Some Arithmetic PhL 6.25.20

Exercise 1:

Consider

$a^2a^3 = (a^2)*(a^3) = (a*a)*(a*a*a) = (a*a*a*a*a) = a^5$

Now generalize a bit,

$a^m a^n = (a*a*....)*(a*a*....) = (a*a*... *a*a*...) = a^{m+n}$ // you ADD the two powers
m copies n copies m+n copies

Note that a could be any complex number.

Now this statement that $a^m a^n = a^{m+n}$ is at first valid for m and n being positive integers. This is not a major step. But then we can analytically continue this to say $a^\alpha a^\beta = a^{\alpha+\beta}$ where now a, α and β are all complex numbers! That is a larger step!

Exercise 2:

Consider

$a^n b^n = (a*a* ....) * (b*b* ....) = (a*b)*(a*b)* ..... = (a*b)^n$
n copies of a n copies of b n copies of (a*b) n copies of a*b reordered

We are just *reordering* terms in a product and multiplication is commutative so order doesn't matter. Here n is a positive integer.

I have now proven these two "rules",

Rule name	Rule
Product rules	$a^n \cdot a^m = a^{n+m}$
	$a^n \cdot b^n = (a \cdot b)^n$

Here are some extended rules:

$a^n * a^m * a^k = a^{n+m+k}$

$a^n * b^n * c^n = (a*b*c)^n$

$((a^n)^m)^k =$ same with any ordering of the letters $= a^{n*m*k}$

Exercise 3:

Prove this rule,

$$(a^n)^m = a^{n \cdot m}$$

Well here I go

$(a^n)^m$ = (a*a*a....)(a*a*a....)

Let $a^n = b$. Then $(a^n)^m = b^m$.

Hmmm. I don't have proof! I am thinking simple induction proofs!

(6.26.20 continue

Start with n= 1. The expression is then $(a)^m = a^m$ and there is nothing of interest,
Next try n=2. We then have $(a^2)^m$ which is $(a*a)^m$.
Try m = 3. We then have $(a^2)^3 = (a*a)^3 = (a*a)*(a*a)*(a*a) = a^6 = a^{2*3}$.
The expression $(a*a)^m$ means "take a*a and do that m times"
Is that the same as a^{2*m} ? Yes it is, but I want a cleaner proof!

Treat n as fixed and m as variable.
Start with m =1. This is then $(a^n)^1 = (a^n) = a^n$. Since this is an identity, it is true,
Now try m=2. This is then $(a^n)^2$.
This is $(a^n)*(a^n)$. Think of it as $[(a^n)]^2 = (a^n)*(a^n)$.
Then we should next have $[(a^n)]^3 = (a^n)*(a^n)*(a^n)$. Like $[x]^3 = (x)*(x)*(x)$ where $x = a^n$.
Then write $[(a^n)]^N = (a^n)*(a^n)*(a^n)$ .... (a^n)
N factors of (a^n)
Can also write this as $(a^n)^N = (a^n)*(a^n)*(a^n)$ .... $(a^n) = a^{n*N}$ with N factors each being (a^n).

Let's now try an **induction proof**. Want to prove that :

1. If $(a^n)^k = a^{n*k}$, then $(a^n)^{k+1} = a^{n*(k+1)}$. [If true for k, then true for k+1.]
2. This is true for k = 1.

So assume that $(a^n)^k = a^{n*k}$ (assume true for k). Multiply both sides by (a^n) to get

$(a^n)^{k+1} = (a^{n*k}) * (a^n) = a^{n*k+n} = a^{n*(k+1)}$

We have thus shown that $(a^n)^{k+1} = a^{n*(k+1)}$. But this is our statement for k+1 !
So we have shown that if our statement is true for k, then it is true also for k+1.
So it remains only to show that the statement is true for k = 1.
But for k = 1, our statement says $(a^n)^1 = a^{n*1}$ or $a^n = a^n$ which is clearly true.

I think this is **my first post-stroke induction proof!** Hurray!

Exercise 4:

I want to do this again.
Here is the induction proof: show this:

1. If $a^n * a^k = a^{n+k}$, then $a^n * a^{(k+1)} = a^{n+(k+1)}$
2. This is true for k = 1.

In words: If this fact is true for k (the fact being $a^n * a^k = a^{n+k}$), then the fact is *also* true for k+1 (in which case the fact says $a^n * a^{(k+1)} = a^{n+(k+1)}$. A shorthand notation would then be
$k \Rightarrow k+1$. We can't just *say* that $k \Rightarrow k+1$, we have to *prove* it!

So assume that $a^n * a^k = a^{n+k}$ (assume true for k). Given the assumption that $a^n * a^k = a^{n+k}$, I then have to prove that $a^n * a^{(k+1)} = a^{n+(k+1)}$. Here is my proof.

Step 1: Write down the statement for k:

$a^n * a^k = a^{n+k}$.

Step 2: Multiply both sides by a. The truth of an equation is not changed by multiplying both sides by the same number (as long as it is not 0 or ∞). We then have,

$a^n * a^k * a = a^{n+k} * a$.
LHS RHS

Step 3: On the LHS we can write $a^k * a = a^{k+1}$. In words, "a to the k times a equals a to the k+1 ". If we now include the factor a^n this says : $a^n * a^k * a = a^n * a^{k+1}$.

Step 4: On the RHS we can write $a^{n+k} * a = a^{n+k+1} = a^{n+(k+1)}$.

Step 5: Then writing LHS = RHS we have $a^n * a^{k+1} = a^{n+(k+1)}$

Step 6: We have now shown that $a^n * a^k = a^{n+k} \Rightarrow a^n * a^{k+1} = a^{n+(k+1)}$. That is, we have shown that if a certain equation is true for k, then it is also true for k+1. The is a key ingredient of an induction proof: we want to show that if something is true for k, then it is also true for k+1. In shorthand, we are showing that $k \Rightarrow k+1$.

Exercise 5:

I am going to back up now and re-prove this claim:

$$(a^n)^m = a^{n \cdot m}$$

Here is an **induction** proof. I will first show that the equation is true for m = 1. For m = 1 it says

$(a^n)^1 = a^{n*1}$

or

$(a^n) = a^n$

so this m = 1 case is trivially shown.

Now let's try showing that m $\Rightarrow$ m+1. We need to show that

$(a^n)^m = a^{n*m} \Rightarrow (a^n)^{m+1} = a^{n*(m+1)}$

Step 1: Start with $(a^n)^m = a^{n*m}$. This is the statement for m.

Step 2: Multiply both sides by a^n which gives [leaves equation still true]

$(a^n)^m * a^n = a^{n*m} * a^n$

LHS RHS

Step 3: On the LHS write,

$(a^n)^m * a^n = a^{m*n}$

This is really the crux statement!

BACK UP.

Consider this equation,

$(a^n)^m = a^{n*m}$

How many powers of a are there on the LHS?
We have a^n multiplied by itself m times. That is

$a^n * a^n * ..... * a^n = a^{n*m}$

Here are some specific examples,

$a^n * a^n = (a^n)^2 = a^{2*n}$

$a^n * a^n * a^n = (a^n)^3 = a^{3*n}$

Try this with some small values of n. Start with n = 1,

$a^1 * a^1 = (a^1)^2 = a^2$

$a^1 * a^1 * a^1 = (a^1)^3 = a^3$

Now go to n = 2,

$a^2 * a^2 = (a^2)^2 = a^4$

$a^2 * a^2 * a^2 = (a^2)^3 = a^6$ // which is a^{2*3}'

I will now make this broader statement:

$(a^n)^m = (a^m)^n = a^{n*m} = a^{m*n}$

And this seems to follow

$((a^n)^m)^k =$ same with any ordering of the letters $= a^{n*m*k}$

I created that new fact out of nothing!

I think this was a good exponent session!

Here is a source of data

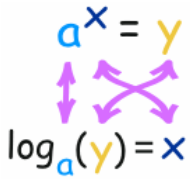
https://www.rapidtables.com/math/algebra/logarithm/Logarithm_Rules.html

Is this useful to me? I will hold judgment .

What do I know about logs?

$\log_a x$

Wow! I know nothing right now! Here is a starting point,



I will write this again:

$a^x = y \quad \Leftrightarrow \quad \log_a(y) = x$

Start with $a^x = y$. How do you *solve* that equation for x? Answer:

$x = \log_a(y)$

How would I prove this claim,

$$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$$

Start from scratch with this:

$a^x = M \quad \Leftrightarrow \quad \log_a(M) = x$
$a^y = N \quad \Leftrightarrow \quad \log_a(N) = y$

We then have that

$a^x * a^y = M*N = a^{x+y}$ // add exponents

Now take the log of both sides

$\log_a(M*N) = \log_a(a^{x+y}) = x+y$

The $\log_a$ of a^{x+y} is in fact x+y.
The log is the power to which you raise a to get x+y.

So we then have

$\log_a(M*N) = x+y = \log_a(M) + \log_a(N)$

This is the very famous rule that the log of a *product* of two numbers is the *sum* of the logs of the two numbers.

Question: What is $\log_a(M^n)$?
Answer: $n \log_a(M)$

Try to follow this development:

3. Proof of Power Rule Law:

$\log_a M^n = n \log_a M$

Let $\log_a M^n = x \Rightarrow a^x = M^n$

and $\log_a M = y \Rightarrow a^y = M$

Now, $a^x = M^n = (a^y)^n = a^{ny}$

Therefore, $x = ny$ or, $\log_a M^n = n \log_a M$ *[putting the values of x and y]*

1. Define x to be $\log_a M^n$.
2. Then $a^x = M^n$.
3. Can I show that this is true:

$\log_a M = y \Rightarrow a^y = M$? yes

4. Can I then show this?

Now, $a^x = M^n = (a^y)^n = a^{ny}$?

This is worse than pulling teeth. Each equation is a battle for me. I don't follow the logic flow. It might be easier for me to start from scratch.

Want to show this:

$\log_a(M^n) = n \log_a(M)$

I know the log of a product is the sum of the logs. So

$\log_a(M^2) = \log_a(M*M) = \log_a(M) + \log_a(M) = 2 \log_a(M)$
$\log_a(M^3) = \log_a(M*M*M) = \log_a(M) + \log_a(M) + \log_a(M) = 3 \log_a(M)$
.....
$\log_a(M^n) = \log_a(M*M*M.....) = \log_a(M) + \log_a(M) + ... = n \log_a(M)$

OK I am happy now. I think I have not done logs for a long time!