This is the Title PhL 3.26.05

These flailings were removed from "Reality of P and Q" and parked in this junk bin. Today I think the doc "the meaning of f(- z) for powers.doc" has the results I was looking for in this long search.

$$p_n^m(0\pm) = (2^m\sqrt{\pi} \ / [\ \Gamma(1/2 - n/2 - m/2)\ \Gamma(1 + n/2 - m/2)\ ]\ ) [(\mp i)^{-m}(\mp i)^m \ + (\pm i)^{-m}\,(\pm i)^m\ ]$$

$$= (2^{m+1}\sqrt{\pi} \ / [\ \Gamma(1/2 - n/2 - m/2)\ \Gamma(1 + n/2 - m/2)\ ]\ )$$

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Why is this the same on both sides? I would have expected a difference in the two sides because there is a cut in the ζ plane. I think I might see why. Go back to:

$$p_n^m(\zeta) = [\ (\mp i)^{-m}\ P_n^m(i\zeta)\ + (\pm i)^{-m}P_n^m(-i\zeta)]$$

$$= [\ (\mp i)^{-m}\ P_n^m(z)\ + (\pm i)^{-m}P_n^m(-z)]$$

Consider the value of this function just above the cut in the z plane, which is just to the right of the cut in the ζ plane, so $z = x + i\varepsilon$

$$p_n^m(\zeta) = [\ (-i)^{-m}\ P_n^m(x+i\varepsilon)\ + (+i)^{-m}P_n^m(x-i\varepsilon)]$$

$$= [\ (+i)^m\ P_n^m(x+i\varepsilon)\ + (-i)^mP_n^m(x-i\varepsilon)]$$

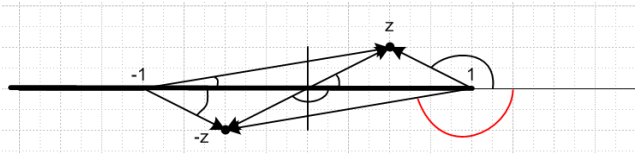
$$= [\ e^{i\pi m/2}\ P_n^m(x+i\varepsilon)\ + e^{-i\pi m/2}P_n^m(x-i\varepsilon)]$$

If we throw in a 1/2 factor (which I have not been doing so far), we find that this is exactly Bateman's on-the-cut function which we know is real on the cut. Does this mean my $p_n^m(\zeta)$ has the ζ = -i cut pulled down and not up?

Consider

$$P_n^m(z)_{0,0L} = (z+1)^{m/2}\,(z-1)^{-m/2}\quad (1/\Gamma(1-m)\ )\quad F(-n, 1+n, 1-m; (1-z)/2)$$

Here we use L world angles as shown here for z-1:

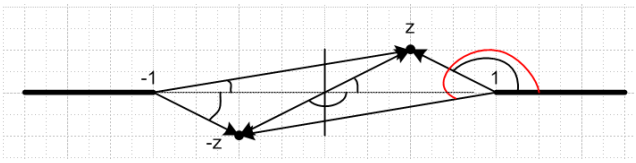


L world

Consider instead:

$$P_n^m(z)_{0,0R} = (z+1)^{m/2}\,(z-1)^{-m/2} \quad (1/\Gamma(1-m)) \quad F(-n, 1+n, 1-m; (1-z)/2)$$

Here we use the R world angles instead for z-1:



R world

So the formula looks the same, but you measure angles differently.

As a reminder about how angles are measured, we can write the above expressions this way:

$$P_n^m(z)_{0,0L} = (z+1)^{m/2}\,(z-1)_L^{-m/2} \quad (1/\Gamma(1-m)) \quad F(-n, 1+n, 1-m; (1-z)/2)$$
$$P_n^m(z)_{0,0R} = (z+1)^{m/2}\,(z-1)_R^{-m/2} \quad (1/\Gamma(1-m)) \quad F(-n, 1+n, 1-m; (1-z)/2)$$

where our subscript L or R reminds us how we are to measure angles.

Correction of a wrong impression: I thought that $P_n^m(z)_{0,0R}$ would be real for z in (-1,1) since there is no cut there. But looking at the above, we have $(z-1)_R^{-m/2}$ where angle is $+\pi$, so

$$(z-1)_R^{-m/2} = |z-1|^{-m/2}\, e^{i\pi(-m/2)} = |z-1|^{-m/2}\,(-i)^m \neq \text{real}$$

It happens to be real if m is an even integer, but in general $P_n^m(z)_{0,0R}$ is complex on (-1,1) !!! Having no cut just means it is continuous there. So $P_n^m(z)_{0,0R}$ is not "real analytic" on this interval. Here is an easy way to see this. When the cut is to the left, just above the cut on (-1,1) we expect $P_n^m(z)_{0,0L}$ to be complex. Rotating the cut CCW by π does not change this fact one iota ! Just above the cut in either case is exactly the same point on the same Riemann sheet.

So all the work between the two bars below just confirms this. It is wasted work, I keep it for now.

Waste Work.

We know that a second way to handle the R situation is this: $(z-1)_R = (z-1)_L\, e^{+i2\pi}$

$$P_n^m(z)_{0,0R} = (z+1)^{m/2}\,(z-1)_R^{-m/2} \quad (1/\Gamma(1-m)) \quad F(-n, 1+n, 1-m; (1-z)/2)$$

$$= (z+1)^{m/2}\,(z-1)_L^{-m/2}\,(e^{-i\pi m})\,(1/\Gamma(1-m)) \quad F(-n, 1+n, 1-m; (1-z)/2)$$

$$= (e^{-i\pi m})\, P_n^m(z)_{0,0L} \qquad \text{Im } z < 0$$

where of course inside $P_n^m(z)_{0,0L}$ we measure the z-1 angle in the L world way. There is no phase if Im(z) > 0. So we might combine these two for all Im(z) by saying

$P_n^m(z)_{0,0R} = [\ e^{-im\pi}(\ 1/2 \mp 1/2) + (\ 1/2 \pm 1/2)\ ]\ P_n^m(z)_{0,0L}$ // upper signs for Im(z) > 0 etc

$P_n^m(-z)_{0,0R} = [\ e^{-im\pi}(\ 1/2 \pm 1/2) + (\ 1/2 \mp 1/2)\ ]\ P_n^m(-z)_{0,0L}$

In the second line above, suppose $z = 4+i\varepsilon$ so that $-z = -4-i\varepsilon$. We expect to have the extra phase in this case, because the argument of the P function on the left has Im(arg) < 0. This confirms the signs in the second equation.

Now let's again consider:

$p_n^m(\zeta) = [\ (\mp i)^{-m}\ P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$

$= [\ (\mp i)^{-m}\ P_n^m(z)_{0,0L} + (\pm i)^{-m}\ P_n^m(-z)_{0,0L}]$

If we change the sign of the phase, we can reverse the equations above and write

$P_n^m(z)_{0,0L} = [\ e^{+im\pi}(\ 1/2 \mp 1/2) + (\ 1/2 \pm 1/2)\ ]\ P_n^m(z)_{0,0R}$ // upper signs for Im(z) > 0 etc

$P_n^m(-z)_{0,0L} = [\ e^{+im\pi}(\ 1/2 \pm 1/2) + (\ 1/2 \mp 1/2)\ ]\ P_n^m(-z)_{0,0R}$

Then we have $e^{+im\pi} = (+i)^{2m}$

$p_n^m(\zeta) = [\ (\mp i)^{-m}\ P_n^m(z)_{0,0L} + (\pm i)^{-m}\ P_n^m(-z)_{0,0L}]$

$= (\mp i)^{-m}\ [(+i)^{2m}\ (\ 1/2 \mp 1/2) + (\ 1/2 \pm 1/2)\ ]\ P_n^m(z)_{0,0R}$

$+\ (\pm i)^{-m}\ [\ (+i)^{2m}(\ 1/2 \pm 1/2) + (\ 1/2 \mp 1/2)\ ]\ P_n^m(-z)_{0,0R}$

The reason I am doing all this work is that $P_n^m(z)_{0,0R}$ is real on the (-1,1) interval so I thought maybe somehow this would show why we are getting $p_n^m(0\pm)$ being the same for + or -. But oddly the above form seems to be complex! Is this just an illusion? Let's take all upper signs:

$= (-i)^{-m}\ [(+i)^{2m}\ (\ 1/2 - 1/2) + (\ 1/2 + 1/2)\ ]\ P_n^m(z)_{0,0R}$

$+\ (+i)^{-m}\ [\ (+i)^{2m}(\ 1/2 + 1/2) + (\ 1/2 - 1/2)\ ]\ P_n^m(-z)_{0,0R}$

$= (-i)^{-m}\ P_n^m(z)_{0,0R} + (+i)^{-m}\ (+i)^{2m}\ P_n^m(-z)_{0,0R}$

$= (-i)^{-m}\ P_n^m(z)_{0,0R} + (+i)^{m}\ P_n^m(-z)_{0,0R}$

Since the P really are real on this range, we are getting $p_n^m(\zeta)$ = complex on the corresponding interval in ζ which is (-i,i). Above we found that $p_n^m(0\pm)$ = real and the same, but here we are finding that each of these quantities is complex. So I have managed to tie myself up in a confused bundle and I don't know what the conclusion is here.

Now once again for good measure, we have:

$$p_n^m(\zeta) = [\,(\mp i)^{-m}\, P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$$

$$= [\,(\mp i)^{-m}\, P_n^m(z)_{0,0L} + (\pm i)^{-m}\, P_n^m(-z)_{0,0L}]$$

If we consider the region (-1,1), we know that $P_n^m(z)_{0,0L}$ is complex. But $P_n^m(z)_{0,0L}$ is real analytic on the range z > 1, so we know that on the cut, the real part is the same above and below and the imaginary part has opposite sign. Therefore we know that

$$\mathrm{Im}[P_n^m(-z)_{0,0L}] = -\,\mathrm{Im}[P_n^m(z)_{0,0L}] \qquad \text{for z just above or below (-1,1).}$$
$$\mathrm{Re}[P_n^m(-z)_{0,0L}] = +\,\mathrm{Re}[P_n^m(z)_{0,0L}]$$

Now let's write

$$(\mp i)^{-m} = (e^{\mp i\pi/2})^{-m} = e^{\pm i\pi m/2} = \cos(\pi m/2) + i\sin(\pi m/2).$$

Then we have

$$p_n^m(\zeta) = (\cos(\pi m/2) + i\sin(\pi m/2))\,(\mathrm{Re}\,[P_n^m(z)_{0,0L}] + i\,\mathrm{Im}\,[P_n^m(z)_{0,0L})]$$

$$+ (\cos(\pi m/2) - i\sin(\pi m/2))\,(\mathrm{Re}\,[P_n^m(-z)_{0,0L}] + i\,\mathrm{Im}\,[P_n^m(-z)_{0,0L})]$$

$$= (\cos(\pi m/2) + i\sin(\pi m/2))\,(\mathrm{Re}\,[P_n^m(z)_{0,0L}] + i\,\mathrm{Im}\,[P_n^m(z)_{0,0L})]$$

$$+ (\cos(\pi m/2) - i\sin(\pi m/2))\,(\mathrm{Re}\,[P_n^m(z)_{0,0L}] - i\,\mathrm{Im}\,[P_n^m(z)_{0,0L})]$$

$$= (c\,\mathrm{Re} + ic\,\mathrm{Im} + is\,\mathrm{Re} - s\,\mathrm{Im}) + (c\,\mathrm{Re} - ic\,\mathrm{Im} - is\,\mathrm{Re} - s\,\mathrm{Im})$$

$$= (c\,\mathrm{Re} - s\,\mathrm{Im}) + (c\,\mathrm{Re} - s\,\mathrm{Im}) = \text{real!}$$

So $p_n^m(\zeta)$ as a function of z is in fact real on the interval (-1,1) and is then real analytic.

So this function is very different from the function $P_n^m(z)_{0,0R}$ which is complex on this interval.

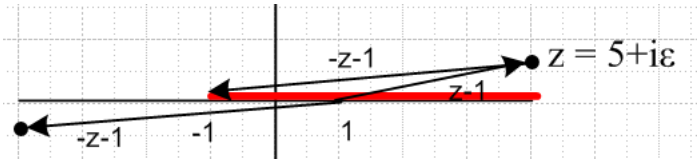
Practice Functions.

<u>Exercise #1</u>: What is the meaning of this function: $f(z) = (-z)^{1/2}$?

One meaning would be this: it has a branch point at z=0 and we pull it off to the right. In this case, for z = -4 we have $4^{1/2} = 2$ and this function is then real for some portion of the real axis, it is real analytic. In this case our angle would range (0,2π). We could swing the cut down and to the left, but then the resulting function would not be real analytic. The function would be real just above the cut on the left.

Exercise #2A: What is the meaning of this function: $f(z) = (-z - 1)^{1/2}$?

One meaning is this: this function has a branch point at z = -1 with the cut going to the right. We are at this branch point when -z-1 = 0 which means z+1 = 0 which means z = -1, hence our conclusion. To the left of the point z = -1, we could regard f(z) as being real, defining thus a principle sheet.

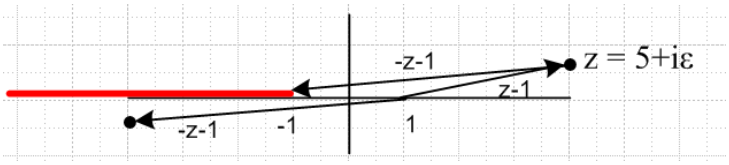


What then is the phase of the vector -z - 1 for the value of z shown above? In cut to right cases, we measure angles (0,2π) relative to the top of the right side real axis, so this vector has phase π. We can translate the vector as we like without changing its phase. Then we claim $-z - 1 = |z+1|e^{i\pi}$. And then we would say $f(z) = (-z - 1)^{1/2} = |z+1|^{1/2} e^{i\pi/2} = i\,|z+1|^{1/2}$. So on top of the cut, this function is pure imaginary with a positive imaginary part. Below the cut we would find pure imaginary with a negative imaginary part. On the left clear axis, f(z) would be real.

This is the $(-z - 1)^{1/2}{}_{0,OR}$ interpretation of the function.

Exercise #2B: What is the meaning of this function: $f(z) = (-z - 1)^{1/2}$?

What happens if we take the cut above and just run it off to the left instead. Then we are not real analytic, true. We would then have this picture:



and now $-z - 1 = |-z-1|\, e^{-i\pi}$ with our usual left cut angle system, so

$$f(z) = (-z - 1)^{1/2} = |-z-1|^{1/2} e^{-i\pi/2} = -i\,|z+1|^{1/2} = -i\sqrt{6}$$

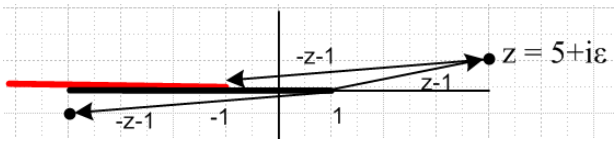
which confirms the idea of not being real analytic. Over at z = -5+iε we get angle 0 for -z-1 and so this then gives are real value just below the cut on the left. This then is $f(z) = (-z - 1)^{1/2}{}_{0,OL}$.

Exercise #3: What is the meaning of this function: $f(z) = (z - 1)^{1/2}$?

This has a branch point at z = 1 which we usually pull off to the left.

Exercise #4: What is the meaning of this function: $f(z) = (z - 1)^{1/2} + (-z - 1)^{1/2}$?

It would be clumsy to try to combine 0,0L and 0,0R functions together, because then you don't really have a well-defined sheet for the sum, so let's take both these functions to be 0,0L functions. Cut structure is then this: (both cuts go off to the left).



Since the $(-z - 1)^{1/2}$ term is not real analytic and the first term is, the sum f(z) is not real analytic and will take non-real values on the real axis.

<u>Exercise #5:</u> What is the meaning of this function: $f(z) = (\mp i)^1 (z - 1)^{1/2}$? Assume 0,0L.

Evaluating we find that

$f(5+i\varepsilon) = (-i)(z - 1)^{1/2} = (-i)\, 2$

$f(0.5+i\varepsilon) = (-i)\, i \sqrt{1/2} = \sqrt{1/2}$ // here the arrow z-1 has phase $+\pi$ and $e^{i\pi/2} = i$

The leading factor $(\mp i)^1$ is a "phase cancellation" coefficient. If we start with $(z - 1)^{1/2}$, we know if we go on top of the cut on the left we have phase i, and then this is cancelled by the -i, and the same for below. So this <u>removes the cut going to the left, but it creates a cut going to the right</u>! *And* on the left we get real. So when we write $f(z) = (\mp i)^1 (z - 1)^{1/2}{}_{0,0L}$ it *looks like* there is a cut to the left, but in fact there is a cut to the right! So the L here does not tell you the direction of the cut.

<u>Question:</u> is this the same as just "rotating the cut to the right CCW" in the function $(z - 1)^{1/2}$?

This is a tricky question and I think the answer is no. We have:

$f(z) = (\mp i)^1 (z - 1)^{1/2} = (\mp i)^1 (z - 1)^{1/2}{}_{0,0L}$

Is it possible that, for some constant K, we have

$f(z) = K\ (z - 1)^{1/2}{}_{0,0R}$?

Our rule is $(z-1)_L = (z-1)_R\, e^{-i2\pi}$ for Im(z) > 0, and no phase otherwise. So let's find the answer to our question. Assume *first* that z = x+iε so there is "no phase" . Then we get

$f(z) = (-i)^1 (z - 1)^{1/2}{}_{0,0L} = (-i)^1 (z - 1)_L{}^{1/2} = (-i)^1 \{ (z-1)_R \}^{1/2} = (-i)^1\ (z-1)^{1/2}{}_{0,0R}$

<u>Note:</u> The function $(z-1)^{1/2}{}_{0,0R}$ has no cut at z = -2, say, but is equal to $|z-1|^{1/2}\, e^{i\pi/2} = i\, |3|^{1/2}$ which we note is not real at z = -2. This value is the same for z = -2 ± iε. The function $(z-1)^{1/2}{}_{0,0R}$ is not real analytic. The function $(1-z)^{1/2}{}_{0,0R}$ is real analytic, but that is not our function here.

Now assume z = x - iε and then we get

$$f(z) = (+i)^1 (z-1)^{1/2}{}_{0,0L} = (+i)^1 (z-1)_L{}^{1/2} = (+i)^1 \{ (z-1)_R e^{-i2\pi} \}^{1/2}$$

$$= (+i)^1 \{ (z-1)_R \}^{1/2} e^{-i\pi} = = (+i)^1 (-1) (z-1)_R{}^{1/2} = -i (z-1)^{1/2}{}_{0,0R}$$

In either case, (z above or below the axis) we seem to get the same result:

$$f(z) = -i (z-1)^{1/2}{}_{0,0R}$$

so, if I have done this right, then K = -i and the answer is in fact "yes". Namely, I just showed that:

$$(\mp i)^1 (z-1)^{1/2}{}_{0,0L} = (-i) (z-1)^{1/2}{}_{0,0R}$$

Need to ponder this more!
(1) on the left, both functions have no cut.
(2) on the right, both functions have a cut
(3) on the left, the LHS is real. $(z-1)^{1/2}{}_{0,0R}$ is imaginary (note above) so RHS is real also.
(4) this function (either LHS or RHS) is then real analytic, and is real for z < 1.

Theorem 1: (a) The function $f(z) = (\mp i)^1 (z-1)^{1/2}{}_{0,0L}$ is the same as $f(z) = (-i) (z-1)^{1/2}{}_{0,0R}$.
(b) This function f(z) is real for z < 1 and has no cut there; there is a cut for z > 1.

<u>Exercise #6:</u> What is the meaning of this function: $g(z) = (\pm i)^1 (-z-1)^{1/2}$? Assume 0,0L.

I think we can say g(z) = f(-z) of Example 5. I would guess the following is true:

Theorem 1A: (a) The function $g(z) = (\pm i)^1 (-z-1)^{1/2}{}_{0,0L}$ is the same as $f(z) = (-i) (-z-1)^{1/2}{}_{0,0R}$.
(b) This function f(z) is real for z > -1 and has no cut there; there is a cut for z < -1.

Let's try to prove this claim. First, let z = x+iε then we have (from Ex 2B above, x > -1)

$$g(z) = (+i)^1 (-z-1)^{1/2}{}_L = (+i)^1 |-z-1|^{1/2} e^{-i\pi/2} = (+i)^1 |z+1|^{1/2} (-i) = |z+1|^{1/2}$$

Next, let z = x - iε. Then we have

$$g(z) = (-i)^1 (-z-1)^{1/2}{}_L = (-i)^1 |-z-1|^{1/2} e^{+i\pi/2} = |z+1|^{1/2}$$

So in either case, we get $|z+1|^{1/2}$. Now let's evaluate for z = x+iε x > -1,

$$(-i) (-z-1)^{1/2}{}_{0,0R} = (-i) |z+1|^{1/2} (e^{i\pi})^{1/2} = (-i)i |z+1|^{1/2} = |z+1|^{1/2}$$

And for z = x - iε we get (draw picture with cut to right, place z at position shown...)

$$(-i)\ (-z-1)^{1/2}{}_{0,0R} = (-i)\,|z+1|^{1/2}(e^{i\pi})^{1/2} = (+i)i\,|z+1|^{1/2} = |z+1|^{1/2}$$ QED

Again, we have the strange result that $f(z) = (-i)\ (-z-1)^{1/2}{}_{0,0R}$ *looks like* it has a cut to the right, but in fact is is real on the right and has a cut to the left.

And we see that for any $x\pm i\varepsilon$, this thing is real. So we are real on $z > -1$. Again, if we just look at the stated form $g(z) = (\pm i)^1\ (-z-1)^{1/2}{}_{0,0L}$, the L makes you think the cut is to the left, and in *this case* that conclusion is correct, despite the $(\pm i)^1$. We have a cut going left from $z = -1$.

<u>Exercise #7:</u> What is the meaning of this function: $f(z) = (\mp i)^1\ (z-1)^{1/2} + (\pm i)^1(-z-1)^{1/2}$?

<u>Note Added:</u> Based on the previous two exercises, we know that the first term is real and uncut when $z < 1$, and we know the second term is real and uncut when $z > -1$. Therefore, on the intersection of these two regions which means (-1,1) we shall find that this f(z) is real and uncut! We could write this as

$f(z) = (\mp i)^1\ (z-1)^{1/2}{}_{0,0L} + (\pm i)^1(-z-1)^{1/2}{}_{0,0L}$ cut to right + cut to left

$= (-i)\ (z-1)^{1/2}{}_{0,0R} + (-i)\ (-z-1)^{1/2}{}_{0,0R}$ cut to right + cut to left

$= (-i)\ [(z-1)^{1/2}{}_{0,0R} + (-z-1)^{1/2}{}_{0,0R}]$ cut to right + cut to left

Here the implication is that we use the upper sign for Imz>0 etc. Let's then example $z = 5+i\varepsilon$ again. We just plug in to get

$$f(5+i\varepsilon) = (-i)\,(z-1)^{1/2} + (+i)(-z-1)^{1/2}$$

$$= (-i)\,2 + (+i)\,i\sqrt{6} = -\sqrt{6} + i2$$ // a complex number

Now let's try our other point:

$$f(0.5+i\varepsilon) = (-i)\,(z-1)^{1/2} + (+i)(-z-1)^{1/2}$$

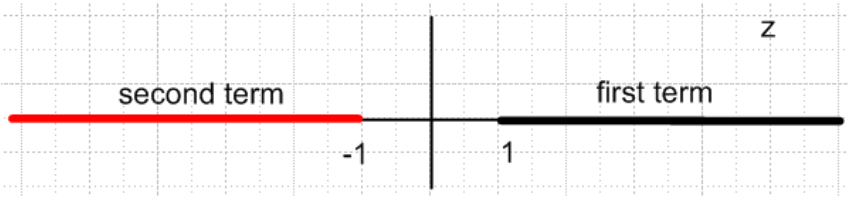
$$= (-i)\,i\sqrt{1/2} + (+i)\,i\sqrt{3/2} = \sqrt{1/2} - \sqrt{3/2} = \text{real}$$

As we vary 0.5 slowly between -1 and 1, we will always get a real result like this. So this unusual example happens to produce a real value in the range (-1,1) for $z = x+i\varepsilon$. If we complex conjugate this fact, we find that the same is true for $z = x - i\varepsilon$. So this function has no discontinuity along (-1,1). If we ignore our tricky sign-changing constants, it appears this function has a cut everywhere along the entire real axis.

<u>Exercise #8:</u> What is the meaning of this function: $f(z) = (\mp i)^m\ (z-1)^{m/2} + (\pm i)^m(-z-1)^{m/2}$?

or $f(z) = (\mp i)^{-m}(z-1)^{-m/2} + (\pm i)^{-m}(-z-1)^{-m/2}$?

where m is an arbitrary real number. This example works the same way as Exercise #7 so that the first term has a cut for z > 1 and the second term has a cut for z < -1 and on (-1,1) the function f(z) is uncut and is real. Here is the cut structure of this f(z) (and the preceding ex 7);



We could write $f(z) = (\mp i)^{m}(z-1)^{m/2}{}_{0,0L} + (\pm i)^{m}(-z-1)^{m/2}{}_{0,0L}$ and the reader is reminded that you cannot tell the direction of the cut of a term just from the "L" sitting there. You have to figure it out.

<u>Exercise #9:</u> What is the meaning of this function: $\mathbf{p_n^m(\zeta) = [(\mp i)^{-m} P_n^m(z) + (\pm i)^{-m} P_n^m(-z)]/2}$

where $z = i\zeta$. Finally we get to the prize. This function has the same cut structure as Exercises 8 and 7, and in fact exactly matches the second line in Ex #8. You find this out by looking at the first Bateman formula for P, p 124 (14). The function is real and uncut on (-z-1). If you were to make the corresponding ζ-plane cut structure, you see the region (-i,i) as being uncut and real. Hurray! That only took a few days.

Comment: At first I thought this p thing was turning out to be a continuation of the Bateman on-the-cut formula for P. It is close, but it is not the same because that formula involves P(x+iε) and P(x-iε), whereas my formula is P(z) and P(-z). The phases are the same.

Next question: What is the cut structure of

$$q_n^m(\zeta) \equiv (-i)^{2m}(\pm i)^{n+1} Q_n^m(z) \qquad z = i\zeta$$

Our sample functions $Q_0(z)$ and $Q_1(z)$ only have a cut from -1,1. To the left, the cuts happen to cancel. But *I think* for general n and m, the cuts do not cancel. But I have no example to show this.

It is *amazing* to me how few sources actually draw and talk about the cut structure! I just hunted around and could really find not much. Maybe only particle physics people do this?

Well we can ask Maple to compute a few values:

$Q_{2.1}^{3.2}(z)$ at $z = -5 \pm i\varepsilon$

```
LegendreQ(2.1,3.2,-5 + .0000000001 * I);
                          .07758761933 + .02520974571 I
LegendreQ(2.1,3.2,-5 - .0000000001 * I);
                          .04795178586 + .06599997110 I
```

So you certain get different values above and below the cut, so the cut must be there! On the other hand, if you go off to the right you get

```
LegendreQ(2.1,3.2,+5 + .0000000001 * I);
                              -.06599997111 - .04795178585 I
LegendreQ(2.1,3.2,+5 - .0000000001 * I);
                              -.06599997110 - .04795178586 I
'
```

and they are the same because there is no cut.

Theorem: In general, the Q function is cut all the way from -1 to -∞. Whether you think of this as one or two cuts superimposed is somewhat academic.

But what about the $q_n^m(\zeta)$ function?

The $q_n^m(\zeta) = (-i)^{2m} (\pm i)^{n+1} Q_n^m(i\zeta)$

Questions still remain:

(1) I thought my p and q functions are designed to be real whenever ζ and m and n are real. In the z plane this makes p and q be real on the imaginary axis. The cuts are still going to be present on the real z axis for each of these functions, but p is cut-free on z in (-1,1) it happens. I don't think q is cut-free there, so I will probably get different values for q(±i0), I have not done this yet. I stopped after doing p(±i0) and I was so amazed to find these were the same. We shall see.

Comment about the p function. Here it is where $z = i\zeta$

$$p_n^m(\zeta) = [(\mp i)^{-m} P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]/2$$

This function, taken as a function of z, is real on two interesting loci. One loci is z = (-1,1) and the other loci is (-i∞, i∞). It does seem a little odd to me that it is possible to have a function be real on this cross shaped domain in the z plane.

But alas, Maple shows this is untrue, so the rock just rolled back down the mountain once again. It is 8:30 AM on 2.12.10, so I think there will be another 12 hours of trying to push it back up again. Something is wrong somewhere, as usual. I get real on the (-1,1) region only when m is integral.