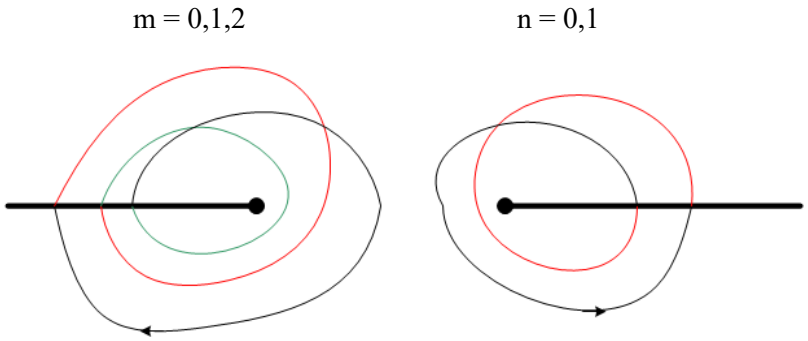
Notes on Contour Integration and Handling Cuts PhL 12.8.08

This is a collection of miscellaneous notes on this general topic.

1. What is the cut structure of $f(t) = t^{1/3}\,(1-t)^{1/2}$.

Here is a useful picture:



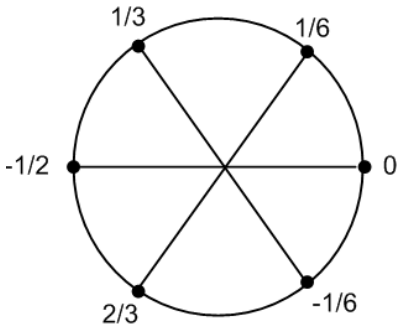
For the branch point on the left which arises from $t^{1/3}$, there are three sheets encountered as you rotate around the point t=0 many times. If we had $f(t) = t^{1/3}$, we would have just the left branch cut, and we would say that f(t) had 3 sheets. Similarly, if we just had $f(t) = (1-t)^{1/2}$, we would only have the branch cut on the right, and the function f(t) would have 2 sheets. For some unknown reason, I have put the "ant path" going CCW on the right, and CW on the left. Should not matter.

We can think of 3 winding numbers possible on the left, and 2 on the right. Call these m and n. Then the phase accrued by moving away from the black-black sheet is this

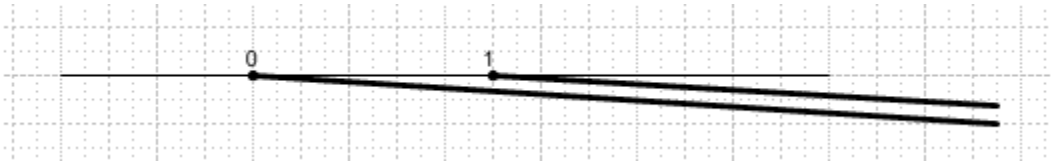
$$\text{phase} = 2\pi(m/3 - n/2)$$

m	n	(m/3 - n/2)	Sheet name:
0	0	0	black-black
1	0	1/3	red-black
2	0	2/3	green-black
0	1	-1/2	black-red
1	1	1/3 - 1/2 = -1/6	red-red
2	1	2/3 - 1/2 = +1/6	green-red

There are 6 sheets for $f(t) = t^{1/3}\,(1-t)^{1/2}$ because there are six different phases, and they are equally spaced around the circle:



There are other ways to draw the cuts, for example



You might think you could somehow join these two branch cuts at infinity and then deform them into this position:



That thought would be incorrect for this example! The two cuts don't "cancel" on the right side. Another way to see this is to suppose the above picture were true. Then you could draw a circle containing this supposed cut, and when you traverse this circle, you pick up a phase $+2\pi/3$ from $t^{1/3}$ and a phase $+2\pi/2$ from $(1-t)^{1/2}$, so total phase is $2\pi(1/3+1/2) = 2\pi(5/6)$ so you don't arrive back where you started, so the function is not single valued, so this cannot be "a sheet".

Example: If we had both fractions the same, say 1/3, then total phase = $2\pi(1/3+1/3) = 2\pi(2/3)$, and still you cannot draw the above picture.

Example: But if both have power 1/2, then the cuts on the right do "cancel" because our phase is then $2\pi(1/2+1/2) = 2\pi = 0$, so yes, single valued. See page 49-49 of Schaum green complex variables book for exactly this case.

Moral: Just because there are two branch points going to infinity, you cannot in general "join them at infinity" and then deform them into an isolated local branch cut. Only in some cases can you do that.

Theorem: In the above case, if you do a little circle around the left branch point so that $t^{1/3}$ changes phase by $2\pi/3$, the phase of the vector (1-t) does not change at all !!! It might deflect slightly as the surface is traversed. (1-t) does NOT pick up a winding number.

Example: Consider the Beta function $B(x,y) = \int_0^1 dt\ t^{x-1}(1-t)^{y-1}$. For general values of x and y, there are two cuts, and you cannot just draw a cut between 0 and 1 for exactly the reasons just discussed.

Example: Consider the integral representation of the confluent hypergeometric function discussed in Messiah's appendix volume 1 page 480:

$$\int_\Gamma dt\ e^{zt}\, t^{\alpha-1}(1-t)^{\beta-\alpha-1}$$

The factor e^{zt} has no branch cuts so we can ignore it. The rest is like our examples above. Suppose we try to make the finite cut from 0 to 1 and wind around the entire thing with a large circle. We pick up the following phase:

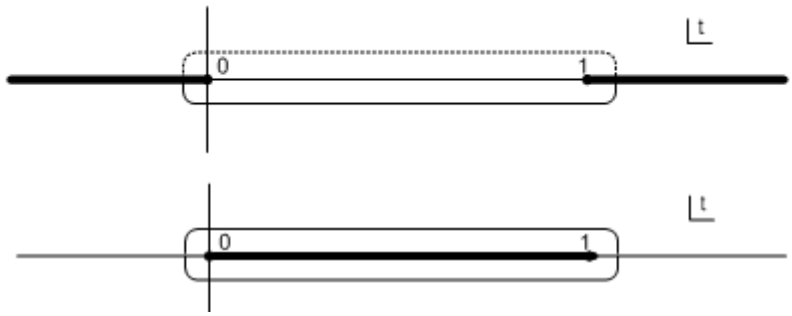
$2\pi\,[\,(\alpha-1) + (\beta-\alpha-1)] = 2\pi\,\beta$ the α cancel

Therefore, if it happens that β = integer b, as top page 481 Messiah assumes, THEN we CAN draw the cut in this manner! So in the picture on page 481 of the contour, you can think of a cut between 0 and 1, and the contour Γ is a closed contour all on one sheet, and the condition of the Laplace Method applies so the "parts" vanish, etc. See separate doc on Laplace's Method for more.

Example: Consider the first term of an e^{zt} expansion in the above integral, so we then have

$$\int_\Gamma dt\ t^{\alpha-1}(1-t)^{\beta-\alpha-1}$$

As just discussed, if β = b = integer, then we CAN draw an isolated cut from 0 to 1 and have Γ be a contour all on one sheet surrounding the cut. We can view the contour in either of these ways:



If α and b are (positive enough) such that the "semicircle" integrals around the branch points vanish, then the upper picture provides an interesting way to evaluate the integral. Notice that it does not matter which way we go to get to the dotted contour when b is an integer. If we go CCW at t=1, we pick up the phase $2\pi(b-\alpha-1) = -2\pi\alpha$. If we go CW at t=0, we pick up phase $-2\pi(\alpha-1) = -2\pi\alpha$. The phase is the same either way, because the dotted portion of the contour is on the same sheet whichever way to get to it.

So, if we think of the contour as the usual CCW, then the lower part gives convergent $\int_0^1$, while the upper part gives $-\,e^{-2\pi i\alpha}\int_0^1$, where the overall minus is because this runs right to left. Thus, we know that

$$\int_\Gamma dt\ t^{\alpha-1}(1-t)^{b-\alpha-1} = [\ 1 - e^{-2\pi i\alpha}\ ] \int_0^1 dt\ t^{\alpha-1}(1-t)^{b-\alpha-1} = [\ 1 - e^{-2\pi i\alpha}\ ]\ B(\alpha,b-\alpha)$$

where we take the Beta function integral representation from GR page 948. We can rewrite this as

$$B(\alpha,b-\alpha) = [\ 1 - e^{-2\pi i\alpha}\ ]^{-1} \int_\Gamma dt\ t^{\alpha-1}(1-t)^{b-\alpha-1}$$

If we were to define $\alpha = x$ and $b-\alpha = y$ this would be written as

$$B(x,y) = [\ 1 - e^{-2\pi i x}\ ]^{-1} \int_\Gamma dt\ t^{x-1}(1-t)^{y-1} \qquad x + y = b = \text{integer}$$

Thus, we have detected a **double typo** in Messiah B.5 which is that he should have - x in the phase exponent there, not y.

Messiah deals with this Beta function for the following reason. Consider in B.4 the leading term "1" of the expansion of e^{zt}. Suppose we assume that

$$K \int_C dt\ e^{zt}\ t^{\alpha-1}(1-t)^{\beta-\alpha-1} = F(\alpha,\beta,z)$$

Then it must be true that, setting $z = 1$ so we keep only the first term in the series F,

$$K \int_C dt\ 1\ t^{\alpha-1}(1-t)^{\beta-\alpha-1} = 1$$

But this integral we know (from above) is $[\ 1 - e^{-2\pi i\alpha}\ ]\ B(\alpha,b-\alpha)$. Thus, we conclude that

$$K = [\ 1 - e^{-2\pi i\alpha}]^{-1} / B(\alpha,\beta-\alpha)$$

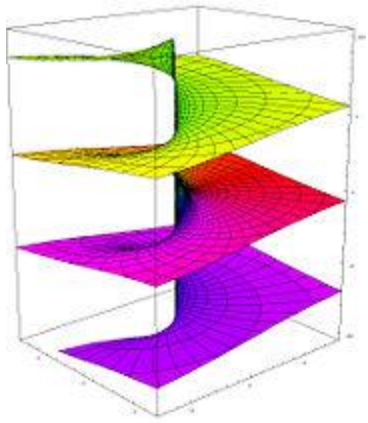
and then we arrive at the fact that

$$F(\alpha,\beta,z) = [\ 1 - e^{-2\pi i\alpha}]^{-1} / B(\alpha,\beta-\alpha)\ *\ \int_C dt\ e^{zt}\ t^{\alpha-1}(1-t)^{\beta-\alpha-1}$$

which is Messiah B.6. He is then on his way to defining the W_1 and W_2 functions which are used in the analysis of Coulomb scattering.

2. How do you draw Riemann sheets:

Here is one attempt.



When we draw our contour pictures, we are drawing a top view of this structure. The branch point is a dot in the middle of such a projection. We can imagine the branch but drawn from the center vertical line to a point on the left cube face. Maybe the first sheet is yellow, then the second sheet is red, and the third sheet is purple.