Contour integration example 1 PhL 6.18.10

Consider this integral

$$I = \int_0^{2\pi} d\theta / \sqrt{\alpha-\cos\theta} \qquad \alpha \geq 1$$

I know we can look this up, but I want to understand this integral as a contour integral, that is the subject of this doc. The usual steps are these

$$z = Re^{i\theta} \qquad dz = izd\theta \qquad d\theta = (1/i)\, dz/z$$

$$z/R + R/z = e^{i\theta} + e^{-i\theta} = 2\cos\theta \qquad \cos\theta = (1/2)(z/R + R/z) = (z^2+R^2)/(2Rz)$$

$$I = (1/i) \oint dz/z \; (1/\sqrt{\alpha-\cos\theta}) = -i \oint dz/z \; (1/\sqrt{\alpha-(z^2+R^2)/(2Rz)})$$

$$= -i \oint dz(1/z)\,(2Rz)^{1/2}\,(1/\sqrt{\alpha 2Rz-(z^2+R^2)})$$

$$= -i\,(2R)^{1/2} \oint dz\,(1/\sqrt{z})\;(1/\sqrt{\alpha 2Rz-(z^2+R^2)})$$

Examine the argument of the radical

$$f(z) = 2\alpha Rz - z^2 - R^2 = (-1)(z^2-2\alpha Rz + R^2) = (-1)(z-z_1)(z-z_2) = (z_1-z)(z-z_2)$$

$$B^2-4AC = 4\alpha^2R^2 - 4R^2 = 4R^2(\alpha^2-1)$$

$$z_{1,2} = \alpha R \pm R\sqrt{\alpha^2-1} = R(\alpha \pm \sqrt{\alpha^2-1}) \qquad z_1 = \text{the } +$$

I claim that

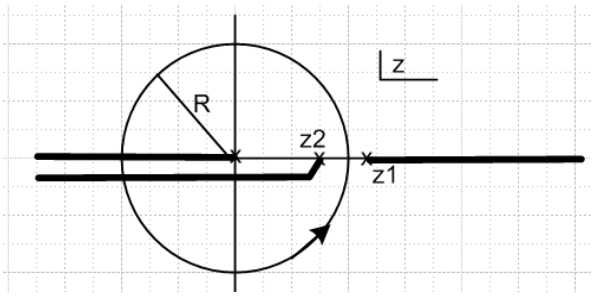
$[\alpha + \sqrt{\alpha^2-1}]$ is ≥ 1 $\quad = 1$ when $\alpha = 1$

$[\alpha - \sqrt{\alpha^2-1}]$ is ≤ 1 $\quad = 1$ when $\alpha = 1$

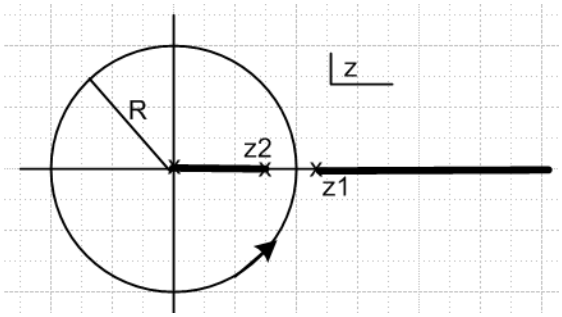
So we now have

$$I = -i\,(2R)^{1/2} \oint dz\,(1/\sqrt{z})\;(1/\sqrt{(z_1-z)(z-z_2)})$$

Now, the whole exercise here is to try to understand what this looks like in the z plane and how to interpret how it looks.



Question: what "sheets" does this circular contour lie on? I guess we realize that we can redraw the cuts like this



and then there is no "mystery" as to which sheet the contour is on. That is to say, at least the contour is all on one sheet, it does not cross any cuts in some strange way. We can then shrink the contour around the little segment shown. Then write for points above the cut

$$z\text{-}z_2 = |z\text{-}z_2|\, e^{i\pi} \qquad \Rightarrow \qquad (z\text{-}z_2)^{-1/2} = (z_2\text{-}z)^{-1/2}\, e^{-i\pi/2} = -i\,(z_2\text{-}z)^{-1/2}$$

So the disc is this

$$\text{above - below} = -i\,(z_2\text{-}z)^{-1/2} - [+i\,(z_2\text{-}z)^{-1/2}] = -2i\,(z_2\text{-}z)^{-1/2}$$

Then we have

$$I = -i\,(2R)^{1/2} \oint dz\, z^{-1/2}\,(z_1\text{-}z)^{-1/2}\,(z\text{-}z_2)^{-1/2}$$

$$= (-i\,)\,(2R)^{1/2} \int_0^{z_2} dx\, x^{-1/2}\,(z_1\text{-}x)^{-1/2}[-2i\,(z_2\text{-}x)^{-1/2}]$$

$$= 2i^2\,(2R)^{1/2} \int_0^{z_2} dx\, x^{-1/2}\,(z_1\text{-}x)^{-1/2}(z_2\text{-}x)^{-1/2}$$

The integral is now of this form

$$\int dx\,(1/\sqrt{x}\,R) \qquad \text{where R is a quadratic}$$

and we can then use page 97 GR7

$$\mathbf{2.266}^{8}\quad \int \frac{dx}{x\sqrt{R}} = -\frac{1}{\sqrt{a}} \ln \frac{2a+bx+2\sqrt{aR}}{x} \qquad [a>0] \qquad \text{TI (137)}$$

$$= \frac{1}{\sqrt{-a}} \arcsin \frac{2a+bx}{x\sqrt{b^2-4ac}} \qquad [a<0, \quad \Delta<0] \qquad \text{TI (138)}$$

$$= \frac{1}{\sqrt{-a}} \arctan \frac{2a+bx}{2\sqrt{-a}\sqrt{R}} \qquad [a<0] \qquad \text{LA 178 (6)a}$$

$$= -\frac{1}{\sqrt{a}} \operatorname{arcsinh} \frac{2a+bx}{x\sqrt{\Delta}} \qquad [a>0, \quad \Delta>0] \qquad \text{DW}$$

$$= -\frac{1}{\sqrt{a}} \operatorname{arctanh} \frac{2a+bx}{2\sqrt{a}\sqrt{R}} \qquad [a>0]$$

$$= \frac{1}{\sqrt{a}} \ln \frac{x}{2a+bx} \qquad [a>0, \quad \Delta=0]$$

$$= -\frac{2\sqrt{bx+cx^2}}{bx} \qquad [a=0, \quad b\neq 0] \qquad \text{LA 170 (16)}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arccosh}\left(\frac{2a+bx}{x\sqrt{-\Delta}}\right) \qquad [a>0, \Delta<0]$$

and we could then finish off the integral this way.

When I started this doc, I thought I was going to have the circular contour stuck crossing some cut, so that the integral did not even make sense. But that did not happen.