A Review of Buck Curve and Surface Integration PhL 4.14.16

The first info is in Section 6.2 with the arc length formula

$$L(\mathbf{r}(t)) = \int_a^b |\, d\mathbf{r}/dt|\, dt = \int_a^b v(t)dt = \int_a^b ds = \int_a^b |d\mathbf{r}| \qquad (6\text{-}20)$$

and here the abs val appears in a fairly obvious manner -- each dr of the curve adds positively to the length of the curve, hard to argue with.

Word "trace" used instead of "image".

Next item of interest is in Section 6.3 about area of a surface,

Area Formula. [338] Area = $\int$ dudv $|\, \mathbf{n}(u,v)|$ where **n** is the non-unit vector normal given in (6-29).

Bucks don't derive this, but I derive it with reference to tensor doc.

$$d\mathbf{A}^3 = J\ d\mathcal{A}'^3\ \mathbf{e}^3 = J\ d\mathcal{A}'^3\ \mathbf{E}_3 = \det(S)\ d\mathcal{A}'^3\ \det(R)\ \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_1 \times \mathbf{e}_2\, d\mathcal{A}'^3 = \mathbf{n}\ d\mathcal{A}'^3$$
$$= \mathbf{n}(u,v)\ dudv$$

$dA^3 = |\, \mathbf{n}(u,v)|\ dudv$ But $|\mathbf{n}|^2 =$ sum of three squared 2x2 matrices as shown (6-30).

I have now (later in the day) derived (6-30) in my wedge Section 10.10. I did not prove that n is really the surface normal, but the Bucks do that on page 336

I went back to the Chapter 6 raw notes and I now have a question.

Puzzler #1.

In the 2D example of wedge 10.10, I get this type of formula

$$\int_\varphi \alpha = \Sigma_{1\le i_1<i_2\le 3} \int_0^1 \int_0^1 f_{i_1 i_2}(\boldsymbol{\varphi}(\mathbf{t}))\, [\, (\partial_1 \varphi^{i_1})\, (\partial_2 \varphi^{i_2}) - (\partial_2 \varphi^{i_1})\, (\partial_1 \varphi^{i_2})]\ dt^1 dt^2$$
$$= \int_0^1 \int_0^1 f_{12}(\boldsymbol{\varphi}(\mathbf{t}))[\, (\partial_1 \varphi^1)\, (\partial_2 \varphi^2) - (\partial_2 \varphi^1)\, (\partial_1 \varphi^2)]\ dt^1 dt^2$$
$$+ \int_0^1 \int_0^1 f_{13}(\boldsymbol{\varphi}(\mathbf{t}))[\, (\partial_1 \varphi^1)\, (\partial_2 \varphi^3) - (\partial_2 \varphi^1)\, (\partial_1 \varphi^3)]\ dt^1 dt^2$$
$$+ \int_0^1 \int_0^1 f_{23}(\boldsymbol{\varphi}(\mathbf{t}))[\, (\partial_1 \varphi^2)\, (\partial_2 \varphi^3) - (\partial_2 \varphi^2)\, (\partial_1 \varphi^3)]\ dt^1 dt^2$$

for the integration of

$$\alpha = f_{12}(\mathbf{x})\ dx^1 \wedge dx^2 + f_{13}(\mathbf{x})\ dx^1 \wedge dx^3 + f_{23}(\mathbf{x})\ dx^2 \wedge dx^3\ .$$

I am integrating over a 2D surface in 3D space using the pullback method of evaluation. At the micro scale, I think a tiny rectangle dt^1dt^2 in R^2 maps into a 2-piped on the surface in R^3 and I should be able to relate things to the discussion in those raw notes concerning things like $\text{cof}(\bar{g}'_{33})$ and so on. In terms of the R matrix, the above reads

$$\int_\varphi \alpha = \int_0^1 \int_0^1 f_{12}(\boldsymbol{\varphi}(\mathbf{t}))\,[R^1{}_1 R^2{}_2 - R^1{}_2\, R^2{}_1]\; dt^1dt^2$$
$$+ \int_0^1 \int_0^1 f_{13}(\boldsymbol{\varphi}(\mathbf{t}))\,[R^1{}_1 R^3{}_2 - R^1{}_2\, R^3{}_1]\; dt^1dt^2$$
$$+ \int_0^1 \int_0^1 f_{23}(\boldsymbol{\varphi}(\mathbf{t}))\,[R^2{}_1 R^3{}_2 - R^2{}_2\, R^3{}_1]\; dt^1dt^2$$

My $\text{cof}(\bar{g}'_{nn})$ only has meaning for a square-R situation. For a tall R matrix, "cofactor" does not mean anything. What appears above are minors of R which I well know.

Bucks in Ch 6 are doing $E^2 \rightarrow E^3$ with a mapping of a planar rectangle to E^3 and that does seem to be the right thing for my α case above. I could "extend" the domain space here to E^3 where the domain just happens to sit in the x-y plane, then can maybe apply the $\text{cof}(\bar{g}'_{33})$ idea to a piece of area. That must be why I was able to replicate the Buck k expression top of their page 299 using Maple and

$$k^2 = \text{cof}\,(g'_{33}) = \text{cof}[(S^TS)_{33}]$$

I tried this calculation by hand and I could not get easily to that same result.

Plan B. $\text{cof}(\bar{g}'_{33}) = \bar{g}'_{11}\;\bar{g}'_{22} - \bar{g}'_{12}\;\bar{g}'_{21}$

$$= (S^TS)_{11}(S^TS)_{22} - (S^TS)_{12}(S^TS)_{21}$$

$$= \Sigma_n S_{n1}{}^2\;\Sigma_m S_{m2}{}^2 - \Sigma_n S_{n1}S_{n2}\;\Sigma_m S_{m1}S_{m2}$$

The mystery here is that this thing is quartic in S, whereas my $\int_\varphi \alpha$ is only quadratic in R. I don't understand that fact. I do admit that the factors like $[R^1{}_1R^2{}_2 - R^1{}_2\,R^2{}_1]\;dt^1dt^2$ are the inverse of what the Bucks are doing. so maybe $1/\text{cof}(\bar{g}'_{33})$. Bucks are mapping rectangle from $E^2 \rightarrow E^3$ whereas I am back-mapping $d\mathbf{A}$ in E^3 on the surface to E^2.

I will let this Puzzler #1 rest for now. I don't have an answer.

Puzzler #1 remains unresolved, but here is some related matter:

1. Buck integral $\mathbf{n}(u,v)\,dudv$ integral on page 338 is an integral of scalar surface area, just the way the ds integral give scalar arc length. Later on page 368 Bucks show a f $\mathbf{n}(u,v)\,dudv$ integral of a scalar function f weighted by this scalar area thing. It is analogous to Buck page 367 7-1 for integrating a function f against arc length.

2.Later on page 403 in the discussion of 2-form integration, Bucks get the same f **n**(u,v) dudv as I get now in my Section 10.10 of wedge, which is

$$\int_{\varphi} \alpha = \int_0^1 \int_0^1 [\, f_{12}(\boldsymbol{\varphi}(\mathbf{t})) \frac{\partial(\varphi^1, \varphi^2)}{\partial(t^1, t^2)} + f_{13}(\boldsymbol{\varphi}(\mathbf{t})) \frac{\partial(\varphi^1, \varphi^3)}{\partial(t^1, t^2)} + f_{23}(\boldsymbol{\varphi}(\mathbf{t})) \frac{\partial(\varphi^2, \varphi^3)}{\partial(t^1, t^2)} \,]\; dt^1 dt^2$$

But they in effect define $F_3 = f_{12}$, $F_2 = f_{13}$ $F_1 = f_{23}$ and in which case my thing above is

$$\int_{\varphi} \alpha = \int_0^1 \int_0^1 [\, f_{23}(\boldsymbol{\varphi}(\mathbf{t})) \frac{\partial(\varphi^2, \varphi^3)}{\partial(t^1, t^2)} + f_{13}(\boldsymbol{\varphi}(\mathbf{t})) \frac{\partial(\varphi^1, \varphi^3)}{\partial(t^1, t^2)} + f_{12}(\boldsymbol{\varphi}(\mathbf{t})) \frac{\partial(\varphi^1, \varphi^2)}{\partial(t^1, t^2)} \,]\; dt^1 dt^2$$

$$= \int_0^1 \int_0^1 [\, F_1(\boldsymbol{\varphi}(\mathbf{t})) \frac{\partial(\varphi^2, \varphi^3)}{\partial(t^1, t^2)} + F_2(\boldsymbol{\varphi}(\mathbf{t})) \frac{\partial(\varphi^1, \varphi^3)}{\partial(t^1, t^2)} + F_3 \frac{\partial(\varphi^1, \varphi^2)}{\partial(t^1, t^2)} \,]\; dt^1 dt^2$$

$$\mathbf{n} = \left(\frac{\partial(\varphi^2, \varphi^3)}{\partial(t^1, t^2)}, \frac{\partial(\varphi^1, \varphi^3)}{\partial(t^1, t^2)}, \frac{\partial(\varphi^1, \varphi^2)}{\partial(t^1, t^2)} \right)$$ exactly as shown in Buck p 403 B

Puzzler #2.

Bucks have this formula

Area Formula. [338] Area = $\int$ dudv | **n**(u,v)| where **n** is the non-unit vector normal given in (6-29).

$$\text{Area} = \int dt_1 dt_2 \, |\, \mathbf{n}(t_1, t_2)|$$

This certainly looks like what I am doing if I set the f_{ij} perhaps to 1. In 10.10(b) I am integrating a general form, whereas they are only doing area. There is a distinction which I recall writing about in later notes I have not yet gotten to today.

So Puzzler #2 is to show how the above | **n**(t_1,t_2)| object is somehow the same as something in my long result above from 10.10(b) [but I have now shown this above in the added section]. I don't know how to make α correspond to an area integral yet. Integrand looks a little like **f** • d**A** so I am integrating a function with three components over the surface. But in Ch 6 the Bucks are really integrating | d**A** | = $\hat{\mathbf{n}}$ • d**A** to get the surface area, so this is a completely different integral. I will return to this soon.

I do verify from tensor doc that the dA^3 = | **n**(u,v)| dudv is correct, however, but I have to think more about the connection, since this thing concerns 3-pipeds and not surfaces.

I think Bucks having nothing more to say about surface integrals in Chapter 6. Remember, they have just done arc length and surface area in this chapter.

OK, we now come to Bucks section 7.2 on differential forms. My surface case is this

$$\omega(\text{2-form}) = A_{12}(\mathbf{r})dx_1dx_2 + A_{13}(\mathbf{r})dx_1dx_3 + \ldots + A_{1n}(\mathbf{r})dx_1dx_n$$
$$+ A_{23}(\mathbf{r})dx_2dx_3 + A_{24}(\mathbf{r})dx_2dx_4 + \ldots + A_{2n}(\mathbf{r})dx_2dx_n$$
$$+ \ldots + A_{(n-1)n}dx_{n-1}dx_n \qquad\qquad n! \text{ terms}$$

$$= A_{ij}dx_idx_j \qquad i,j = 1,2\ldots n \qquad // \; dx_2dx_2 = 0 \text{ so OK to include}$$

Fact: For integrating a curve, Bucks get

$$\int_\gamma \omega = \int_{t_1}^{t_2} [\, A(\boldsymbol{\gamma}(t))\gamma'_x(t) + B(\boldsymbol{\gamma}(t))\gamma'_y(t)] \; dt = \int_\gamma [\, A(\mathbf{r})dx + B(\mathbf{r})dy \,]$$

and there are NO absolute values on anything. This is the exact result I get in 10.10, and of course Sjamaar also does this case in his Chapter 4 and he too has no abs values. I think those only arise in the arc length and area integrals. So I am good with the Bucks on my integration of 1-forms.

What about integration of 2-forms? They stall a bit before doing an example,

Well they first do $E^2 \rightarrow E^2$ and get this little rule

$$du \wedge dv = \frac{\partial(u,v)}{\partial(x,y)}\, dx \wedge dy \qquad \text{or} \qquad dt^1 \wedge dt^2 = \frac{\partial(t^1,t^2)}{\partial(x^1,x^2)}\, dx^1 \wedge dx^2$$

but this seems upside down from what I want. In my pullback thing I really have instead

$$d\varphi^1 \wedge d\varphi^2 = \frac{\partial(\varphi^1,\varphi^2)}{\partial(t^1,t^2)}\, dt^1 \wedge dt^2 = [\, (\partial_1\varphi^1)\,(\partial_2\varphi^2) - (\partial_2\varphi^1)\,(\partial_1\varphi^2)]\, dt^1 \wedge dt^2$$

so going $2 \rightarrow 2$ you can have this either way, no problem. Note that on page 384 there is no abs value, it is just the determinant thing. So Buck p 384 Theorem 1 is the same as my (10.9.16). Note that both these forms are in the same space, you cannot write the left side as $dx^1 \wedge dx^2$ even though you can think of the function φ^1 as being called x^1.

They get around to my case on page 384. The tall 3x2 R matrix is implied by xformation p 384 A. But then they are off on a particular specific example. But page 386 looks promising. In my raw Ch 7 notes I do the k = 2 problem in E^3 and I get this result

$$\omega = [\, A(\mathbf{r})\frac{\partial(y,z)}{\partial(u,v)} + B(\mathbf{r})\frac{\partial(z,x)}{\partial(u,v)} + C(\mathbf{r})\frac{\partial(x,y)}{\partial(u,v)}\,]dudv$$

$$\int\int_\Sigma \omega = \int\int_\Sigma [\, A(\mathbf{r})\frac{\partial(y,z)}{\partial(u,v)} + B(\mathbf{r})\frac{\partial(z,x)}{\partial(u,v)} + C(\mathbf{r})\frac{\partial(x,y)}{\partial(u,v)}\,]dudv$$

and this then agrees with my Section 10.10 !! But this is not an external result because Buck's never derive it, only I derive it . But I claim that Ara confirms it, and I do quote from Ara later in Ch 7 raw notes [But Bucks have this same result on page 403 after all!]

DEFINITION 11.1 *The integral of a 2-form on S is given by*

$$\iint_S F dx\wedge dy + G dy\wedge dz + H dz\wedge dx = \iint_D \left[F\frac{\partial(x,y)}{\partial(u,v)} + G\frac{\partial(y,z)}{\partial(u,v)} + H\frac{\partial(z,x)}{\partial(u,v)} \right] dudv$$

So I really should add Ara to my references. φ [no longer needed, but maybe review Ara]