Conformal Mapping and Potential Theory PhL 11.3.09

Preface. I have no doc on this subject. Jackson's course did some of this, notes are lost. Ahlfors has a chapter exactly on this subject (Chap 6) which I have not read. I have a Schaum book with a section on the subject. And Stakgold has a brief section that only deals with Green's Functions, Vol II p 164 (he does not use the word conformal). So here I will try to collect the basic facts which I seem to be quite confused about right now.

Preliminaries

1. I know that the key idea of an analytic function f(z) is that the derivative at some point z_0 must be the same no matter in which direction you approach z_0 and this derivative is then called $f'(z_0)$. I then know that the real and imaginary parts of f(z) satisfy the Cauchy Riemann equations, and then I know the real and imaginary parts of f(z) are harmonic, and then so is complex f(z) harmonic, meaning satisfies Laplace. I know that any power of z is analytic, so any polynomial is analytic. All this is from Ahlfors Chap 2 on complex functions.

2. If I am given real harmonic $f_r(x,y)$, I know how to compute $f_i(x,y)$ [use the C-R equations] and therefore I know how to obtain the analytic function f(z) of which f_r is the real part of. It is possible to compute f(z) directly from $f_r(x,y)$, something Ahlfors shows. So given $f_r(x,y)$, the imaginary part $f_i(x,y)$ of f(z) is completely determined, it is not arbitrary, it cannot be set to 0, for example.

Development of Schaum Theorem 2 (p 242)

We consider a mapping R to R' (z in R, w in R') where w = f(z), and f(z) is analytic. We write,

$$w = f(z) \quad \text{means} \quad w = u + iv = u(x,y) + i\,v(x,y) \quad (u,v) = (u(x,y), v(x,y))$$

where we are sort of overloading the symbol "u" to mean the real coordinate of w, and to mean a function of two variables x and y implied by the mapping. Since functions u and v are the real and imaginary parts of an analytic function f, they are both harmonic, eg, $\nabla^2 u(x,y) = 0$. Notice the fact that u and v are REAL functions.

Page 233 Theorem 1 says that as long as $f'(z) \neq 0$ in R, we can invert f to get $z = f^{-1}(w)$. I shall return to this subject below, where I will show that $f^{-1}(w)$ exists and is analytic in R' subject to a few small extra conditions. If w = u + iv then we write the inverse version of our line above,

$$z = f^{-1}(w) \quad \text{means} \quad z = x + iy = x(u,v) + i\,y(u,v) \quad (x,y) = (x(u,v), y(u,v))$$

where now we are overloading the symbol "x" to mean the real part of z, and to mean the name of a function of the two variables u and v. Notice the fact that x and y are REAL functions.

Statement of Theorem 2. Assume we have a mapping f:R→R' where f(z) is 1-to-1, and onto. We know from elsewhere (outlined in section below) that (1) this means $f^{-1}(w)$ exists; (2) if we further assume that f(z) is analytic, and that $f'(z) \neq 0$ in R, then we know that $f^{-1}(w)$ is analytic on R'. Since $f^{-1}(w)$ exists, we

know that the functions x and y shown above exist. So, suppose $\varphi(x,y)$ is real and harmonic in R. Then we can write $\varphi(x,y) = \varphi(x(u,v), y(u,v)) \equiv \Phi(u,v)$. We clearly end up with a real function $\Phi(u,v)$ defined on region R'. Theorem 2 claims that this function $\Phi(u,v)$ is *harmonic* on R', something that is not immediately obvious.

The Schaum proof of Theorem 2 is non-trivial and is given on page 242. Most of the labor goes into this computation:

$\nabla_{x,y}^2 \varphi(x(u,v), y(u,v))$ = sum of 10 terms, all with plus signs, shown as (1) mid page 242.

Things are written so that the only derivatives acting on φ are ∂_u and ∂_v things, so we are just doing a change of variables from x,y to u,v in the usual curvilinear coordinates sense. Lots of $(\partial_x u)$ and similar factors appear. The 10 terms are grouped into 5 pairs where I have labeled certain factors circle 1,2,3,4,5. Factors 1 and 2 are 0 because u and v are harmonic, being real and imaginary parts of analytic function f(z), as noted above. Factor 4 vanishes due to Cauchy Riemann equations for u and v. Finally, factors 3 and 5 are in fact the same and are each in fact equal to $|f'(z)|^2$ as is clearly shown. Our final result is this:

$$\nabla_{x,y}^2 \varphi(x(u,v), y(u,v)) = |f'(z)|^2 \nabla_{u,v}^2 \varphi(x(u,v), y(u,v)) = |f'(z)|^2 \nabla_{u,v}^2 \Phi(u,v)$$

Now with all this preparation, Theorem 2 is trivial. As long as $f'(z) \neq 0$, if $\varphi(x,y)$ is harmonic in R, then $\Phi(u,v)$ is harmonic in R', and vice versa.

Now we return to the existence of $f^{-1}(z)$ and its nature. First, on page 65 of Ahlfors we restrict our interest to functions $f:R \rightarrow R'$ which are one-to-one and which are ONTO, which means all of R' gets hit in the mapping (once) as R is scanned. For such restricted f(z) functions, $f^{-1}(w)$ exists and reverse maps all of R' back into R. If in addition f and f^{-1} are continuous, then he says f(z) is a **topological mapping**. Here Ahlfors is just talking about mappings, not necessarily analytic functions. From page 24 of Ahlfors, we know that if f(z) is analytic, it is continuous, as are its real and imaginary parts. So our only uncertainty at this point is whether $f^{-1}(w)$ is continuous. Now on page 75 of Ahlfors -- and first I quote my own notes --

$w = f(z)$	$z = f^{-1}(w) \equiv g(w)$	$1 = \partial_z z = \partial_z g(w(z)) = g'(w)\, \partial_z w = g'(w)\, \partial_z(f) = g'(w) f'(z).$
=>	$g'(w) = 1/f'(z)$	so if $f'(z) = 0$, $g'(w)$ there does not exist, g not analytic

we can see that $g'(w)$ always exists as long as $f'(z) \neq 0$. If we restrict to f(z) which are one to one and onto, we know g(w) exists (the inverse of f(z)), and if we further restrict to $f'(z) \neq 0$, then we see that not only does g(w) exist, it is analytic in w. So $f^{-1}(w)$ is analytic in this case on w in region R'. And then it is of course also continuous.

Phil Theorem 1. Suppose real $\varphi(x,y)$ is harmonic on R on whose boundary σ we have $\varphi(\sigma) = A$. Given an analytic mapping $w = f(z)$ that is one to one and onto and $f'(z) \neq 0$ in R, we construct $\Phi(u,v)$ as noted above, and we observe that Φ is harmonic on R' which has boundary σ'. We know that every point on boundary σ maps into some point on boundary σ'. If $\varphi(z) = A$ for every z on σ, then $\Phi(w) = A$ for every w on σ'. Note that we don't claim to know which points on σ go onto which on σ', but if φ is constant on boundary σ, then Φ is the same constant on σ'.

(a) R is simple. Suppose we want to solve a Laplace problem on region R' with $\Phi = A$ on σ'. Perhaps region R' is "ugly" and the problem is hard to solve. Suppose we can find a mapping $w = f(z)$ which maps R into R', where R is a nice simple region we know how to handle. We note in passing that f^{-1} exists and

$z = f^{-1}(w)$ means $z = x + iy = x(u,v) + i\, y(u,v)$ $(x,y) = (x(u,v), y(u,v))$

so knowing f(z) means we know $f^{-1}(w)$ which means we know functions x(u,v) and y(u,v). So the plan is to go ahead and solve the problem on R and determine $\varphi(x,y)$. Then we know at once that the solution to the problem on R' is $\Phi(u,v) = \varphi(x(u,v), y(u,v))$ and we are all done!

This idea may now be stated in the equivalent reverse sense. I quote the above and edit.

(b) R' is simple. Suppose we want to solve a Laplace problem on region R with $\varphi = A$ on σ. Perhaps region R is "ugly" and the problem is hard to solve. Suppose we can find a mapping $w = f(z)$ which maps R into R', where R' is a nice simple region we know how to handle. We note in passing that

$w = f(z)$ means $w = u + iv = u(x,y) + i\, v(x,y)$ $(u,v) = (u(x,y), v(x,y))$

so knowing f(z) means we know functions u(x,y) and v(x,y). So the plan is to go ahead and solve the problem on R' and determine $\Phi(u,v)$. Then we know at once that the solution to the problem on R is $\varphi(x,y) = \Phi(u(x,y), v(x,y))$ and we are all done!

Question: Is there some simpler way to prove Theorem 2 ?

In the above work, suppose we start off with real harmonic function $\varphi(x,y)$ being the real part of some analytic function I will call h(z). We know that, given φ, we can find ψ and h such that

$h(z) = \varphi(x,y) + i\psi(x,y)$ = analytic in R

We can then think about our mapping $w = f(z)$ and say

$h(z) = h(f^{-1}(w)) = H(w) = A(u,v) + i\, B(u,v)$

Since h and f^{-1} are both analytic, certainly H(w) is analytic, so A and B are harmonic on R'. Of course A and B and φ and ψ are all real functions. Comparing the two lines above, we see that

$A(u,v) = \varphi(x,y) = \varphi(x(u,v), y(u,v)) = \Phi(u,v)$ above!
$B(u,v) = \psi(x,y) = \psi(x(u,v), y(u,v))$

So here is an easier proof of Theorem 2: (1) realize that real harmonic $\varphi(x,y)$ is the real part of some analytic function I have arbitrarily called h(z). (2) take analytic (+other properties) mapping $w = f(z)$ and construct $h(z) = h(f^{-1}(w)) = H(w)$ which is analytic in w. The real part of H(w) is $\varphi(x(u,v), y(u,v)) \equiv \Phi(u,v)$ which, being the real part of an analytic function H, must be harmonic in R' . So Theorem 2 is now completely obvious, no math needed. One other detail:

$H(f(z)) = h(z)$ $\quad dH/dz = dH/df \; df/dz = dh/dz \quad$ $H'(w)\, f'(z) = h'(z)$

So we end up with $H'(w) = h'(z) / f'(z)$ so again we must insist that $f'(z) \neq 0$ on R.