+1) + (1-w)\,] / [\,(w+1) - (1-w)\,]$$

$$= [\,2\,] / [\,2w\,] = 1/w$$

Thus, we get

$$w^* = 1/w$$

which seems a pretty simple result. If we now scale everything up by the radius, we get

$$w^* = R^2/w$$

because this is the power of R that makes the dimensions work right. [On dimensional grounds alone, it is hard to imagine how $w^* = f(R,w)$ could be any function other than this one!]

If we move the origin of the circle to "a" on the real axis, we get

$$(w^*-a) = R^2 / (w-a)$$

and if we move off the real axis, this becomes

$$|w^*-a| = R^2 / |w-a|$$

This then agrees with the red underline on page 81 where he calls the points z and z*.

So here is the answer to the problem stated:

(1) the three points a,w and w* are collinear, where a is the center of the circle on which the three points z_1 z_2 z_3 all lie.

(2) the rule $|w^*- a| |w-a| = R^2$ tells you how to find the location of w* if you know where w is, and vice versa. If one distance is > R, the other is < R, which says w and w* are on opposite sides of the circular boundary.

(3) Here is how you take the $R \rightarrow \infty$ limit to get the straight line case. Let

$w = R-d$
$w^* = R+d'$

then $w^*w = R^2$ says

$(R+d')(R-d) = R^2$

$(d'-d)R - dd'= 0$

$(d'-d)R = dd'$

$(d'-d) = dd'/R$

As $R \rightarrow \infty$ for finite d and d', we conclude that

$d = d'$

[See raw notes for a more elaborate version of this argument.]

(4) I have verified the "construction" shown on Ahlfors page 81.

- draw the line that passes through z and a (the circle center)
- draw a perp line to the above line that passes through z
- draw circle tangent lines at the two points where that perp line intersects the circle.
- where these tangent lines intersect is the point z*