Buck Chapter 1 PhL 12.23.14

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About the Buck couple.

Author of this book is R. Creighton Buck (1920-1998) "with collaboration of" his wife Ellen. Creighton went to UC in Cincinnati and then PhD at Harvard in 1947. Going through Brown, he ended up at U Wisconsin, professor in 1954. Went emeritus in 1990 after many years there! He was a pianist and sci fi writer!

Ellen R. Buck (Fedder, 1919-2011) did Rockford then math MS at Radcliffe. She met hubby at the Harvard Math Club, they married in 1944, no divorce as far as I can tell. They coauthored two math books, this being one and the other Introduction to Differential Equations 1976. Advanced Calculus went through three editions:

1st Ed was 1956, my 2nd edition was 1965. A third came out in 1978.

Interesting parallels with Ellen Edes: Cincinnati and Harvard and Sci Fi for him, Radcliffe for her, both interested in math. She did many other things I don't mention here.

Is this 3rd edition available on the web? A search shows only fake attack sites, so answer is no (same conclusion on 8/23/15). There is a google books entry which seems to have lots of pages, however, though not all. Pick the book with the colored cover! Search for google books with preview only. There is a red covered book there with preview (not complete).

I read this entire first chapter during Math 105a in 1972 or so, and it is easy to read now, at least so far.

Chapter 1: Sets and Functions [pp 1-54]

This opening chapter is about all the topics which I have bolded below, and there are many!

1.1 Main advanced feature is **more than one variable**. **Geometry** is useful as a visualization tool.

1.2 The Geometry of En Cartesian space.[2] **Algebraic** refers to the rules of the E^n as a vector space. Topology comes later and relates to "ordering" of points in a space. **Vector space** rules in (1-2). Fact that the midpoints of *any* quadrilateral form a **parallelogram**, easy to prove (I never gave it a thought before).

Interesting problem on antiparticles right at the book start p 8. Authors refer to E^n as **n-space**. I think these are the only spaces that will appear in this entire book! I call it Cartesian space in n dimensions.

1.3 Distance and Norms. [9] We get the **norm** in E^n but does not bother with normed linear space phrase or Banach. Then we get the **triangle inequality** in E^n in terms of norms (= distances). Then the **inner product** and the **Schwarz Inequality** proved: $\mathbf{p} \bullet \mathbf{q} \leq |\mathbf{p}|\,|\mathbf{q}|$ [Theorem 1]. They use the symbol ▌ in place of Q.E.D, claiming popularity at the time. They then express a line passing through points p and q as

$p + \lambda(p-q) = p(1-\lambda) + \lambda q$ segment p to q is then $0 \leq \lambda \leq 1$

Final def is **convexity** where any segment must be in the convex set. Phrase "metric space" not used.

1.4 Sets and Functions. [15] First comes set definitions, seems strange ordering of info. Use of a ε S set notation. Subsets. Contains = covers. **Union** and **intersection**. **Disjoint** sets, **complement** of a set. Then comes the function as a **mapping** between sets (so you need sets first!). **Domain** of mapping (range is not mentioned). How to make a parametric curve in E^2 and E^3. Then on to sequences. **Sequence** as mapping from integers I to points in some space, and that set of points is called a **trace**. Range is snuck in bottom page 19, oops. Then notion of the **graph** of a function f: A→B and notation AxB (they call this a Cartesian Product) to sort of draw that graph (but in multivariable space). Various misc facts stated, words increasing and convex. Complex numbers treated just as E^2. Fast glimpse of a vector space whose elements are functions. No mention of the phrase Hilbert Space.

1.5 Topological ideas. [24] Notion of **neighborhood**, might or might not be a round **ball**. Notion of a set being **open** or **closed**. It is open if you can fit a neighborhood around any point which lies in the set -- there is no **boundary** for such a set. Closed includes all boundary points. **Complement** of open set is closed and vice versa. **Closure**, **interior**, **exterior**. Notion of a **cluster point** = **accumulation point** = **limit point**: any ball has ∞ elements. Otherwise point is isolated. **Infiniteness** of set refers to number of elements, not metric size of the set. The latter relates to being **bounded** or **unbounded**. Set can be **connected** or **disconnected**. Formally a topology on a set of points is an **enumeration** of all the open sets. Notion of set **D being open relative to set S**, I am wobbly on that idea, page 31. Nice examples of sets on page 30. List of facts about open and closed sets on page 28. The authors are encouraging geometric visualization I think and which I like.

Notes on Open Relative to Another Set. Look at Fig 1-21 on page 33. An open disk drawn in the full plane is obviously an open set in E^2 , the openness indicated by the dotted line boundary meaning that boundary is not part of the set . But here we have a truncated disk where the truncating line segment shown as a dark line is part of the set = truncated disk. With respect to the full plane, since boundary points are part of set D, the set D is *neither* open nor closed. For a set to be open, the set can contain no boundary points (p 25). For a set to be closed, every boundary point must be in the set. So set D would be open were it not for that left edge segment. The set D is "open relative to S" simply because D is "formed as the intersection of an open set in E^2 (the open disk) with the set S. Yes, D is not open relative to E^2, but by this strange definition, it is open relative to the half plane S which includes the vertical axis.

Theorem 2. Set S = not connected $\Leftrightarrow$ S = A$\cup$B where A and B are disjoint and are each open relative to S. [page 32]

Let's parse the proof of this theorem now in the $\Rightarrow$ direction to see how this "relative" business comes in. Saying that S is "not connected" means it is "disconnected", and by the definition on page 29 top, this means that S is the union of disjoint non-empty sets. Since A and B are disjoint, there are no boundary points of A located in B and vice versa. Thus, for example, A has no points in $\overline{B}$, the closure of B. A then lies in the complement of B, so far so good. I agree that every point in S must lie either in A or B. And I agree that A = S $\cap$ comp($\overline{B}$). Ah! Now since $\overline{B}$ is closed, comp($\overline{B}$) is open, and thus A is "the intersection of an open set with S" so A is open relative to S. There you have it. I guess this theorem runs the other way with no major extra mystery but I will not study that right now.

1.6 Ordering for R ≡ E^1 and grab bag of other topics [34]. The Bucks list off (p 34) some obvious inequality properties of elements of R (the real numbers). Notation [a,b] indicates a closed interval, so I assume that [a,b) would be open on the right end, but they don't use that combined notation at this point. Comment on symbol ∞ as not being an element of the set, just a symbolic way to say something. They then define upper and lower bounds for a set in an obvious way, and of course there are then a **LUB** and a **GLB**, all for a set in R.

Theorem 3. R (the line) is a connected set.

This seems fairly obvious, but they do it by definitions. Suppose disjoint so R = A$\cup$B as in theorem 2. The proof is well stated, I don't need to repeat it, and it makes use of the lub concept.

[36] Notion of $\mathbf{R}_0$ **(rationals)** being **dense** in R -- which just means that R = $\overline{R_0}$. They note that R_0 is countable whereas R is not. One page 37 they claim that Theorem 3 proof fails for R_0 because the lub idea "fails". They write A$\cup$B again, but now the special point "c" is $2^{1/3}$ which is not in R_0. Thus, you see that there is a "break" in things at point c, and so R_0 is disconnected at that point. In fact R_0 is disconnected at every non-rational real number and there are many such numbers! The idea of at least one point being "missing" means that such a set is incomplete. If there are no "gaps" in a set, then they say it is **complete** on page 37. I think we shall see another definition of complete later on, where a complete set is one which contains all its limit points for sequences, but we don't have the sequence idea at this point.

[37] We are then on to the **nested set** concept as shown in the figure. There is a theorem here which is not given a number: If the sets of a nested set are all closed and bounded, THEN there must be a point which lies in all the sets. Reader is told to prove this

Theorem 4. (**Bolzano-Weierstrass**). Every bounded infinite set S in E^n has at least one cluster point.

This is a very famous theorem, though it is often stated in other equivalent ways. Their proof uses the fig bottom of page 38. They get S into an initial square because the set S is stated to be bounded. Then keep dividing the square into 4 parts. Each time this is done, at least one of those 4 parts must be an infinite set. It is not hard then to show that for any little disk neighborhood you pick, you can get an infinite set square inside that disk and that is the definition of a cluster point: any disk you pick (no matter how small) has an infinite number of points in it . Notice that "sequence" is not involved here.

Notice that we are now building up a set of Theorems. The last one in this Chapter will be Theorem 14. The theorem numbers then start over in each new chapter.

Theorem 5. (**Heine-Borel**). For any closed and bounded set in E^n, any cover of open sets has a finite subcover.

The issue here is that the initial covering contain an *infinite* number of open sets and your problem is to show that you can get a full covering with only a finite subset of that infinite set of sets. A proof is given and it uses facts just presented.

Now suppose you have a set such that all infinite coverings by open sets have finite subcovers. Such a set is said to "have the Heine-Borel property" which is a form of being **compact** (by definition). The theorem then says that any closed and bounded set has the **Heine-Borel property**, or any closed and bounded set is compact. This is stated only for E^n and I know there are issues for other Hilbert Spaces though Bucks do not discuss this.

1.7 Sequences [40] A sequence is a mapping from integers I to E^n and the set of points in E^n that you get is the trace = image = range. It is implied that the sequence is infinite. A sequence converges if for any tiny ball you pick, you can pick N high enough that the entire tail is in the ball. $\lim_{n\to\infty} p_n = p$ is the notation. Think of $| p_n - p | < \varepsilon$. The example on page 41 (showing the trace) is a bounded sequence which does not converge because it is always jumping back and forth. It diverges (although it is bounded).

Theorem 6: A convergent sequence is bounded. This seems pretty obvious and proof is tiny. Note that this means that the trace of the mapping which is a_n is bounded in the range of the mapping.

Theorem 7: Sum of two convergent sequences converges to the obvious point.

Theorem 8: The set $\overline{S}$ consists of points which are all limit points of convergent sequences.

It seems this would be true for any closed set.

Comment on the **Axiom of Choice**.

Definition of a **subsequence** of a sequence.

Theorem 9. Any bounded sequence has a convergent subsequence.

A nice example is page 41 with the jumping back and forth. The sequence is bounded but does not converge. But we know there has to be a least one cluster point from BW Theorem and if we take just the points in the trace going to that cluster point, they form a convergent subsequence.

Definition: A convergent sequence has $| p_k - p | < \varepsilon$ for k large enough. A **Cauchy sequence** on the other hand has $| p_k - p_j | < \varepsilon$ for both k and j being large enough.

It turns out that if a sequence converges, then it is a Cauchy sequence, though I would have to do the exercise to show that. The converse is also true:

Theorem 10. A Cauchy sequence is always a convergent sequence (in E^n).

It is not sufficient that $| p_{j+1} - p_j | < \varepsilon$ to get a Cauchy sequence (special case where k = j+1). However, if it is true that $| p_{j+1} - p_j | < c^j$ where $|c| < 1$, then p_j converges, and that is the next theorem:

Theorem 11. If $| p_{j+1} - p_j | < c^j$ where $|c| < 1$, then $\{p_j\}$ converges.

The proof given first shows that $\{p_j\}$ is a Cauchy sequence, then we use Theorem 10.

1.8 Real sequences [48] In general E^n the points in the trace don't have any notion of $p_i > p_j$ since there is no obvious ordering of points in E^n. But in $R = E^1$ there is such ordering, and that allows some new facts to come out. **Monotonic** is defined.

Theorem 12: In R, a bounded monotonic sequence must converge.

Theorem 13: For any element x of R, there is always an integer n such that n > x.

Both these theorems seem obvious but proofs are given. The Bucks then to into four interesting examples. Here they are

1	$a_n = (n+1)/n$	monotone decreasing	$\rightarrow$ 1 from above
2	$a_n = (n)^{1/n}$	monotone decreasing after 3rd term	$\rightarrow$ 1 from above
3	$a_n = (1 + 1/n)^n$	monotone increasing	$\rightarrow$ e from below
4	$a_n = (a_{n-1} + A/a_{n-1})/2$	monotone decreasing	$\rightarrow \sqrt{A}$ from above

In the last case, where sequence is specified as a recursion, Bucks show that convergence is very rapid, and this is used as a simple computer algorithm for computing square roots.

Finally: we know that a bounded sequence might have several cluster points. In R, there must then be as max and a min cluster point, and these have special names: lim sup a_n and lim inf a_n. (page 51). These limits exist for a bounded sequence even though the sequence may not converge! So sort of a consolation prize in the world of limits.

Theorem 14: [52] For any starting point, (1-25)'s recursive definition of x_n converges to $\sqrt{x}$.

Notion of rate of convergence of a sequence.