Complex ODEs PhL 2.24.10

This is an oddball topic, just another that has me totally confused.

On the one hand, one thinks of an ODE as being in a real variable x on some interval (a,b) of the real axis. There is nothing "complex" about it.

On the other hand, there are regular singular points, branch points, cuts, Riemann sheets, and all the apparatus of the complex z plane. Usually the singular points are on the real axis, however. But to wrap around a branch point, you have to have the complex plane.

Stakgold's ODE discussion has no complex aspects to it as I recall. In Vol I we do single real variable, then in Vol II we do multiple real variables. His book is not about "complex functions" really.

Ahlfors has an ODE chapter at the end of his book which I have not read yet.

Hypothesis #1. You start with any ODE in variable x. $L_x u(x) = 0$. Forget BC's for the moment. There are two independent solutions, call them $u_1(x)$ and $u_2(x)$. We then sort of "continue" everything to the complex z plane and we then magically have $L_z u(z) = 0$ with solutions $u_1(z)$ and $u_2(z)$. Away from singularities, these $u_i(z)$ are magically analytic functions. One way to accept that is that the $u_i(x)$ had power series representations with real coefficients, and these then naturally become analytic functions when you replace x by z. In the complex ODE $L_z u(z) = 0$ the derivatives like ∂_z are now really complex function derivatives which are "the same in all directions". For me, I think of f(z) as the complex function, and I think of f '(z) as that complex derivative. So according to this hypothesis, all ODE's can be thought of as "complex ODE's", an example being the Legendre ODE.

When we do a problem like the oblate hyperboloid with boundary conditions in 3D space, each of the three variables is treated as a real variable, so this is back in the Stakgold or Smythe bailiwick.

Example 1: $-\partial_x^2 f(x) = k^2 f(x)$ // changing from u to f = u + iv.

In the real ODE world, we have solutions $e^{\pm ikx}$ for $f_i(x)$. (Perhaps I should have used sin(kx), cos(kx) here to be fully in the real world and not have $f_i(x)$ be complex. But this does show the intermediate world where $f_i(x)$ is a complex function of a real variable.) Now let's jump to complex:

$-\partial_z^2 f(z) = k^2 f(z)$ $\qquad f(z) = e^{\pm ikz} = e^{\pm ik(x+iy)} = e^{\pm ikx}\, e^{\mp iky}$

How then do we compute a derivative $\partial_z f(z)$. You do it "in any direction" and Ahlfors on page 25 shows perhaps the simplest way to do it:

$f(z) = u(z) + iv(z)$

$f(x,y) = u(x,y) + iv(x,y)$

$\partial_z f = \partial_x u(x,y) + i\, \partial_x v(x,y)$

This is "doing it in the x direction". So let's try it:

$f(z) = e^{\pm ikx}\, e^{\mp iky} = \cos(ikx)\, e^{\mp iky} \pm i \sin(ikx)\, e^{\mp iky} = u + iv$

$\partial_x u(x,y) = -ik\sin(ikx)\, e^{\mp iky}$

$\partial_x v(x,y) = \pm ik\cos(ikx)\, e^{\mp iky}$

$\partial_z f = \partial_x u(x,y) + i\, \partial_x v(x,y) = -ik\sin(ikx)\, e^{\mp iky} + i\, \{ \pm ik\cos(ikx)\, e^{\mp iky} \}$

$= \mp k\cos(ikx)\, e^{\mp iky} - ik\sin(ikx)\, e^{\mp iky} = \mp k\, e^{\mp iky}[\cos(ikx) \pm i \sin(ikx)]$

$= \mp k\, e^{\mp iky}\, e^{\pm ikx} = \mp k\, e^{\pm ikz}$

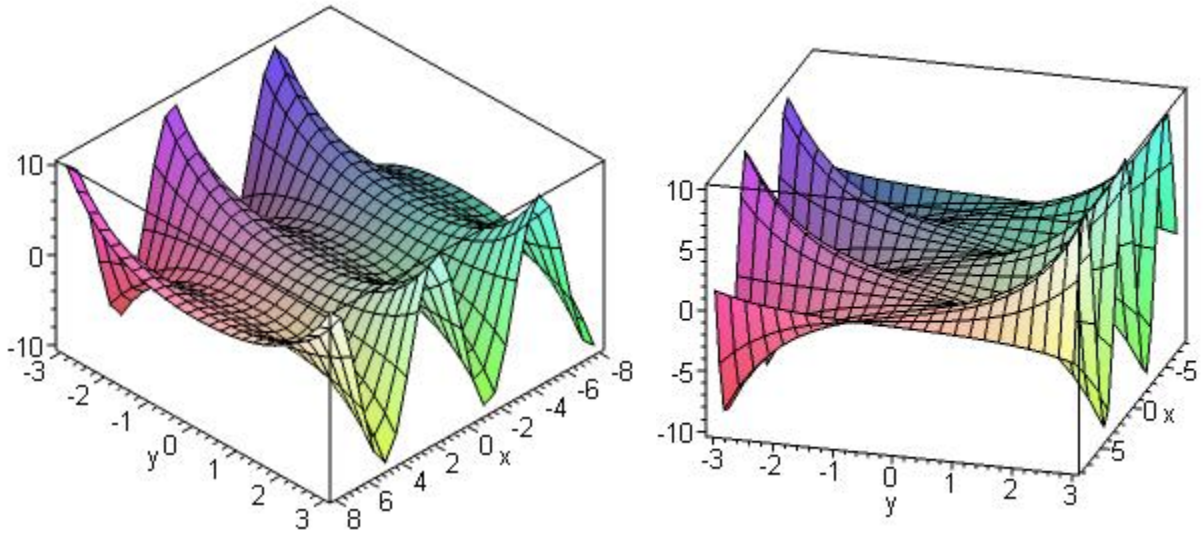
Sort of the long way around, but the point is that you really can do it using the x and y notation only. Of course we would get the last result instantly just applying ∂_z to $f(z) = e^{\pm ikz}$. So we have then shown, using x and y notation only, that this complex function f(x,y) really does solve $-\partial_z^2 f(z) = k^2 f(z)$.

Here is another way to view the above. Let $f(x) = \Sigma_n a_n x^{n+r}$ be the Frobenius solutions about point x = 0 which we assume is non-singular. Then a_n are just right so that $-\partial_x^2 f(x) = k^2 f(x)$. The a_n are real. Then define the new complex function $f(z) = \Sigma_n a_n z^{n+r}$. We know the rule for powers, and it is completely obvious that this thing must be a solution of $-\partial_z^2 f(z) = k^2 f(z)$.

What about BC's ? Suppose we solve our $L_x u(x) = 0$ subject to some BC's. We just "continue" this equation to complex z in the same magical fashion away from singular points of the ODE. Then a BC has the form $BC[\, u(z_1)] = 0$ where it happens that z_1 = real. Same idea for $L_x u(x) = f(x)$. The notion of "continue" is just this: to the extent something is a power series with real coefficients, that is what you continue by replacing x by z.

Example 2: Consider the string problem $-\partial_x^2 f(x) = k^2 f(x)$ where f(0) = f(b) = 0. The solution we know is f(x) = A sin(kx). The complex solution is then f(z) = Asin(kz).

plot3d(Re(sin(x+I*y)), x = -8..8, y = -3..3); Im over here ↓



The physical string lies along the center line of the two pictures. On the left we get sin(kx) along that line, and on the right we get 0 along that line. So the surfaces are what you get by "continuing" the string off the real axis. We show here THE solution to $-\partial_z^2 f(z) = k^2 f(z)$ with $f(0) = f(\pi/k) = 0$. In all directions except the real axis, we get exponential growth.

Question #1: Suppose f(z) is a solution to $L_z f(z) = 0$. Is f(z*) also a solution? Do we have $L_z f(z^*) = 0$?

Answer #1: If we assume that our solution f(z) is analytic in some mirrored region about the real axis (which might not be true if there are somehow complex branch points that make this not so) then when we say that $L_z f(z) = 0$ we mean this is true for all z in complex domain D. Then z* is also in that domain so definitely we have $L_z f(z^*) = 0$.

Answer #2: If we start with $L_z f(z) = 0$ and if we assume that L_z is "real", meaning that written as L_x the various functions inside L are real functions of real x, then we apply * to find that $L_{z^*} f(z^*) = 0$. To get this form, we just star all occurrences of z in L_z, and we assume that power series form for f(z) with real coefficients. So this shows that $L_{z^*} f(z^*) = 0$, NOT that $L_z f(z^*) = 0$. So this answer is inconclusive and tends toward the negative.

Example 3: $-\partial_z^2 f(z) = k^2 f(z)$

We have our complex solutions of a complex variable:

$$f(z) = e^{\pm ikz} = e^{\pm ikx}\, e^{\mp iky}.$$

So we want to test out this function:

$$f(z^*) = e^{\pm ikz^*} = e^{\pm ikx}\, e^{\pm iky}.$$ // sign change on y

We can manually compute the derivative as we did above. Imagine writing all the steps as above, but then we edit everything and replace y → -y. Our derivatives were all x derivatives, so nothing changes except for this replacement. Our last step would then be:

$$\partial_z f(z^*) == \mp k\, e^{\mp ik(-y)}\, e^{\pm ikx} = \mp k\, e^{\mp ik(-y)}\, e^{\pm ikx} = \mp k\, e^{\pm ikx}\, e^{\pm iky} = \mp k\, f(z^*)$$

It seems to work, and we seem then to have shown that $L_z f(z^*) = 0$ and Answer #1 seems right. Again, our solution is for all z, so we just examine the solution at the particular point $z = z^*$.

This is a bit tricky. We always have f(z) being a function of z, not z*. So in some sense $\partial_{z^*} f(z) = 0$. Ahlfors talks this way sometimes, imagining functions F(z,z*) where these are two independent variables, replacing f(x,y). In this sense, we would say

$$L_z f(z^*) = [-\partial_z^2 - k^2] f(z^*) = -k^2 f(z^*) \neq 0.$$

Similarly, $L_{z^*} g(z) = 0$. I guess the key is that, when you declare a function, you have to clearly state what independent variables it is a function of: Is it x,y or is it z,z*.

Example 4: $P_n^m(z)$

We know that $P_n^m(2 + i3)$ solves the Legendre ODE. That is to say, $[L_z P_n^m(z)]|^{2+i3} = 0$. We loosely write this as $L_z P_n^m(2 + i3)] = 0$. This is true for any z in the complex plane. So $L_z P_n^m(2 - i3)] = 0$ as well.

Theorem: In the sense of z = (x,y) as the variables,

$$[L_z f(z)]^{z=z1} = 0 \quad => \quad [L_z f(z)]^{z=z1^*} = 0$$

or

f(z) is a solution of $L_z f(z) = 0$ at point $z = z_1$ => f(z) is a solution of $L_z f(z) = 0$ at point $z = z_{1^*}$

or in very sloppy notation

$$L_z f(z_1) = 0 \quad => \quad L_z f(z_1^*) = 0$$

Question #2: Suppose $[L_z f(z)] = 0$ is an ODE and f(z) is a solution. Are either of the following also solutions of the ODE?

(a) $g(z) = A\, f(z) + B\, f(-z)$

(b) $h(z) = A\, f(z) + B\, f(z^*)$

Clarification: We have to make the *distinction* between a functional form and a function evaluation. Here is what we know:

If $[L_z f(z)] = 0$, we say that f(z) is an ODE solution at any point z. In (a) or (b) above, if our meaning of the second term is "functional evaluation", then f(z) involves the point z, and f(-z) involves a completely different point -z. So the function g(z) is a linear combination of solutions to the ODE evaluated *at different points* in the z plane! So in this sense, we cannot say that we are linearly combining solutions at the same point, so the whole thing makes no sense. Therefore, when we write things like (a) and (b), we have to be talking about the "functional form" interpretation of the second terms.

Question #3. Suppose we write $f(z) = \Sigma_n a_n z^n$. How do we express our two different meanings for f(-z)?

functional form:

$f(-z) = \Sigma_n f_n (-z)^n = \Sigma_n f_n (-1)^n z^n = \Sigma_n F_n z^n = F(z)$, some *new function* of z.

function evaluation:

$f(-z) = \Sigma_n f_n (-z)^n$, evaluation of f(z) at the point z = -z.

Example 5. So we can ask about the validity of this equation: $-\partial_z^2 F(z) = k^2 F(z)$
where $F(z) \equiv f(-z)$ in the functional sense.

For this particular ODE, it turns out that $L_z = L_{-z}$ since

$$d/dz \; d/dz = d/d(-z) \; d/d(-z)$$

so we can write:

$$-\partial_z^2 F(z) = -\partial_{[-z]}^2 F(z) = -\partial_{[-z]}^2 f(-z) = -\partial_w^2 f(w) \qquad w = -z, \text{ new complex variable}$$

$$= k^2 f(w) = k^2 f(-z)$$

Therefore: $-\partial_z^2 f(-z) = k^2 f(-z)$ so that

$$L_z f(-z) = k^2 f(-z)$$

in functional sense, meaning really that

$$L_z F(z) = k^2 F(z).$$

This would also be true if the ODE had a term $z\partial_z$ since $z \; d/dz = (-z) d/d(-z)$. But it would not be true if the ODE had a term $5\,\partial_z$.

So for this example, the linear combination $g(z) = A f(z) + B F(z)$ *would* in fact also be a solution of the ODE *at the point z*. We are no longer talking about functions at two different points. Both f(z) and F(z) are evaluated at the point z. So this is an example of the (a) case above.

What about the (b) case? Let's try to mimic the previous example.

Example 6. So we can ask about the validity of this equation: $-\partial_z^2 G(z) = k^2 G(z)$
where $G(z) \equiv f(z^*)$ in the functional sense.

Let's paste and then edit stuff from above:

functional form:
$f(z^*) = \Sigma_n f_n (z^*)^n \neq \Sigma_n G_n z^n = G(z)$, some *new function* of z. // Note the NOT equal.

function evaluation:
$f(z^*) = \Sigma_n f_n (z^*)^n$, evaluation of f(z) at the point $z = z^*$.

So things are now quite different. There exists no G(z) that we can even talk about! So we cannot really ask whether $L_z G(z) = 0$ since there is no G(z). We can say all these things, however:

$L_z f(z) = 0$ $L_w f(w) = 0$ $L_{z^*} f(z^*) = 0$

But we cannot claim that $L_z = L_{z^*}$. For example, we do not have d/dz d/dz = d/dz* d/dz* . This is because we do not have $(dz)^2 = (dz^*)^2$. That is to say,

$(dz)^2 = (dx + idy)^2 \neq (dx - idy)^2$

Therefore, if we write

$h(z) = A f(z) + B f(z^*)$

this cannot be represented in a functional form sense as

$h(z) = A f(z) + B G(z)$

so we cannot even talk about this being a linear combination of two ODE solutions *at the same point*. All we can say is that it is the sum of solutions at two different points.

Example 7: $P_n^m(z)$.

The Legendre ODE has the property that $L_z = L_{-z}$ so we can then make these claims:

$F(z) \equiv P_n^m(-z)$ is a solution of $L_z F(z) = 0$ at z, the Legendre ODE

$g(z) = AP_n^m(z) + P_n^m(-z)$ is a solution of $L_z g(z) = 0$ at z, the Legendre ODE

In contrast, we **cannot say** that

$F(z) \equiv P_n^m(z^*)$ is a solution of $L_z G(z) = 0$ at z, the Legendre ODE

$h(z) = AP_n^m(z) + P_n^m(z^*)$ is a solution of $L_z h(z) = 0$ at z, the Legendre ODE

Combining solutions on different Riemann sheets.

Let f(z) be an analytic function that has multiple sheets. Then $f(z)_n$ refers to the solution on the nth sheet, and perhaps n = 0 is the principle sheet. If there are three branch points we might write $f(z)_{n1,n2,n3}$ to fully describe the function, but we just say $f(z)_n$ as a symbolic notation:

The ODE Sheet Theorem: if $L_z f(z)_{n1} = 0$, then if $L_z f(z)_{n2} = 0$ for any sheet n_2 .

I am not quite sure how to prove this, but I know it is true. The ODE is a local operator object that acts on a tiny patch of the z plane somewhere. It does not know which "sheet" that patch lies on, so it is valid for any sheet. The sheet structure is a global property, not a local property. So that is my first argument. The second is that I know that when you start with some $f(z)_0$ and cycle around a branch point to an adjacent sheet, the result is some linear combination of the two independent solutions of the ODE, and we write this as $f(z)_1$. We generally don't have $f(z)_1 = f(z)_0$. For example, $\sqrt{z} \neq -\sqrt{z}$ to pick a very simple example. One more try: The ODE's $L_z f(z)_{n1} = 0$ and $L_z f(z)_{n2} = 0$ look exactly the same! We are just talking two different sheets. The solution to each equation is some linear combination of some set of two independent solutions that you select. Notice that our theorem is not telling us how to find $f(z)_{n2}$ if we are given $f(z)_{n1}$, it is just saying that whatever $f(z)_{n2}$ comes out to be, in will be an ODE solution. But I will go on to show below how to actually find $f(z)_{n2}$ if we are given $f(z)_{n1}$.

The solution to $L_z f(z)_0 = 0$ has a Frobenius form $\Sigma_n a_n (z-a)^{n+r}$ where a is a branch point and r is a fixed Frobenius exponent or index. If we wind once around this branch point we get

$$f(z)_0 = \Sigma_n a_n (z-a)^{n+r} \rightarrow f(z)_1 = \Sigma_n a_n [(z-a)e^{2\pi i}]^{n+r} = \Sigma_n a_n e^{2\pi i(n+r)} (z-a)^{n+r}$$

$$= \Sigma_n a_n e^{2\pi i r} (z-a)^{n+r} = e^{2\pi i r} \Sigma_n (z-a)^{n+r} = e^{2\pi i r} f(z)_0$$

But suppose we started with some linear combination:

$$F(z)_0 = A \Sigma_n a_n (z-a)^{n+r1} + B \Sigma_n b_n (z-a)^{n+r2}$$

then when we go around the branch point once we get

$$F(z)_1 = A e^{2\pi i r1} \Sigma_n a_n (z-a)^{n+r1} + B e^{2\pi i r2} \Sigma_n b_n (z-a)^{n+r2}$$

$$= A e^{2\pi i r1} \Sigma_n a_n (z-a)^{n+r1} + B [e^{2\pi i r2} - e^{2\pi i r1} + e^{2\pi i r1}] \Sigma_n b_n (z-a)^{n+r2}$$

$$= e^{2\pi i r1} [A \Sigma_n a_n (z-a)^{n+r1} + B \Sigma_n b_n (z-a)^{n+r2}] + B [e^{2\pi i r2} - e^{2\pi i r1}] \Sigma_n b_n (z-a)^{n+r2}$$

$$= e^{2\pi i r1} F(z)_0 + B [e^{2\pi i r2} - e^{2\pi i r1}] \Sigma_n b_n (z-a)^{n+r2}$$

So this is what I wanted! Here I show that when you wrap around a branch point, you get a multiple of the original function, plus a new term which is some linear combination of whatever two independent solutions you picked. If we happen to pick the exact Frobenius solutions as the two independent solutions, then the wrap causes us to pick up a multiple of the second solution. I don't know if this is a theorem with someone's name on it. I don't know where to find this result even stated, though I have read it perhaps in Messiah or elsewhere.

So this pretty much firms up my ODE Sheet Theorem stated above. As you go to a nearby sheet, the result you get is still an ODE solution. This is what "analytic continuation" to nearby sheets means.

Of course if the exponents are just right, we have to deal with the log case and all that, but I think the point is clearly made without getting involved in those details. When we wrap a log branch point, we pick up an additive term, but I know it will still be an ODE solution. For example, when we start with $Q_1(z)$ on its principle sheet, we pick up an additive term Az and we know $z = P_1(z)$ which is an ODE solution. In fact, recall these facts from ODE notes 1 doc, for the case of a branch point at z = 0,

"(6) What happens if $r_1 = r_2 + s$ with s = positive integer ? In this case the solutions are p 195 B:

$$w_1(z) = z^{r1} [1 + \Sigma_{n=1}^{\infty} A_n z^n]$$

$$w_2(z) = z^{r2} [- 1/s + \Sigma_{n=1}^{\infty} h_n z^n] + g_s w_1(z) \ln(z) "$$

So you see if z wraps, we pick up a multiple of $w_1(z)$ which is a solution.

Corollary #1 : If if $L_z f(z)_{n1} = 0$, then $L_z \{ A f(z)_{n1} + B f(z)_{n2} \} = 0$.

This says that you can linear combine solutions at the same point on different sheets and what you get will still be an ODE solution.

Corollary #2 : If if $L_z f(x+i\varepsilon) = 0$, then $L_z \{ A f(x+i\varepsilon) + B f(x-i\varepsilon) \} = 0$

Here we might imagine that $f(x+i\varepsilon) = f(x+i\varepsilon)_0$ where $z = x + i\varepsilon$ is on Sheet 0. We are of course imagining that we are on a cut. So $f(x-i\varepsilon)$ means $f(x+i\varepsilon)_1$. So we are just applying Corollary #2 to the particular location $z = x + i\varepsilon$. What we really mean by the above is this:

$$L_z \{A f(z)_0 \}|^{z = x+i\varepsilon} + L_z \{B f(z)_1 \}|^{z = x+i\varepsilon} = 0$$

or

$$L_z [A f(z)_0 + B f(z)_1] |^{z = x+i\varepsilon} = 0$$

Of course it will also be true in general that

$$L_z [A f(z)_0 + B f(z)_1] = 0 \qquad \text{for all z on the complex z plane.}$$

Notice please that our Corollary does not say $L_z \{ A f(z) + B f(z^*) \} = 0$ for general z. We have already shown above how this makes no sense.