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Chapter 4: Convergence **[pp 158-220]**

4.1 Infinite Series **[158]**

An **infinite series** involves terms $\{a_n\}$ and partial sums $\{A_n\}$, both of which are sequences. These two sequences together compose the "infinite series" object.

Page 159 shows that you can add **two** of these infinite series (be careful not to rearrange anything, the definition says "add term by term"), how to mult a series by a **const**, and how to "**multiply**" two series in a particular sense that seems quite strange to me and they call it a **Cauchy product** since there are other kinds of products. Examples are given. We see much later on p 172 that this product's nth term c_n is the sum of the nth diagonal of the $a_i b_j$ matrix.

[160] If the $\{A_n\}$ partial sum sequence converges, then one says "the series converges". So we may say,

$A_n \to A \quad \Leftrightarrow \quad$ series [$\{a_n\},\{A_n\}$] "converges" // else "diverges"

Note that diverges does not imply $A_n \to \infty$ since you could have a jumping situation where A_n is finite (bounded), but won't stabilize. It is possible for two series to diverge but their sum converges, and an example is given.

Both a series and the analogous integral are the same in the theory of Stieltjes integrals, again my R-N book has that stuff. Comment about using Σa_n to denote a series *and* to denote the sum of that series if it converges, two different animals. I think Bucks will use Σa_n to denote a series and not its sum.

Cauchy Product of two series**:** Consider these two power series (from end of this chapter),

$$(\Sigma_{n=0}^{\infty} a_n x^n)\,(\Sigma_{m=0}^{\infty} b_m x^m) \equiv [\, a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots][\, b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots]$$

Suppose x lies in within the radius of convergence of both series which includes the point x = 1. We shall later learn that you can rearrange the series (perhaps some conditions) without changing it. So lets rearrange according to powers of x to get:

$= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)\,x^2 + \ldots \equiv c_0 + c_1x + c_2x^2 + \ldots$

Maybe this rearrangement is so simple it is OK without conditions, I am not sure. Now evaluate at the point x = 1:

$(\Sigma_{n=0}^{\infty} a_n)\,(\Sigma_{m=0}^{\infty} b_m) = \Sigma_{k=0}^{\infty} c_k$

and this is the Cauchy Product! So there is a lot of reasonableness to this product. If the series converge at x = 1 and if rearrangement is allowed, then this MUST be the product, and is the only product.

Cauchy Product Application. Let $a_n = a^n/n!$ and $b_n = b^n/n!$ Then

$c_0 = a_0b_0 = 1$

$c_1 = a_0b_1 + a_1b_0 = 1 * b + a * 1 = (a+b)$

$c_2 = a_0b_2 + a_1b_1 + a_2b_0 = 1*b^2/2 + ab + a^2/2*1 = (1/2)(a^2+2ab+b^2) = (1/2)(a+b)^2$

$c_3 = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0 = 1*b^3/3! + a*b^2/2! + \ldots = (1/3!)\,(b^3 + 3b^2a + 3a^2b + a^3)$

$= (a+b)^3/3!$

.....

$c_k = (a+b)^k/k!$

Thus we find that

$(\Sigma_{n=0}^{\infty} a_n)\,(\Sigma_{m=0}^{\infty} b_m) = \Sigma_{k=0}^{\infty} c_k$

or

$(\Sigma_{n=0}^{\infty} a^n/n!)\,(\Sigma_{m=0}^{\infty} b^m/m!) = \Sigma_{k=0}^{\infty} (a+b)^k/k!$

If we define the series as $f(a) \equiv \Sigma_{n=0}^{\infty} a^n/n!$ then we have shown that f(a)f(b) = f(a+b). Since this is in fact the expo series, this is consistent with $e^a e^b = e^{a+b}$. This subject arises on page 202.

Theorem 1: [161] (**Term Limit Test**) If series Σa_n converges, then $a_n \to 0$. Proof is totally trivial using the A_n. Contrapositive gives a very easy test for series convergence!

[Note: not true for analogous $\int_0^{\infty} \cos(x^2)\,dx = (1/4)\sqrt{2\pi}$, p 144 since integrand $\not\to 0$ as $x \to \infty$]

Theorem 2: [161] (**Comparison Test**). If $a_n \leq b_n$ and things non-negative, then Σb_n conv $\Rightarrow \Sigma a_n$ conv.

The theorem is trivial since a bounded monotone sequence converges by Thm 12 Section 1.8. Theorem is analogous to Comparison Test for integrals Theorem 14 p 140.

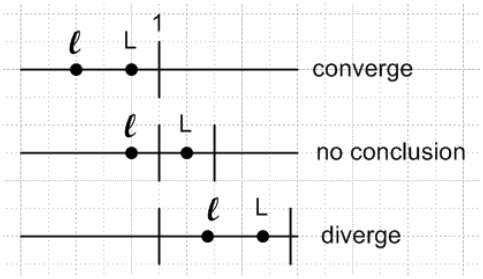
Corollary: [161] (**Two Series Ratio Test**) If $(a_n/b_n) \to L$ (positive and finite) with both a_n and b_n non-negative, then when you consider Σa_n and Σb_n, either both converge or both diverge.

Analogous to Ratio Test for integrals p 140. Since they give no proof, I will try to reroll the integral proof adjusted for the series. Given that $\lim_{n\to\infty} (a_n/b_n) = L$, we can find some large N where the limit is quite close to L and for n > N we can say $L/2 \le (a_n/b_n) \le 2L$. Then $a_n \le [2Lb_n]$ and since all is positive, the comparison test says if b_n converges, so does a_n. But we also have $b_n \le (2/L)a_n$ so if a_n converges then so does b_n. QED !

Theorem 3. [161] (**Ratio Comparison Test**) Again we have both a_n and b_n non-negative and $a_{n+1}/a_n \le b_{n+1}/b_n$ for large n. This says that the A ratio is smaller than the B ratio in the limit. Theorem says: Σb_n conv $\Rightarrow$ Σa_n conv. Proof is very simple.

Theorem 4: [161] Consider $\Sigma_{n=0}^{\infty} x^n$. For |x| < 1, sum is 1/(1-x), otherwise series diverges (real x). Proof shown is very simple.

Theorem 5. [161] (**Bounce Ratio Test**) If $a_n > 0$ and allow that the ratio a_{n+1}/a_n might bounce around for large n, so this ratio itself might not have a limit, but assume the ratio is bounded so lim sup ratio = L and lim inf ratio = ℓ. That is to say, the ratio is at least bouncing around inside $\ell \le a_{n+1}/a_n \le L$.
Obviously we have $\ell \le L$. Theorem says that $L < 1 \Rightarrow \Sigma a_n$ converges and $\ell > 1 \Rightarrow \Sigma a_n$ diverges, otherwise no conclusion.



Corollary: [162] (**No-Bounce Ratio Test**) Now assume ratio a_{n+1}/a_n does not bounce around but has a simple limit r. Then in the previous theorem, we have L and ℓ both equal r. Then $r < 1 \Rightarrow$ converge, $r > 1 \Rightarrow$ diverge, and $r = 1 \Rightarrow$ no conclusion.

Theorem 6 [162] (**The Root Test**). Again $a_n \ge 0$ and assume lim sup $(a_n)^{1/n} = r$. Then same conclusion as last theorem: $r < 1 \Rightarrow \Sigma a_n$ converges and $r > 1 \Rightarrow \Sigma a_n$ diverges

The above theorem makes use of the root and not the ratio.

Example 1 p 162. A simple but weird series has a ratio that jumps back and forth between 2 and 1/4, so Theorem 5 ends up with L > 1 and $\ell < 1$ which is the "no conclusion" case shown above. *But* the root test looks at

n = even $\quad a_n = \frac{1}{2^{n/2}} \qquad (a_n)^{1/n} = \frac{1}{\sqrt{2}}$

n = odd $\quad a_n = \frac{1}{2^{(n+3)/2}} \qquad (a_n)^{1/n} = \frac{1}{\sqrt{2}^{(1+3/n)}}$

As n→ ∞, both cases have limit r = $1/\sqrt{2}$ = 0.707 < 1. So by Corollary, series converges! One says that this series "escapes the Bounce Ratio Test, but is nailed by the Root Test." So you can have the root test say series converges, while ratio test does not know.

Theorem 7 [163] (**Sum/Integral Compare Test**) The sum 1 to ∞ of f_n = f(n) and the integral of f(x) from 1 to ∞ both converge or both diverge if f is continuous and monotone decreasing (to 0, meaning f ≥ 0) on [1,∞).

Proof is simple. This I think is the first test in the book relating integrals to series.

Corollary. [163] Series Σ_1^{∞} $(1/n^p)$ and Σ_1^{∞} $(1/n) / (\ln n)^p$ converge p > 1 else diverge.

To see why, use f(x) = $(1/x^p)$ for the first and f(x) = $(1/x) / (\ln x)^p$ for the second with Thm 7. Basically the 1/x integral diverges since integral is lnx. For p = 1+ε, you get both to converge. Int of 1/(xlnx) diverges just as Int of 1/x does.

Theorem 8 [p 163] (**Raabe's Test**). Assume $a_n > 0$ as usual, look at ratio $r_n \equiv a_{n+1}/a_n$. Suppose $r_n \leq 1 - p/n$ for p > 1 and large n. Then series converges.

This type of series escapes the No Bounce Root Test, because the ratio approaches 1 from below, and we cannot say ratio → L < 1. In fact we have L = 1 so The NBRT has no conclusion.

Proof involves a use of the MVT and a simple lemma.

Example 1: [164] A series is given where $r_n \to 1$, so cannot use the NBRT test Thm 5. But it passes the Raabe test since $r_n = 1 - (3/2)/(n+1) \leq 1 - (3/2)/n$ so p = 3/2. So this series escapes the NBRT but is captured by the Raabe Test which shows it converges.

We now start allowing negative valued terms.

Theorem 9: [164] (**Abs Value Test**) If $\Sigma |a_n|$ converges, so does Σa_n.

Analogous to Thm 15 p 141 which says $\int_a^b |f|$ converges ⇒ $\int_a^b f$ converges . Simple proof given.

Fact: It is possible for $\Sigma |a_n|$ to diverge without Σa_n diverging. In this case, Σa_n is **conditionally convergent**, and if both converge you have **absolute convergence**. This is the same idea as with integrals on page 141.

Example 1 [p 165] The abs converges, so then does the series shown
Example 2 and Example 3: same thing.
Example 4: Has $a_n \to \pm 1/4$ and not 0, so diverges.

Example 5: Series is $\Sigma_n(-1)^n$ 1/n. We now that the abs val series Σ_n1/n diverges using Theorem 7 above, but this of course does not say $\Sigma_n(-1)^n$ 1/n diverges. We need more tools for this series!

Theorem 10 [165]: (very obscure) Here $a_n \rightarrow 0$, $\Sigma|a_{n+1} - a_n|$ converges, B_n is bounded. Then the product sequence $\Sigma a_n b_n$ converges. Says nothing about whether Σa_n or Σb_n converges.

Corollary if a_n is also monotone decreasing it turns out that $\Sigma|a_{n+1} - a_n|$ converges, so you get this alternate version :

Theorem 10A = Corollary 1: (very obscure) If monotone $a_n \rightarrow 0$ and B_n is bounded, then the product sequence $\Sigma a_n b_n$ converges. Says nothing about whether Σa_n or Σb_n converges.

The motivation is to have $b_n = (-1)^n$ and then these theorems might comment on alternating series.

Corollary 2 [166] (**Leibniz Alternating Series Test**) : If a_n monotone $\rightarrow 0$, then $\Sigma (-1)^{n+1} a_n$ converges.

Proof: Let $b_n = (-1)^{n+1}$. Obviously B_n is bounded. So apply Thm 10A, QED.

Notice that this does not seem to have a simple integral analog.

Theorem (Exercise 12 on p 179): For a convergent alternating series, if you stop at some term, the |error| is $\leq$ the |next term|. Also, the approach to the limit alternates between above and below.
I have not proven this, but it gets used later on page 187. Proof seems would be very easy.

My proof: If series is $\Sigma (-1)^n a_n$ so *if we knew* that a_n was monotone decreasing, I think the above theorem would be obvious. You just draw a little bar chart of the terms and the conclusions fall out. We do know that $a_n \rightarrow 0$ by the very first theorem we studied. I guess by definition of "alternating series" we know that the a_n are all positive, so yes, then we do know monotone $a_n \rightarrow 0$ and then we have it.

Example 5: [167] By this test, $\Sigma (-1)^{n+1}(1/n)$ converges.

$\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4$
$\ln(2) = 1 - 1/2 + 1/3 - 1/4 .... = .693$

Example 1 [167]. Has alternating signs but a_n is not monotone: 1/3,1/2,1/5,1/4... so diverges

Example 2 has b_n = (1,1,-2) repeated. This is not $(-1)^m$ but is caught by Thm 10A above, converges.

Example 3 [167] has $\sin(nx)/\sqrt{n}$ and meets the 10A test, but you have to show that B_n for $b_n = \sin(x)$ is bounded. They show this with a Lemma p 167, and then this series converges by Thm 10A with $a_n = 1/\sqrt{n}$.

Rearrangement (p168). Bucks show ways to rearrange the ordering of a series. If you do this, you should not expect to get the same sum, and you might even convert a convergent series to a divergent one. In

Example 1, Bucks show that if rearrangement did not change a sum, then you end up with S = 3/2 S for the ln2 series appearing above, which is a contradiction.

Theorem 11: [169] Only if a series is absolutely convergent are you allowed to rearrange the series without changing the sum.

Parenthesis Insertion (Regrouping). Example 1 page 169 shows how you might rearrange a series by doing this, sort of grouping the terms of a series. It turns out that if the series converges, then so does any such regrouping. However, if the original series diverges, this regrouping may make it converge. Example 2 gives a *trivial* example of this happening!

Theorem A1 [p 170] This is a very obscure section, I will just summarize it. I derived (4-2) in line which shows that the difference between a sum and its corresponding integral is bounded from below and above as shown there. Note that f(n) = f_n must be *positive and decreasing* (as for example f_n = 1/n is).

(4-2) says $f(n) \le [\sum_{k=1}^{n} f(k) - \int_1^n f(x)dx] \le f(1)$.

If we express $\Sigma - \int$ as an error C_n, then $\Sigma = \int + C_n$ where $f(n) \le C_n \le f(1)$

The first example is f(n) = 1/n which results in $1/n \le C \le 1$ and $\int = \ln(n)|_1^n = \ln(n) - 0 = \ln(n)$.

So the result is that $\Sigma = \ln(n) + C$ where $1/n \le C \le 1$. Since $0 < 1/n$ we get $0 < C \le 1$ which is close to what they claim at **result A** [170]. But the **main point** of this example is this:

$$\sum_{k=1}^{n} f(k) = \ln(n) + C_n$$

so you would say this sum is logarithmically divergent. This then estimates the "**rate of divergence**" . The divergence rate is not affected by C_n since it is bounded to lie within [0,1]. This also seems to be a good way to "estimate" the value of a painful summation if the integral is simple. Here the sum and the integral are always within ±1 of each other for any n. I know that sums of the same form as integrals are very often much harder to do just because of the discreteness factor.

Note: if f_n is positive and increasing, they claim you get the same result but *reverse inequalities*. So

Theorem A2. $f(1) \le [\sum_{k=1}^{n} f(k) - \int_1^n f(x)dx] \le f(n)$ if f(n) increasing

In Example 2 [170], they then consider $f(n) = n^{1/2}$ which is the increasing case, and I note that $\int = (2/3)[n^{3/2} - 1]$ in this case. The two inequalities above then say

$$f(n) \ge [\Sigma - \int] \ge f(1)$$

or

$$n^{1/2} \geq \Sigma - (2/3)[n^{3/2} - 1] \geq 1$$

or

$$n^{1/2} - 2/3 \geq \Sigma - (2/3)n^{3/2} \geq 1 - 2/3$$

or

$$n^{1/2} - 2/3 \geq \Sigma - (2/3)n^{3/2} \geq 1/3$$

This gives the two results

$$\Sigma - (2/3)n^{3/2} \geq 1/3$$
$$\Sigma - (2/3)n^{3/2} - n^{1/2} \leq -2/3$$

or

$\Sigma \geq (2/3)n^{3/2} + (1/3)$	$\equiv r(n)$	red	
$\Sigma \leq (2/3)n^{3/2} + n^{1/2} - 2/3$	$\equiv g(n)$	green	//checked all this on 1.14.15

Note that $g(n) - r(n) = n^{1/2} - 1 > 0$ for large n. Here is a plot of the two functions

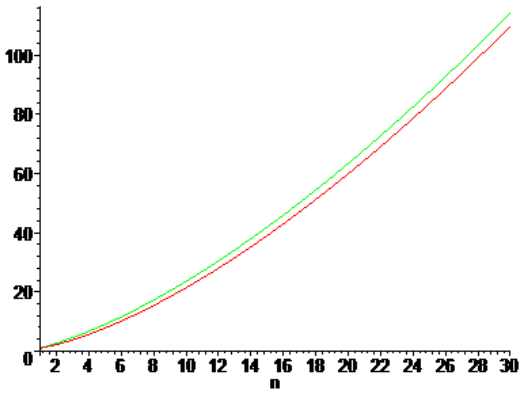
```
r := (2/3)*n^(3/2)+1/3;
```

$$r := \frac{2}{3}n^{\left(\frac{3}{2}\right)} + \frac{1}{3}$$

```
g := (2/3)*n^(3/2) + n^(1/2) - 2/3;
```

$$g := \frac{2}{3}n^{\left(\frac{3}{2}\right)} + \sqrt{n} - \frac{2}{3}$$

```
plot([r,g],n=1..30);
```



Thus, we have a rather tight looking upper and lower bound for our series. $\Sigma\, n^{1/2}$ -- it has to fit between the two curves which are quite close.

For large n you would ignore the constants and say

$\Sigma \geq (2/3)n^{3/2}$	$\equiv r(n)$	red	
$\Sigma \leq (2/3)n^{3/2} + n^{1/2}$	$\equiv g(n)$	green	//checked all this on 1.14.15

The second line here agrees with result B on page 170. But the first result is the one that tells you the series diverges as $n^{3/2}$ but they don't even quote this first inequality.

The Bucks are interested in getting a cleaner description of the series $\Sigma \sqrt{n}$ as it diverges, and to this end they prove the following:

Theorem 12: The quantity $[\Sigma - \int - f(n)/2]$ is bounded above and below provided that f is positive *and* has negative curvature in the entire range 1 to ∞ (see p 171 figure). What this says is

$$a \leq [\Sigma - \int - f(n)/2] \leq b \qquad \Sigma \equiv \Sigma_1^n f(k) \qquad \int \equiv \int_1^\infty f(x)dx$$

where a and b are real numbers independent of n. You can of course write this as $|\Sigma - \int - f(n)/2|$ being bounded, but that is not really the useful part of the theorem. They show in the (long) proof that

$$\int \geq \Sigma - f(n)/2 - f(1)/2 \qquad \text{page 171 A}$$

$$\int \leq \Sigma - f(n)/2 + f(2)/2 - f(1) \qquad \text{page 171 B}$$

This then says that

$$0 \geq [\Sigma - \int - f(n)/2] - f(1)/2 \qquad \text{page 171 A}$$

$$0 \leq [\Sigma - \int - f(n)/2] + f(2)/2 - f(1) \qquad \text{page 171 B}$$

or

$$f(1)/2 \geq [\Sigma - \int - f(n)/2] \qquad \text{page 171 A}$$

$$f(1)-f(2)/2 \leq [\Sigma - \int - f(n)/2] \qquad \text{page 171 B}$$

or

$$f(1) - f(2)/2 \leq [\Sigma - \int - f(n)/2] \leq f(1)/2$$

which has the form I quoted above,

$$a \leq [\Sigma - \int - f(n)/2] \leq b \; . \qquad a = f(1) - f(2)/2 \qquad b = f(1)/2 \qquad \textbf{Theorem 12}$$

Example 2A [172]: We apply the above to $f(n) = n^{1/2}$

$$f(1)-f(2)/2 \leq [\Sigma - \int - f(n)/2] \leq f(1)/2$$

or

$$1-1/\sqrt{2} \leq [\Sigma - \int - n^{1/2}/2] \leq 1/2$$

Inserting the integral $(2/3)[n^{3/2} - 1]$ this says

$$1-1/\sqrt{2} \leq [\Sigma - (2/3)n^{3/2} + 2/3 - n^{1/2}/2] \leq 1/2$$

or

$$1-1/\sqrt{2} - 2/3 \leq [\Sigma - (2/3)n^{3/2} - n^{1/2}/2] \leq 1/2 - 2/3$$

or

$$1/3 - 1/\sqrt{2} \leq [\Sigma - (2/3)n^{3/2} - n^{1/2}/2] \leq -1/6$$

Now since RHS is negative, this time define

$$-C \equiv [\Sigma - (2/3)n^{3/2} - n^{1/2}/2]$$

so that

$$1/3 - 1/\sqrt{2} \leq -C \leq -1/6$$

which then means

$$1/6 \leq C \leq 1/\sqrt{2} - 1/3$$

so we end up with

$$[\Sigma - (2/3)n^{3/2} - n^{1/2}/2] = -C$$

or

$$\Sigma = (2/3)n^{3/2} + (1/2)\, n^{1/2} - C \qquad \text{where } 1/6 \leq C \leq 1/\sqrt{2} - 1/3$$

which agrees with page 172 A.

Comments: Previously using Theorem A2 we showed that our sum was sandwiched between two functions in this manner for large n :

$$(2/3)n^{3/2} \leq \Sigma \leq (2/3)n^{3/2} + n^{1/2}$$

The difference between the two curves is $n^{1/2}$ which gets very large for large n, so in absolute terms the uncertainty of the sum lies in a band that is ever increasing. Theorem 12 however is able to nail things down much more tightly. The uncertainty rather than being $n^{1/2}$ is just a small constant on the order of unity, and we see that the correct approximate function is $(2/3)n^{3/2} + (1/2)\, n^{1/2}$. This happens to lie in some sense half way between my red and green curves. Thus, this result is certainly consistent with the previous less strict result.

Example 3 page 172. Here f(n) = ln(n). Requote Theorem 12 from above,

$$f(1) - f(2)/2 \le [\Sigma - \int - f(n)/2] \le f(1)/2$$

and assume f(n) = ln(n) so we are summing logs. How does this Example 2 work out?

$$f(1) = \ln(1) = 0$$
$$f(2) = \ln(2)$$

```
Int(ln(x),x=1..n);
```

$$\int_1^n \ln(x)\,dx$$

```
value(%);
```

$$n\ln(n) - n + 1$$ // in agreement with p172B

So we then have

$$- \ln(2)/2 \le \Sigma - [n\ln(n) - n + 1] - \ln(n)/2 \le 0$$

or

$$- \ln(2)/2 \le \Sigma - n\ln(n) + n - 1 - \ln(n)/2 \le 0$$

or

$$1 - \ln(2)/2 \le \Sigma - n\ln(n) + n - \ln(n)/2 \le 1$$

or

$$1 - \ln(2)/2 \le [\Sigma - (n+1/2)\ln(n) + n] \le 1$$

Define

$$C = [\Sigma - (n+1/2)\ln(n) + n]$$

so we get

$$\Sigma_{k=1}^{n} \ln(k) = (n+1/2)\ln(n) - n + C \quad \text{where} \quad 1 - \ln(2)/2 \le C \le 1 \quad \text{or} \quad .653 \le C \le 1$$

which agrees with page 172 C. Once again, they are merely trying to characterize the manner in which a divergent some diverges (AND they are coming up with a very tight approximation to the sum!). Now of course

$$\Sigma_{n=1}^{n} \ln(n) = \ln(n!) = (n+1/2)\ln(n) - n + C$$

which says

$$n! = \exp[(n+1/2)\ln(n) - n + C] = e^{(n+1/2)\ln(n)}\, e^{-n}\, e^{C}$$

$$= n^{(n+1/2)}\, e^{-n}\, e^{C}$$

$= n^n \sqrt{n}\ e^{-n} e^C$

which agrees with their result p 172D, but they have e^{Cn} as a typo I think. No, they just have this thing meaning exp(C_n), so let's not flag a typo.

Here is Stirling

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

which suggests that $e^C = \sqrt{2\pi}$ so C = ln($\sqrt{2\pi}$) = .919

```
evalf(ln(sqrt(2*Pi)));
                                    .9189385330
```

This is indeed in the range .653 ≤ C ≤ 1 given above. Here is fancy Stirling

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \cdots\right)$$

So our derived Stirling approx is "weak" only because we could not show that C = ln($\sqrt{2\pi}$), we could only show it was in a certain range, fine.

Double Series Stuff p 172

Bucks give you no heading to indicate a change of topic, always trying to save space I suspect.

Here we look at $\Sigma_{ij}a_{ij}$ which is called a **double series**. If you write out the elements in a 2D array, you realize that you can do the sum in many different ways to work toward an double infinite sum. One way is to go down the columns one at a time (doing the column sums first, then adding those sums), or you can do the rows first, or you can do the obvious **zig zag** ordering which seems reasonable, in which case you are working down a triangle. For a fixed n,n you can do rows or columns first because you are doing a finite sum (the zig zag only then does the upper left half of the square).

The **convergence definition** they choose on page 172 is Cauchy like in that the abs value shown has to → 0 as shown. In the general Cauchy you would have separate N and M, but here it is N and N as shown. They are keeping the matrix square NxN as they increase N to infinity.

Theorems are just quoted without proof and the reader can find proofs in Knopp 1928 or other sources.

Theorem A: [173 top] $\Sigma_{ij} |a_{ij}|$ converges ⇒ $\Sigma_{ij}\, a_{ij}$ converges [like Theorem 9 p 164]
If the abs sum converges, so does the other sum, series is **absolutely convergent**.

Theorem B: [173] $\Sigma_{ij}\, a_{ij}$ is absolutely convergent ⇔ you can rearrange arbitrarily and maintain the sum. This is like Theorem 11 p 169.

Corollary C: [173] If $\Sigma_{ij}\, a_{ij}$ is absolutely convergent, you can rearrange as the zig-zag ordering.

Then we have

Theorem 13. If Σa_j and Σb_j are each absolutely convergent to A and B, then the zig zag ordered double sum Σc_n converges to AB. This zig-zag ordering is in fact the Cauchy product defined earlier.

I wrote in pencil some unknown time ago that also $\Sigma|c_n|$ converges, but now that does not seem right. This claim is not made by the Bucks, and I don't see what it should be true. It would mean that Σc_n series was absolutely convergent, but that claim is NOT made. Restate things:

Theorem 13: [173] $\Sigma a_j \to$ A and $\Sigma b_j \to$B and $\Sigma|a_j| \to$ A' and $\Sigma|b_j| \to$B' => $\Sigma_{zz} a_i b_j \to$ AB.

The theorem says nothing at all about $\Sigma_{zz}|a_i b_j|$. The zz means zigzag.

Theorem 13A: [173] If either Σa_j and Σb_j is absolutely convergent, and the other is conditionally convergent, then Thm 13 is still valid.

$\Sigma|a_j| \to$ A ($\Sigma b_j \to$ B but $\Sigma|b_j|$ diverge) => $\Sigma_{zz} a_i b_j \to$ AB

Restate once again

Theorem 13A: [173] $\Sigma a_j \to$ A and $\Sigma|a_j| \to$ A' (abs conv) and $\Sigma b_i \to$ B but $\Sigma|b_j|$ diverges => $\Sigma_{zz} a_i b_j \to$ AB.

If $\Sigma|b_j|$ were to converge, then we would have Theorem 13, so here we are relaxing one of the four premise conditions and obtaining the exact same conclusion as Theorem 13.

I updated my scribd to email and ginger 88 and was able to download the 1951 **Knopp book** on series, (2nd Ed) and in it I found theorem 13A on page 321 from which I quote:

° Theorem of ***Mertens*** [23]. *If at least* ***one*** *of the two convergent series* **188.**
$\Sigma a_n = A$ *and* $\Sigma b_n = B$ *converges absolutely,* Σc_n *converges and* $= A \cdot B$.

When Knopp writes Σb_n = B he is of course implying that Σb_n converges. The book also treats infinite products, by the way. Each Theorem gets a huge fat number on the right which I think let's you find things quickly in this book from later references, a good idea. There are about 300 theorems!! Wow. He has a wiki page, was at Tubingen for most of his professorial career. This book went Dover in 1956, and there is another Dover in 1990 of a similar title. Knopp was 1882-1957, a little before Harry.

Theorem 13B: [174] If both Σa_j and Σb_j are conditionally convergent, then Σc_n may diverge.

Theorem 14. [174] If $a_{ij} \geq 0$, then if the columns-first sum converges, so does the rows-first sum.

I skip the proof which looks straightforward.

Counterexample: The a_{ij} shown on page 175 are NOT positive, and in this case you get different numbers for the columns-first and rows-first summations! I wonder what the analogous 2D integration situation would be, something involving a delta function added to its derivative?

Parameters in a Single Sum [175]

The topic of double series is suddenly ended and here Bucks consider 5 examples of single series which contain a parameter. The problem is to determine for what range of the parameter(s) the series diverges or converges. Examples 1 and 2 we have already seen. Examples 3 and 4 also have a single parameter. The final Example 5 [176] has two parameters α and β, and it is shown that you get convergence in the half plane above $\beta = \alpha+1$, else divergence.

Then comes a long set of exercises. This section is well suited to "exercises" since you can throw lots of series at the student. But some ask for proofs and some are "word problems". I see that when I took the Math 105a Buck course as a college sophomore in 1968, I had to do the red circled problems. I do regret that all my work done then is lost forever. At that time I was a poor documenter, I had not learned of its value "later on". Maybe something is in the MRL attic? The APL attic is gone forever.

4.2 Uniform Convergence **[180]**

Suddenly the topic is changing from the study of convergence of 1D and 2D series of *constants*, to the study of the convergence of 1D series of *functions* of one variable x.

Note: Uniform *continuity* for a function f(x) was treated in Section 2.3. Recall that you need a δ that works for *all* x in the domain D. Here we are talking uniform *convergence* of a sequence of functions or of a series whose terms are functions of x.

Def: [180A] A *series* $\Sigma f_n(x)$ **converges pointwise** to f(x) on domain D if it converges for each x in D. You might find that δ depends on x, and there is no specific δ that works for all x.

Def: [180B] A *sequence* $\{f_n(x)\}$ **converges pointwise** to f(x) on domain D if it converges for each x in D. That is to say, for any given x in D, you can find N such that $\| f_n(x) - F(x) \| < \varepsilon$ for $n > N$. Again, you might find that N depends on x, so you really have N(x).

I now jump ahead and will then come back and fill in.

Def: [182A] A *sequence* $\{f_n(x)\}$ **converges uniformly** to F(x) on domain D if it converges for all x in D "with the same N". That is, there is an N such that $\| f_n(x) - F(x) \|_D < \varepsilon$ for $n > N$ *and* for all x in D. The point is that the same large integer N works for all x in D, you don't need N(x).

Important point: Here $\| f_n(x) - F(x) \|_D$ is the max norm, sometimes called the infinity norm. I have much more on this later. Earlier definitions just have a normal E^m geometric norm.

Def: [183A] A *series* $\Sigma f_n(x)$ **converges uniformly** iff the *sequence* of partial sums $F_n(x)$ converges uniformly to F(x) as per the above definition.

Def: [183B] A *series* $\Sigma f_n(x)$ **converges uniformly** iff the tail of the series converges uniformly to 0. Can write this last as $\| \Sigma_{k=n}^{\infty} f_n(x) \|_D \to 0$. This is stated as if it were an alternate definition.

My proof that two Defs are same: If the tail → 0 for all x, then $F_n(x) \to F(x)$ for all x. But Bucks use the word "pointwise" after the iff, which seems perhaps wrong. Their notation with the sub D suggests they mean uniform. *They* give no proof that the two definitions are the same.

We now do a little **digression** then resume back on page 180.

Page 180 makes the distinction between pointwise convergence and uniform convergence.

For a series with a parameter, maybe that parameter is p in a set E. Pointwise means you converge for any single point p in E you pick. This means you can find a δ(p,ε) so partial sum is within some ε of its limit. It may be that as p approaches some p_0 in E, δ(p,ε) has to be smaller and smaller for the same ε, but that is OK.

Now, uniform convergence says there is some δ(ε) so partial sum is within some ε of its limit for all points p in E. The key point is that the same δ has to work for ALL points p. It cannot be a function of p as it could be for pointwise convergence.

Side Question and Digression from Text: Consider $f(x) = \Sigma_{n=0}^{\infty} x\, b_n(x)$ where x is a parameter. Are you allowed to extract x from this sum?

Answer: Write

$$f(x) = \lim_{N\to\infty} \{ \Sigma_{n=0}^{N} x\, b_n(x) \} \ .$$

Inside {..} we have x times a (presumably) finite quantity, so we can extract the x from Σ^N ,

$$f(x) = \lim_{N\to\infty} \{ x \Sigma_{n=0}^{N} b_n(x) \} \ .$$

Now I think $\lim_{x\to a} [f(x)g(x)] = [\lim_{x\to a} f(x)] * [\lim_{x\to a} g(x)]$. But how do I know this? We know this is true from p 78 Thm 14 provided the two limits on the right exist. If one of them does not exist, then the limit on the left may still exist, so the equation then does not mean much. An example of the latter is to choose f(x) = x and g(x) = 1/x and a = 0 so then $\lim_{x\to 0} [x * 1/x] = \lim_{x\to 0} [x] * \lim_{x\to 0} [1/x]$. The left side is 1, while the right side is 0 times a limit which does not exist, sort of $1 = 0 * \infty$.

We shall assume that the series $\Sigma_n b_n(x)$ converges so the partial sums B_n have a limit. So we apply this theorem noting that $\lim_{N\to\infty} x = x$ and we find

$$f(x) = x \lim_{N\to\infty} \{ \Sigma_{n=0}^{N} b_n(x) \} = x \Sigma_{n=0}^{\infty} b_n(x)$$

So the **answer** is that you are allowed to extract the x as long as the residual sum converges.

Exercise PL 1**.** Show that $f(x) = \Sigma_n^{\infty} x(1-x)^n$ converges for $0 < x < 1$. For the moment, we shall assume that the sum $\Sigma_n^{\infty}(1-x)^n$ converges, so as above we can extract the x. So $f(x) = x\,\Sigma_n^{\infty}(1-x)^n$ and we now consider $g(x) = \Sigma_n(1-x)^n$. Let $y = 1-x$ so this becomes $G(y) = \Sigma_n y^n$. We know from Thm 4 p 161 that this converges for $|y| < 1$, it is just a geometric series. Thus g(x) converges for $|1-x| < 1$ which is the range $0 < x < 2$. This includes (0,1], so we conclude that g(x) converges for $0 < x \leq 1$ and then so does f(x).

What does f(x) converge to? $G(y) = 1/(1-y) = 1/x$, so it converges to $f(x) = x\,G(y) = x * 1/x = 1$. Thus, f(x) converges to 1 for $0 < x < 2$.

Now go back to $a_n(x) = x(1-x)^n$. It certainly is true that $a_n(x \to 1) \to 0$ for fixed n so each term $a_n \to 0$ and you might think therefore the whole series $\to 0$ for that reason: add up zeroes to get zero. But we just showed that for x at or near 1, the series adds up to 1, not 0. How do we *explain* this? Well,

$$\lim_{x\to 1} \{ \lim_{N\to\infty} \Sigma_{n=0}^{N} x(1-x)^n \} = \lim_{x\to 1} \{ 1 \} = 1$$
$$\lim_{N\to\infty} \{ \lim_{x\to 1} \Sigma_{n=0}^{N} x(1-x)^n \} = \lim_{N\to\infty} \{ 0 \} = 0$$

The fact is that when you reverse the order of the two limits, the answer is different! This is a key concept brought out by this example. It seems that there is some issue going on at x = 1. [Also at x = 0.]

End digression and resume, going back to page 180.

Example 1 (p 180). Infinite series with x as a parameter. $F(x) = \Sigma_n^{\infty} x(1-x)^n$ [as studied above]. We have $a_n(x) = x(1-x)^n$ and it is true that $a_n(x \to 0) = 0$ and $a_n(x \to 1) = 0$ for any finite n. In either of these cases we can write out the series of interest

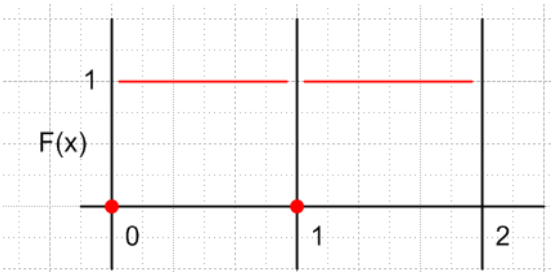
F(0) = 0 + 0 + 0 ..... = 0 // moving x→ 0 limit through the sum
F(1) = 0 + 0 + 0 ..... = 0 // moving x→ 1 limit through the sum

But this logical thinking is wrong, because adding up an infinite number of 0 things may not give you 0. So I would *not* claim that F(0) = 0 or that F(1) = 0, but it seems that way by writing things out.
Notice that we are not "extracting the x from the sum", we are leaving the x where it is. In the above, we are writing out the series with the parameter preset to two different values.

Now according to the Exercise above, we know that

$F(x) = \Sigma_n^{\infty} x(1-x)^n = 1$ for $0 < x < 2$ don't know about x = 0 and x = 1

Here is a plot where I show the two naive 0 values as red dots



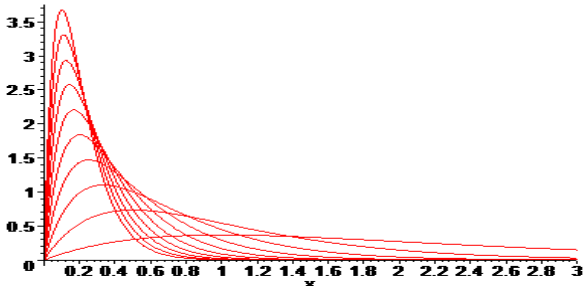
The function F(x) is in fact continuous at x = 1, and has limit 1 as you approach x = 0 from above.

Example 2. In this example, $a_n(x) = nx^2/(n^3+x^3)$. Certainly $a_n(x\to\infty) = 0$. If you were to take this limit first and add up the terms, they would add to 0 so you would naively argue $F(\infty) = 0$. On the other hand, for any $0 < x < \infty$ (finite x) the series goes as $1/n^2$ and we know this converges. Their question is: what is lim F(x) as $x\to\infty$? Bucks show that $F(x) \geq x/12$, though they don't know exactly what F(x) is (and Maple gives a horrible Ψ function mess). Thus, the series diverges, you could say $F(x\to\infty) = \infty$. This then differs from doing the other ordering to get $F(\infty) = 0$. Again we have an order interchange disagreement.

Example 3 (p 181). Here $f_n(x) = n^2x\, e^{-nx}$. We are not doing a series here, we are instead considering the integral $I_n = \int_0^1 f_n(x)\, dx$. We can see that $f_n(x) \to 0$ as $n \to \infty$, so you might logically think that the integral $I_n \to 0$ as well, since just a finite integral. We ponder $0 \leq x \leq 1$ as a range of x. By directly computing the integral, we find that $I_n = 1$, so all these curves have unit area:

```
f := (n,x) ->n^2*x*exp(-n*x);
plot( [seq(f(n,x),n=1..10)],x=0..3, color=red);
```

$$f := (n, x) \to n^2\, x\, e^{(-n\,x)}$$



So again if you first compute the area then take the limit, you get $I_n = 1 \to 1$. But if you go inside the integral and first take the limit $n\to\infty$ and then do the integral, you conclude that $I \to 0$ which is wrong. It is again an order interchange issue.

Example 4 (p 181). Slightly different $f_n(x) = x\, e^{-nx^2}$. Again $f_n(x) \to 0$ as $n \to \infty$, so this approaches what appears to be the "flat" curve f = 0 which of course would have zero slope. So naively you might then argue that $df_n(x)/dx \to 0$. But Maple shows that $[df_n(x)/dx]^{x=0} = 1$ for all n, so in the limit $n\to\infty$ you get that the slope is 1.

```
f := x*exp(-n*x^2);
```

$$f := x\,e^{(-n x^2)}$$

```
diff(f,x);
```

$$e^{(-n x^2)} - 2 x^2 n\, e^{(-n x^2)}$$

```
eval(%,x=0);
```

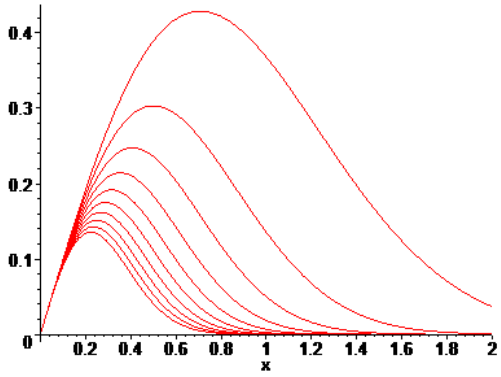
$$1$$

or just plotting

```
f := x*exp(-n*x^2);
```

$$f := x\,e^{(-n x^2)}$$

```
plot({seq(f,n=1..10)},x=0..2, color = red);
```



So the point here is that order interchange between differentiation and some other limit might not be allowed. ok to here

Notation : $|\ f\ |_E$ refers to the max value of function f evaluated on the set E (they use lub).
And then $|\ f - g\ |_E$ is the "max distance" between f and g over E. For f:R→R this would just be the maximum size of the vertical gap between the two functions over some interval.

[182] Let f_n be a sequence of functions. Here I repeat Def 183B already stated above:

def: $|\ f_n - F\ |_E \to 0$ ⇔ sequence f_n converges uniformly to F over E

Reconsider Example 3 [called Ex 3A on 182]. For any particular x, as n→∞ you do get $f_n \to 0$ so f_n converges pointwise to function F = 0. But $|\ f_n - 0\ |_E \to \infty$ as the plots above suggest. They compute where the max is, and then show that max(f_n) = n/e which occurs at x = 1/n, so in fact $|\ f_n - 0\ |_E = n/e \to \infty$.

def: Bottom of page 182 gives the standard ε,δ definition of pointwise convergence of a sequence. They point out that δ is a function of point p in general, but for uniformity, δ must work for all p. I have already noted this definition as Def 180B and the point about δ stated above in these notes.

So far, then, Bucks have discussed "uniform and pointwise convergence of a sequence of functions".

If you think of the sequence of partial sums associated with a series, then :

def: *series* is uniformly convergent $\Leftrightarrow$ partial sum *sequence* is uniformly convergent

[This appears above as Def 183A stated above in these notes.]

Bucks spring something here, suddenly we have the sequence tail appearing out of nowhere. [This also is noted above, but I will leave these notes here. Still unclear about pointwise or uniform.]

PL Theorem. Suppose $| f_n - F |_E \to 0$ where f_n is partial sum of series a_n . Then

$$| \Sigma_{k=1}^{n} a_k - F |_E \to 0 \qquad \text{or} \qquad | A_n - F |_E \to 0$$

Now I guess write $\Sigma_{k=1}^{n} a_k + \Sigma_{k=n+1}^{\infty} a_k = \Sigma_{k=1}^{\infty} a_k$ = pointwise convergent [with $a_n(x)$] .
Then we know $\Sigma_{k=n}^{\infty} a_k = F$, so we can write

$$\Sigma_{k=1}^{n} a_k + \Sigma_{k=n+1}^{\infty} a_k = F$$

or

$$\Sigma_{k=1}^{n} a_k - F = - \Sigma_{k=n+1}^{\infty} a_k$$

We can then claim that

$$| \Sigma_{k=1}^{n} a_k - F |_E = | - \Sigma_{k=n+1}^{\infty} a_k |_E = | \Sigma_{k=n+1}^{\infty} a_k |_E$$

Therefore

$$| \Sigma_{k=1}^{n} a_k - F |_E \to 0 \qquad \Leftrightarrow \qquad | \Sigma_{k=n+1}^{\infty} a_k |_E \to 0 \quad \text{"tail goes to 0"}$$

and we get these alternate equivalent definitions: (as usual, over domain E)

Definition and Theorem (no number, top page 183)

series is uniformly convergent $\Leftrightarrow$ partial sum sequence is uniformly convergent
series is uniformly convergent $\Leftrightarrow$ series is pointwise convergent and $| \text{tail}_n |_E \to 0$

def: [183] **Cauchy Property for sequence**:

For any $\varepsilon>0$, you can find N such that $| f_n - f_m |_E < \varepsilon$ for n,m both > N .

Theorem 15 (and precursor p 183)

Sequence f_n has Cauchy property $\Leftrightarrow$ sequence f_n is uniformly convergent.

In all these things it is understood that there is some domain E which then labels the |..| thing.

A long proof is given I think it is basic so as usual I skip it.

Corollary [p 184] :

series is uniformly convergent $\Leftarrow$ $\| \text{tail}_{n,m} \|_E \to 0$ // Cauchy tail

This gives what looks like a simple test for uniform convergence if you know about the tail.

Theorem 16 (**Weierstrass Comparison M-Test**). [184] Consider a series $\Sigma_k u_k(x)$. Suppose you know that $| u_k(x) |_E \leq M_k$ for all k (perhaps in an infinite tail) and suppose you also know that the series $\Sigma_k M_k$ converges. Here M_k are not functions of E points. Then $\Sigma_k u_k$ is *uniformly* convergent on E.

A small proof is given. Compare this to Theorem 2 page 161 stated earlier in this doc with u_k constants:

Theorem 2: [161] (**Comparison Test**). If $u_k \leq M_k$ and things non-negative, then ΣM_k conv $\Rightarrow \Sigma u_k$ conv.

So you see that when this is generalized from u_k constants to $u_k(x)$ functions over E, you use $| u_k(x) |_E \leq M_k$ in place of $u_k \leq M_k$.

Example 1: [184] $| u_k |_E = |\sin(kx)/k^2|_E \leq 1/k^2 = M_k$, so series is uniformly convergent for all x.

We have not heard the word "continuous" for a while now, and here it makes a stage appearance:

Theorem 17: [184] Suppose $f_n(x) \to F(x)$ uniformly on E. If the $f_n(x)$ are *continuous*, so is F.

The δ ε proof requires *uniform* convergence in order to arrive at a δ that works over all p in E.
Note that Thm 17 concerns a sequence of functions, not a series. But as usual we can think of the sequence of partial sum functions which is associated with a series. Restate the above

Theorem 17A: Suppose $\Sigma_{k=1}^{n} u_k(x) \to F(x)$ uniformly on E. If the $u_k(x)$ are continuous, so is F(x).
[Corollary 1 top of page 185]

Pl Mini Theorem. A discontinuous function in effect has an infinite slope at any discontinuity since the function "jumps". In the ε,δ world, there is no δ >0 so $| f(x) - f(y)| < \varepsilon$ for $|x-y| < \delta$. Let f(x) be some series $\Sigma_{k=0}^{\infty} u_k(x)$ having such a discontinuity. Can this series be uniformly convergent on a set E which contains a discontinuous point? Consider $|\Sigma_{k=0}^{n} u_k(x) - F |_E < \varepsilon$. You need do find a δ >0 for *all* x in interval E, but if x = 0 is a discontinuity with infinite slope, you cannot get such a δ > 0 because you must have δ = 0. I guess I am hazy on this but OK for now.

My little theorem says if a series is uniformly convergent over set E, then its limit is continuous over E. Thus, if the limit is discontinuous somewhere in E, then you cannot have uniform convergence on E. I guess this is really what Theorem 17A says.

Example 1A visit on page 185. This is $u_k = x(1-x)^k$ as usual on the 0,1 region. Theorem 17 A concludes that, since x = 0 is a place where the limiting function F is discontinuous, the series Σu_k cannot be uniformly continuous on [0,1]. You might wonder if it is uniformly continuous on (0,1) since you have removed the problem point(s). The answer is NO from following theorem:

Theorem 18: [185] Let E be closure of S which is open, like (0,1). Suppose f_n converges uniformly on S, and the f_n are continuous on E. Then f_n converges uniformly on E.

This says that you cannot have uniform convergence on the interior of a set *without also having* uniform convergence on the boundary. Thus, in our example above, since we do not have uniform convergence at a certain boundary point, we know we cannot have uniform convergence inside $0 < x < 1$.

Corollary 2 [185 bottom]. This is a bit confusing in its relation to Theorem 18. I think this is the idea. The open set S (my notation) is $c < x < \infty$ let's say. We assume series Σu_n converges uniformly on this set S. We argue that the terms $u_n(x)$ are continuous at $x = \infty$ since we assume each term has a finite limit so that $u_n(\infty) = b_n$. Theorem 18 then says Σu_n converges uniformly on the closure of S which includes the point $x = \infty$ and in fact converges to Σb_n. The claim is that you can then do termwise evaluation of the series at $x = \infty$. They have a proof, but I must admit, it leaks for me. It is just not clean.

Example 2A. [186] This is Example 2 from p 180, where we have $F(x) = \Sigma_{n=1}^{\infty} nx^2/(n^3+x^3)$. We showed on page 180 that F(x) converges for any [a,b] but diverges for $x = \infty$. Thus, F(x) does not converge on the entire real axis, and therefore it is not uniformly convergent on set E where E means $a \le x < \infty$, where this means the "whole unbounded interval".

Note: It is now 6:30PM 1/15/15 and I have finally crawled back up to this point in the notes where I was on my first push. That is, I had to do a second "review push" to feel "comfortable" with all these million things. As part of that review push I created meta notes for Chapters 1,2,3.

Theorem 19. [186] (**order interchange for sequence and integral in E^2**). The domain D of the integration is closed and bounded (compact) in E^2. The sequence of functions $\{f_n(x)\}$ converges uniformly on D to some F(x), so $\lim_{n\to\infty} f_n(x) = F(x)$.

Then: $$\lim_{n\to\infty} \left[\int_D f_n(x,y)dA \right] = \int_D \left[\lim_{n\to\infty} f_n(x,y) \right] dA \quad \text{which is} \int_D F(x)\, dA$$

This says you can take the $n\to\infty$ right through the integral if premise conditions are met. In other words, you can interchange the order of "sequence limit" and "integration" which itself is a certain limit.

Now take $f_n(x)$ in the above theorem to be the partial sum $f_n(x) = \Sigma_{k=1}^{n} u_k(x)$. We assume that this series "converges uniformly on D" and that each of the $u_k(x)$ is continuous (so that $f_n(x)$ is).

$$\lim_{n\to\infty} \left[\int_D \Sigma_{k=1}^{n} u_k(x)dA \right] = \int_D \left[\lim_{n\to\infty} \Sigma_{k=1}^{n} u_k(x) \right] dA$$

On the left we can extract the finite sum $\Sigma_{k=1}^{n}$ from the integral to get

$$\lim_{n\to\infty} \Sigma_{k=1}^{n} \left[\int_D u_k(x)dA \right] = \int_D \left[\lim_{n\to\infty} \Sigma_{k=1}^{n} u_k(x) \right] dA$$

and we normally write this as

$$\Sigma_{k=1}^{\infty} \left[\int_D u_k(x)dA \right] = \int_D \left[\Sigma_{k=1}^{\infty} u_k(x) \right] dA \quad \text{which is } \int_D F(x)\, dA$$

term by term integration

I have just proved:

Corollary [186] : (**order interchange for series and integral in E^2**) If $u_k(x)$ are continuous on D and if $\Sigma u_k(x) \to F(x)$ converges uniformly on D, then if you need to integrate F(x) over D, you may do so "term by term".

For me, this is a very important theorem. I just dimly recall it from the Buck math course. The Bucks have invested a lot of effort to get the reader to this theorem!

Example 1 [187]. Bucks show that $\Sigma_n(-1)^n t^n$ converges uniformly on [0,1] to 1/(1+t) and of course the terms are all continuous in this range. So

$$1/(1+t) = F(t) = \Sigma_n(-1)^n t^n$$

Integrate term by term over D = [0,x] to get

$$\ln(1+x) = x - x^2/2 + x^2/3 + \ldots \quad = \text{the famous series}$$

$$\ln(2) = 1 - 1/2 + 1/3 \ldots. \text{ etc}$$

Regarding term-by-term integration of a uniformly convergent series (Theorem 19 above), it is possible to remove the requirement of "uniform" provided certain other premises are met. Thus:

Theorem 20. [188] (**Arzela-Osgood-Lebesgue**) Assume $f_n(x) \to F(x)$ *pointwise* on bounded D and f_n and F are integrable on D. Suppose *also* that $\| f_n(x) \|_D \leq M$ (a constant) for all n. Then

$$\lim_{n\to\infty} \left[\int_D f_n(x) \right] = \int_D \left[\lim_{n\to\infty} f_n(x) \right]$$ perhaps D is in E^n ?

and you can do order interchange between sequence limit and integration.

If f_n is a series partial sum, then I guess there is a corollary:

$$\lim_{n\to\infty} \left[\int_D \Sigma_{k=1}^{n} a_k(x) \right] = \int_D \left[\lim_{n\to\infty} \Sigma_{k=1}^{n} a_k(x) \right]$$

or

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n}\int_D a_k(x)\right] = \int_D\left[\lim_{n\to\infty}\sum_{k=1}^{n} a_k(x)\right]$$

or

$$\sum_{k=1}^{\infty}\left[\int_D a_n(x)\right] = \int_D\left[\sum_{k=1}^{\infty} a_n(x)\right] = \int_D F(x)$$

and again we are allowed to do term-by-term integration. Bucks claim the proof is a nightmare.

This is usually now called **Arzela's Bounded Convergence Theorem**. He did this in 1885 and Osgood rediscovered it in 1897. We now back up

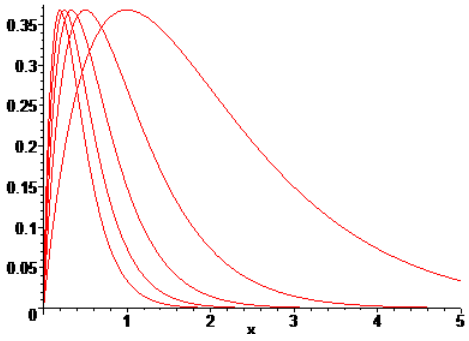
Example 3B: [187] Reconsider $f_n(x) = n^2xe^{-nx}$ which is Ex 3 on page 181. There it was shown that the area under the curve on $[0,\infty]$ is 1 for any n, so $\lim_{n\to\infty}\int f_n = \lim_{n\to\infty} 1 = 1$, while $\int \lim f_n = \int 0 = 0$, so you cannot interchange order of integration and $n\to\infty$ limit. Then later on page 182 when we make the plots, we see that the max of the curve $f_n(x)$ is n/e while $\lim_{n\to\infty} f_n(x) = 0$. Thus $|f_n(x) - 0|$ cannot be $< \varepsilon$ for n>N unless you make it be N(n). This is exactly the definition of a lack of uniform convergence. So this example is a sort of prototype function for non-uniform convergence but yes pointwise convergence.

Example 4: [187] Consider $f_n(x) = nxe^{-nx}$, very similar to the above. For this function, the peaks are < 1/e for all n and again $f_n \to 0$ on $[0,\infty]$. Here are some plots showing the 1/e peaks.

```
f := n*x*exp(-n*x);
```

$$f := n\,x\,e^{(-n\,x)}$$

```
plot( {seq(f,n=1..5)},x=0..5, color = red);
```



Even though the peaks are finite, w always have $|f_n(x) - 0| \approx 1/e$ and you cannot drive this to $< \varepsilon$, so this function is also pointwise convergent but not uniformly convergent on $[0,\infty]$ as $n\to\infty$.

The difference between these two examples is this: In the first, it is non-uniformly convergent and you cannot interchange ordering between integration and limit $n\to\infty$, Doing that gives 0 = 1. But in the second case you can do this interchange! Either order gives 0. You know that $f_n \to 0$, and you can see that the integral area $\int f_n \to 0$, so both orders give the same result, even though f_n is not uniformly convergent. This is then an example of the Arzela-Osgood theorem where M = 1/e shown above.

Example 1 [188]. Here $f_n = \exp(-nx^2)\, g(x)$ where g(x) is continuous on [-1,1] and therefore bounded on that interval (continuity implies cannot have points of infinity!) Note that $f_n \to 0$ for $x > 0$ and $f_n \to g(0)$ at $x = 0$. Assuming $g(0) \neq 0$, f_n is NOT uniformly continuous on [-1,1]. But, f_n is bounded by some M (I guess that would be M = max g(x)) and so Arzela applies and you can interchange order.

Differentiation ordering.

Theorem 21 [188] (**Series/Derivative Order Interchange**)

Suppose $\Sigma u_n(x) \to F(x)$ pointwise on [a,b].

Suppose $u'_n(x)$ exists and is continuous on [a,b].

Suppose $\Sigma u_n'(x) \to G(x)$ uniformly on [a,b].

Then G(x) = F'(x). That is to say, $\lim_{n\to\infty} [\Sigma_{k=1}^{n} \partial_x u_n(x)] = \partial_x [\lim_{n\to\infty} \Sigma_{k=1}^{n} u_n(x)]$

$$G(x) = \partial_x [F(x)]$$

So under the premises of this theorem, you can interchange order of $\lim_{n\to\infty}$ and ∂_x for a series.

Example 2 : [188] $f_n(x) = \exp(-n^2 x)$ and consider $\Sigma_n f_n(x)$. Since f_n is continuous and monotone decreasing as a function of n for any fixed x, this series has the same convergence nature as $\int \exp(-n^2 x)$ which is a gaussian and which we know converges. This is from Thm 7 [163] above. Thus, for each x in $(0,\infty]$ $\Sigma_n f_n(x)$ converges, but it is not obvious to what function it converges. This ∞ series does not appear in GR7 [aside: GR7 list off the various series convergence tests we are studying in this chapter.]

Side Problem: Show that $\Sigma_n f'_n(x)$ is uniformly convergent on $(0,\infty)$. We know $f'_n(x) = -n^2 \exp(-n^2 x)$, ' so restate the problem:

Show that $\Sigma_n g_n(x)$ is uniformly convergent on $x < 0$ where $g_n(x) = -n^2 \exp(-n^2 x)$. Well, we can see that as $n\to\infty$ we have $g_n(x) \to 0$, as in most of our previous examples (but this is a new example). Let $n^2 = m$ so write this as - m exp(-mx). But this is Example 4 [187] which I show plots for above. So Arzela applies and we could integrate term by term. But that is NOT what we seek here. Here we really want to show that $\Sigma_n g_n(x)$ is uniformly convergent on $x < 0$. How do I even know it is pointwise convergent? I guess I would appeal to the same integral comparison where I know that $\int dn\, n^2 \exp(-xn^2)$ converges for $x > 0$. So this tells us that $\Sigma_n g_n(x) \to G(x)$ where G(x) exists. So that is why we are pointwise convergent.

Contradiction with Thm 18 p 185. Suppose $\Sigma_n g_n(x)$ were uniformly convergent on $0 < x$. That theorem then implies that $\Sigma_n g_n(x)$ is uniformly convergent on $[0,\infty]$ where we close the domain. But at $x = 0$ I can see that $g_n(x) \to -n^2$ as $n\to\infty$ and the series is divergent. But for $x > 0$ I get $g_n(x) \to 0$. So how can our series possibly be uniformly convergent at x = 0 as required by Them 18?

OK, I think that is correct and I have misinterpreted the Bucks claim. For a given δ, they are saying that the series $\Sigma_n g_n(x)$ is uniformly convergent on $x > \delta$. That is a different story. Thus we have G(x) existing as the limit for $x > \delta$ and we have uniform convergence. What about the full ε δ argument? We have to show that for any ε we can get $| \Sigma_n g_n(x) - G(x) | < \varepsilon$ if we make δ small enough. Well, I guess I won't take

the time to drive this through, Bucks left me hanging on page 189 with this example. Their point is that since you can take $\delta = .00001$ if you like, you "in effect" have uniform convergence on $(0,\infty)$ and somehow, since $\Sigma_n g_n(x) = \Sigma_n f'_n(x)$, we meet the conditions of Theorem 21 [188] and we are then allowed to do term by term differentiation. And we can then continue to second and third derivative and finally get to the kth derivative which gives result A on page 189.

Footnote: We see that $|n^2 \exp(-n^2x)| \leq n^2 \exp(-n^2\delta)$ for $x > \delta$. By the integral comparison test, we know that $\Sigma\, n^2 \exp(-an^2)$ converges (to something). Then by the comparison test, we know that $n^2 \exp(-n^2x)$ converges for $x > \delta$.

Example 1 : [189] (**Euler constant**) I get down to this step B

$$\gamma = \lim_{N\to\infty} [\, \Sigma_{n=1}^{N} \{\, 1/n - \log(n+1) + \log(n) \,\} \,]$$

Consider:

$$\Sigma_{n=1}^{N} \{ - \log(n+1) + \log(n) \} = - \log(2) + \log(1) - \log(3) + \log(2) - \log(4) + \log(3) \ldots$$

$$= \log(1) - \log(N+1) = 0 - \log(N+1) = - \log(N+1)$$

where we get pairwise cancellation of terms. Since a finite sum, no convergence issue. Then we have

$$\gamma = \lim_{N\to\infty} [\, \Sigma_{n=1}^{N} (1/n) \,] + \lim_{N\to\infty} [-\log(N+1) \,]$$

$$= \lim_{N\to\infty} \{ \Sigma_{n=1}^{N} (1/n) - \log(N+1) \}$$

We can rewrite this with no change as

$$\gamma = \lim_{N\to\infty} \{ \Sigma_{n=1}^{N} (1/n) - \log(N) \}$$

or

$$\lim_{N\to\infty} \{ \Sigma_{n=1}^{N} (1/n) - \log(N) - \gamma \} = 0$$

or

$$\lim_{N\to\infty} \{ \sigma_N \} = 0 \qquad \sigma_N \equiv \Sigma_{n=1}^{N} (1/n) - \log(N) - \gamma$$

Therefore

$$\Sigma_{n=1}^{N} (1/n) = \log(N) + \gamma + \sigma_N \qquad \text{where } \sigma_N \to 0 \qquad \text{// Buck result C}$$

and then

$$\lim_{N\to\infty} [\, \Sigma_{n=1}^{N} (1/n) - \log(N)] = \gamma$$

Comments: We already knew that $\Sigma_n(1/n)$ is log divergent, maybe from integral test. This shows that for large N, the sum is almost exactly equal to log(N) ! You miss by a small constant $\gamma = .57721$ which is the famous Euler constant. However, they don't show here how you compute the value of this constant!

This example begins by writing a certain series $\Sigma_n f_n(x)$ and showing it is uniformly convergent on [0,1] and that allows termwise integration to get the integral of the limiting F(x). The termwise sum of these integrals is what is defined as the γ constant.

Example 2 : [189] A new series f_n which they show (I agree) is uniformly convergent for $x > \delta$ and the series adds up to F(x). Look at Cor 2 [185 bot] noted above. It says you are allowed to add up the limits to get the limit of F(x), sort of term by term addition. But $f_n(x) = 1/(1+n^2x^2)$ so at $x \to \infty$ this term by term sum is a sum of all 0's, so we get $F(\infty) = 0$. Bucks then show what seems obvious, that $F(0) = \infty$. They prove this is in fact the case, and the proof depends on the fact that all terms are *positive*.

So this example has $F(\infty) = 0$ and $F(0) = \infty$. The sum is doable in Maple and I get

$$\Sigma_{n=1}^{\infty}\ 1/(1+n^2x^2) \ = \ \frac{1}{2}\,\frac{\sqrt{-x}\,(\Psi(1-\sqrt{-x})-\Psi(1+\sqrt{-x}))}{x}$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z) = \partial_z \ln \Gamma(z)$ = the digamma function. I can have Maple compute this sum and then plot the result

```
sum( 1/(1+(n*x)^2),n=1..infinity): factor(%);
```

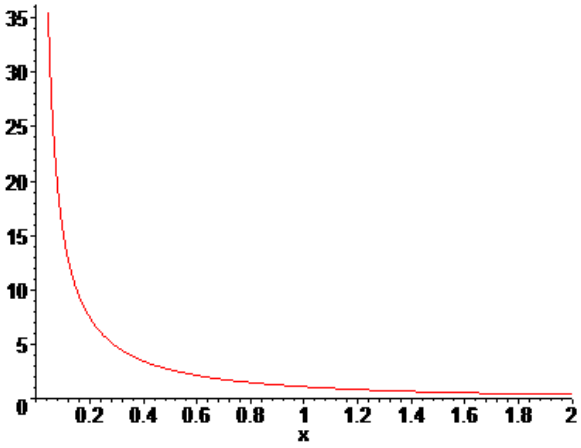
$$\frac{1}{2}\,\frac{I\left(\Psi\left(-\frac{-x+I}{x}\right)-\Psi\left(\frac{x+I}{x}\right)\right)}{x}$$

The imaginary parts cancel I guess so this is real, and we can plot it like so

```
f := unapply(%,x);
```

$$f := x \to \frac{1}{2}\,\frac{I\left(\Psi\left(-\frac{-x+I}{x}\right)-\Psi\left(\frac{x+I}{x}\right)\right)}{x}$$

```
plot(f(x),x=0..2);
```



and this confirms that $F(\infty) = 0$ and $F(0) = \infty$ (more or less).

Example 3 : [190] Now $f_n(x) = [\ 1 + n(1-x)]x^n$ on (-1,1). We know that Σx^n converges for $|x| < 1$, and I guess $\Sigma n x^n$ does as well which is $\Sigma n\ e^{-n\log(x)} = \Sigma n\ e^{-na}$ = converges by integral test. Now look at the point x = 1. Clearly $f_n(1) = 1$ so 1 + 1 + 1...should diverge. But this is a tricky telescoping series and it turns out that $\Sigma_n f_n(x) = 1$ for all $|x| < 1$, so at x = 1 and x = -1 you also get 1, not ∞. The terms are not all positive in this example over the range of x, and that allows the telescoping of the series.

Question: On line A we have order interchange between $\lim_{x\to\infty}$ and $\Sigma_{n=1}^{\infty}$. This is order interchange for a series and the limit of a parameter. Have the Bucks treated this case? Here are the theorems we have had relating to interchange:

Theorem 21 = deals with interchange of series and derivative
Theorem 20 = deals with interchange of sequence and integral
Corollary [186] = deals with interchange for series and integral
Theorem 19 = deals with interchange for sequence and integral
Corollary 2 [185] = deals with interchange of series and $\lim_{x\to\infty}$ of parameter

OK, it is this last item which is invoked in Ex 4 below.

Example 4 : [190] Instead of $f_n = 1/(1+n^2x^2) \to F(x)$ as Example 2, here we consider
$f_n = x^2/(1+n^2x^2) \to x^2F(x) = G(x)$. They show that $G(\infty) = \Sigma(1/n^2)$ = converges = $\pi^2/6$ says GR7. It is this fact that requires line A which is based on Cor 2 [185]. I think they are using the same Cor in some form to argue that G(0) = 0. Bucks just seem to be interested in the limits of series when the parameter takes values x = 0 and $x = \infty$. Seems OK.

Ah, now here is some distinction. In Example 2 they showed that $\Sigma 1/(1+n^2x^2)$ is uniformly convergent only for $x > \delta$ for any $\delta > 0$. Here they claim that $\Sigma x^2/(1+n^2x^2)$ is uniformly convergent on the entire real axis for x due to the inequality. So having a power of x in the summand makes a difference in the nature of the convergence!

Example 5 : [190] Now we do $\Sigma x/(1+n^2x^2)$ with yet another power of x inside the sum. Bucks claim right off the bat that this series converges uniformly for $x \geq \delta$, as in some other examples

Pause to prove this claim. I can see that $f_n = x/(1+n^2x^2) \leq x/(1+n^2\delta^2)$ for fixed $\delta > 0$, but this does. But this does not seem to permit the comparison test as used in Ex 2 p 188 or Ex 2 p 189, because x is sitting there on the right side and RHS gets very large when x gets large. So we need some other way to argue that this sum of this f_n is uniformly convergent on $x > \delta$. Why is it even pointwise convergent for some $x > 0$? It is true that $f_n \to 0$, so it has a chance.

Now I have to scan all the pointwise convergence theorems and look for one that works here: How about Cor 172 the No-Bounce Ratio Test? We have

$f_{n+1}/\ f_n = [\ 1+x^2n^2]/\ [\ 1+x^2(n+1)^2] \quad < 1$ for any $x > 0$.

OK, so this says we at least have pointwise convergence. If we have convergence by this test for ALL x in the set $[\delta,\infty)$, we still have to do the ε and δ stuff to show that δ not a function of ε. Perhaps we can show that we can find some $\delta > 0$ such that $\text{Tail}\{ \Sigma x/(1+n^2x^2)\} < \varepsilon$ for any ε.

OK, I don't get it, sorry. I do not know how to prove their claim, so I will just accept it for now and move to the next nanostep. Maybe the argument is that we know $1/(1+n^2x^2)$ converges uniformly on $x > \delta$ according to p 189 Ex 2, and just multiplying by x does not changes this fact.

Moving on, they now ask about the part of the range $[0,\delta]$. If you can take limit through the sum (as I think you can for uniform), you could get $G(x) = 0$ due to that x on the top in $G(x) = \Sigma x/(1+n^2x^2)$. They show that in fact $G(0) = \pi/2$ and not 0, so they conclude we do NOT have uniform convergence of the series over any range that includes $x = 0$.

These examples are all a bit vague to me in their purpose.

The continuous function that is nowhere differentiable. [191] The function is constructed as an infinite series $\Sigma_j u_j(x)$ where $u_j(x) = 10^{-j} K(10^j x)$ where K is the simple sawtooth function shown in Fig 4-4 on page 192. The Bucks discuss a proof that it the infinite series represents a continuous yet nowhere differentiable function. The idea is clear graphically and I could follow all their proof if I wanted to take the time. The continuity is intuitively clear because you are just adding continuous sawtooth functions of smaller amplitude and higher frequency. But at any point $x = b$ you pick, you are going to "be at the apex of some sawtooth" so you won't be differentiable at $x = b$! If you consider just a finite series of terms N, you can find lots of OK points, but as $N \to \infty$, the set of available differentiable points gets less and less and eventually goes away.

I don't really see what this special series has to do with what went before. It is of course an interesting fact and I found a thesis listing off functions like this historically. I think the one here is Van Der Waerden's function of 1930. The oldest known one was found in 1830 but not published I think.

Tietze's Extension Theorem [193]

Another grab bag item I guess. You have E closed and bounded in E^n and f(p) [real] is continuous and bounded on E (so $|f(p)| \le M$). Then there is a way to extend f(p) to all of E^n (call it F(p)) such that $F(p) = f(p)$ on E *and* such that F(p) is continuous and bounded on all of E^n.

The Bucks approach the proof by explicitly constructing a function F(p) that works. In 1D we know graphically how to do this, and for certain cases in 2D (like a circle) we know as well. In general, it is non-trivial to show this, hence their proof. The construction depends on the metric of E^n d(p,C) which is the distance between a point p and a set C in E^n. The distance is defined as 0 if p is inside C, and it is the distance of closest approach for p outside C. That is, it is the distance to the closest point in C from p. This thing is defined as $\varphi(p) = d(p,C) = \min | p-c |$ for c ranging over C. The lemma shows $\varphi(p)$ as a function is continuous and $\varphi(p) > 0$ for p outside of C. C is regarded as closed to include its boundary.

This is a Lemma on page 193. The next step is to divide up the initial given region E into three parts A,B and C based on the size of f(p) in relation to -M and M which it must lie between in value. Then these regions define things like d(p,A), and $F_1(p)$ is then constructed as in p 194 item a.

In some manner they similarly define $F_2(p)$ and so on so we end up with a sequence $F_n(p)$ of functions. Then the solution constructed function is $F(p) = \Sigma_n F_n(p)$. They show that this F(p) does in fact match f(p) on E, and that it is continuous on all of E^n , and that it has the same bound as f(p).

I have not parsed this in detail, but it would certainly be interesting to plot the resulting F(p) in Maple and see how it tapers off outside region E (I suspect that is what it does) for some simple region E shapes and some simple f(p) functions.

There is something I do like about this proof and the fact that they threw it in. The problem is to find a handle on the solution function, how do you build it? It turns out to be an infinite series, and that of course has been the topic of this subsection.

It is now 1.16.14 at 6 PM.

4.3 Power Series **[196]**

This is a relatively easy 9 pages (at last). Here are the main ideas:

Theorem 22: [196] Power series $\Sigma a_n x^n$ has some R **radius of convergence** so $|x| < R$ gives convergence. You can find R from either the ratio test or root test, as shown top p 197. Could find R = 0 or $R=\infty$.

You could restate this replacing x by g(x) so $|g(x)| < R$ gives you some convergence domain in x.

Corollary [197] You can generalize trivially to $\Sigma a_n(x-c)^n$.

Theorem 23. [197] The power series $\Sigma a_n x^n$ with R converges *uniformly* on $|x| \le b < R$ any b.

Theorem 24. [198] For $|x| < R$, you can differentiate $f(x) = \Sigma a_n x^n$ term by term to get f'(x).
The series f'(x) has the same R as f(x), and the same uniform convergence region.

Corollary 1 [198] For $f(x) = \Sigma a_n(x-c)^n$, you can identify $a_n = f^{(n)}(c)/n!$.

You can keep diffing using Thm 24 and then just set x = c in the result to prove this corollary.

Corollary 2 [198] $\Sigma a_n(x-c)^n = \Sigma b_n(x-c)^n$ in ball near x, then $a_n = b_n$.

This says that a function analytic near x = c can have only one power series there, and it is the Taylor series we studied earlier.

Example 1: $f(x) = \Sigma_n x^n/n$. Bucks find convergence range for f(x), f'(x) and f''(x) with the main attention to the points x = 1 and x = -1. Both pointwise and uniform convergence are considered for each.

Theorem 25 [199] (**Abel**) If $f(x) = \Sigma a_n x^n$ has radius of convergence R, and if f(x) converges at x = R, then f(x) is *uniformly* convergent on [0,R]. Same idea if you do $R \to -R$.

Corollary [200]. If $f(x) = \Sigma a_n x^n$ has radius of convergence R, and if f(x) converges at x = R to some value S, then you must get this same value S if you take the $\lim_{x \to R}[\Sigma a_n x^n]$.

I think this is saying that f(x) is continuous from below at x = R. Recall

Theorem 17: [184] Suppose $f_n(x) \rightarrow F(x)$ uniformly on E. If the $f_n(x)$ are *continuous*, so is F.

So this theorem would then provide a proof of the Corollary, given that Thm 25 gives uniform.

We now have a lot of **example series**. The first three on page 200 come from f(x) = 1/(1-x) and its first two derivatives. The fourth comes from integrating the first term by term, which is allowed since we have uniform convergence, as was shown in Corollary [186] to Theorem 19 [186]. Then there are four more samples on top page 201. One comes from integration. One leads to arctan series and a series for π, another leads to the ln(x) series.

Example 2 after those is the **expo series**. [201,202] If you define E(x) by the series, then you can show (using my Cauchy Product result earlier) that E(x)E(y) = E(x+y). If you then define e ≡ E(1) = some number, then you know that E(2) = E(1)E(1) = e * e = e^2 and then E(n) = e^n . You then analytically continue the fact that e^n = series off the integers to general x, though Bucks are quiet about how such continuation might work. It certain seems reasonable. So this is a great way to introduce the exponential function from scratch, as it were. You could add the series and thus compute E(1) to find out what the number is.

Final examples. Replace x by $-x^2$ in the expo series to get series for $\exp(-x^2)$ as shown where integral goes only up to finite t. But cannot take $t \rightarrow \infty$ so this particular series not useful for that purpose. But we already know the infinite integral from page 150.

I perused the Exercises.

4.4 Improper Integrals with a Parameter **[204]**

I have to provide lots of preliminary explanatory notes here just to get out of the starting gate. In the notes that follow, I define three distinct meanings for the phrase "uniform convergence" :

(uniform convergence)^p = involves convergence of a sequence of *points* $f(p_n)$ in E^m
where $p_n \rightarrow p$ in $E \subset E^n$ and $f(p_n) \rightarrow f(p)$ in E^m with the E^m norm
where same δ works for all p in E, associated with uniform continuity

(uniform convergence)^f = involves convergence of a sequence of *functions* $f_n(p)$ in $\mathcal{L}_{max}$
where $f_n \rightarrow f$ with the $\mathcal{L}_{max}$ norm over set $E \subset E^n$

(uniform convergence)^{p2} = involves convergence of a sequence of *points* $f(t_n,p)$ in E^m
where $t_n \rightarrow t_0$ in for all t_0 in $T \subset E^k$ and for all p in $E \subset E^n$,
and $f(t_n,p) \rightarrow f(t_0,p)$ in E^m with the E^m norm

In my notation p2, the p means we are talking about a sequence of points, not functions, and the 2 means that the function has 2 variables and is uniform in two senses at once. We have $f : E^k \times E^n \rightarrow E^m$. It is this

last concept that is introduced in this section. Here following are my notes leading to the above three meanings:

(a) A Review of Continuity and Convergence

In Section 2.3 Bucks discuss continuity and uniform continuity for f: $E^n \rightarrow E^m$. The two notions are very closely related and we say

For any $\varepsilon > 0$ and for a specific p_0 in D which lies in E^n,
we can find $\delta(p_0) > 0$ such that $| f(p) - f(p_0)| < \varepsilon$ when $|p-p_0| < \delta$
=> f(p) is continuous at the point p_0

For any $\varepsilon > 0$ and *all* p_0 in D which lies in E^n,
we can find $\delta > 0$ such that $| f(p) - f(p_0)| < \varepsilon$ when $|p-p_0| < \delta$
=> f(p) is uniformly continuous on E

In both the above we have the idea that $\lim_{p \rightarrow p0} f(p) = f(p_0)$. In Theorem 1 [57] Bucks showed that if we have this kind of continuity at some point p_0, then for any point sequence $p_n \rightarrow p_0$ we have $f(p_n) \rightarrow f(p_0)$. When the word "sequence" appears in the discussion, so does the word "convergence". In the above, we are saying that if p_n converges to p_0 in the domain, then $f(p_n)$ converges to $f(p_0)$ in the range. The phrase is that "continuity preserves convergence". This convergence is $f(p_n) \rightarrow f(p_0)$ in both cases stated above, the difference is in the nature of the convergence with respect to whether δ depends on p_0 or not. It is true that this type of convergence does involve *functions* (like f).

Notation anomaly: The symbol p_0 is here overloaded since it represents our point of interest, but it also represents the first point in the sequence $\{p_n\}$. These are completely different meanings, so one must just keep this in mind.

Now, in the above we have $p_n \rightarrow p_0$ being a "sequence of points in E^n"
f(p) is a "function" such that f: $E^n \rightarrow E^m$.

E^n is of course a particular Hilbert Space in the language of Stakgold vol I p 110.

Stakgold discusses another Hilbert Space he calls $\mathcal{L}_2^{(r)}(a,b)$ and $\mathcal{L}_2^{(c)}(a,b)$ where are the spaces of real or complex valued functions on the interval (a,b) which are L_2 integrable over that interval. We could generalize this to talk about $\mathcal{L}_2^{(m)}(D)$ which would be functions taking values in E^m which are defined on a domain D in E^m, which again are square integrable over D. Another choice is $\mathcal{L}_1^{(m)}(D)$ which is functions for which $\int_D d^m x \, \| f(x) \|$ is integrable and then $\| f \|_1 = \int_D d^m x \, \| f(x) \|$, whereas in the previous case we had $\| f \|_2 = (\int_D d^m x \, \| f(x) \|^2)^{1/2}$ and in general $\| f \|_p = (\int_D d^m x \, \| f(x) \|^p)^{1/p}$. Note that E^2 and complex are not quite the same, we let that pass.

Another norm of interest is this one: $\| f \|_\infty = \max(\|f(x)\|)$ on D. I might call this "the max norm" but wiki seems to like the "infinity norm" because it is like the 1 and 2 norms except $p \to \infty$ (though I have not proven that fact). Officially it is the "supremum" norm, but I will call it the max norm.

This max norm is the one the Bucks use for functions, and they write it as $\| f \|_E$ over set E.

Back in Section 4.2 we have the following new concepts. The players are now functions $f_n(p)$ which form a sequence of functions $\{f_n(p)\}$ which perhaps converge to some function f(p)

For any $\varepsilon > 0$ and for a specific p in D which lies in E^n,

we can find $\delta(p) > 0$ such that $| f_n(p) - f(p)|_{En} < \varepsilon$ when $n > N$. // n is overloaded!

=> $f_n(p)$ converges pointwise to f(p) on D

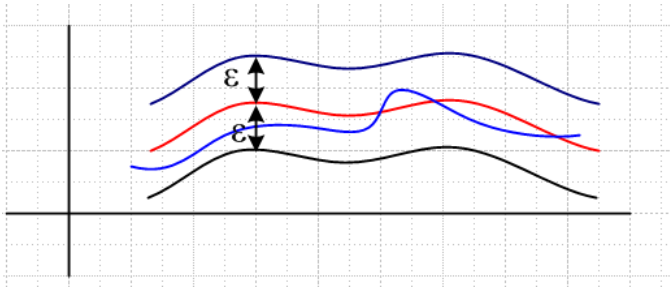
Notice that the norm used here is the geometric norm of E^n , it is not any of the fancy function space norms listed above. As we have learned, this pointwise convergence is not very "interesting" because it doesn't lead to nice facts like order interchange of things. What is really much more interesting is the following notion:

For any $\varepsilon > 0$

we can find $\delta > 0$ such that $\| f_n - f \|_\infty < \varepsilon$ when $n > N$.

=> sequence f_n converges uniformly to f over D

Now we are for the first time using this max norm, and that norm knows about the nature of $f_n(p) - f(p)$ for all values of p in D, because it needs to know that in order to compute the "max". So it seems best not to display the function argument p, since the norm here is for the function itself! The main idea here is that if you have $\| f_1 - f_2 \|_\infty$ very small, then the two functions are close together for ALL values of their arguments. If $\| f_1 - f_2 \|_\infty < \varepsilon$ then each function lies within the 2ε band of the other function:



Recap. In the above, we have discussed two totally different notions of convergence: We have first a convergence of points p_n or $f(p_n)$ which involves the single function f. And we have second a convergence of functions f_n to some function f. In more detail:

1. In the discussion of continuity and uniform continuity, we had $\lim_{p \to p0} f(p) = f(p_0)$ using the E^m norm saying that that $| f(p) - f(p_0)|_{Em} < \varepsilon$. We could then bring into the discussion a sequence of points p_n and we could then say $\lim_{pn \to p0} f(p_n) = f(p_0)$ and we then have convergence of the sequence $\{f(p_n)\}$ which is a sequence of points in E^m. This convergence is associated with the continuity of the function f(p). In the

case that the same δ works for all p_0, we had uniform continuity over E, and you *could* say that we have uniform convergence of $f(p_n) \rightarrow f(p_0)$ where $p_n \rightarrow p_0$ where p_0 is any point in E. In other words, we are saying that $f(p_n) \rightarrow f(p_0)$ converges to p_0 sort of "uniformly" for any p_0 in E. This use of the phrase "uniform convergence" is not used by the Bucks. You might call it (uniform convergence)^p since it involves a sequence of points. Note that, although f(p) is a function, here we are really talking about sequences of points, $\{ p_n \}$ and $\{ f(p_n) \}$. We are *not* talking sequences of function, only one function f is involved in the discussion.

2. In the above subsequent discussion of sequences of functions $f_n(p)$, we talked above about $f_n(p)$ converging pointwise to f(p) on D, which involved the E^n norm, and we talked about f_n converging uniformly to f using the max norm over D. We say that f_n converges uniformly to f over D, and this is the official definition of the phrase "uniform convergence". Maybe call this (uniform convergence)^f since it involves a sequence of functions.

(b) Application to a bystander parameter

Imagine that p lies in some set E, and that t lies in some set T.
Consider a function of both variables f(p,t).
Then consider this ε δ statement:

1. For any $\varepsilon > 0$ and for a specific p in E and for a specific t_0 in T,
we can find $\delta(t_0,p) > 0$ such that $| f(t,p) - f(t_0,p)| < \varepsilon$ when $|t-t_0| < \delta$.

I would describe this as "continuity at a point t_0 in T with a bystander parameter p in E".
I would also say that $f(t,p)$ "converges" to $f(t_0,p)$ as $t \rightarrow t_0$.
This means there are sequences t_n so that $f(t_n,p)$ "converges" to $f(t_0,p)$ as $t_n \rightarrow t_0$.

Our first "upgrade" of this concept would be:

2. For any $\varepsilon > 0$ and for *all* p in E and for a specific t_0 in T,
we can find $\delta(t_0) > 0$ such that $| f(t,p) - f(t_0,p)| < \varepsilon$ when $|t-t_0| < \delta$.

I would describe this as "continuity at a point t_0 in T which is uniform over E".
I would also say that $f(t_n,p)$ (converges uniformly)^p over E to $f(t_0,p)$ as $t_n \rightarrow t_0$.
Notice that there is only one function here, it is called f.

A different upgrade of item 1 might be this:

3. For any $\varepsilon > 0$ and for a specific p in E and for *all* t_0 in T,
we can find $\delta(p) > 0$ such that $| f(t,p) - f(t_0,p)| < \varepsilon$ when $|t-t_0| < \delta$.

I would describe this as "uniform continuity" over T with a bystander parameter p".
One could also say we have (uniform convergence)^p of $f(t_n,p) \rightarrow f(t_0,p)$ as $t_n \rightarrow t_0$.
The sequence involved is a sequence of *points* $\{t_n\}$ or $\{ f(t_n,p) \}$; only function is f.

Now in the next item we combine both the above upgrades:

4. For any $\varepsilon > 0$ and for *all* p in E and for *all* t_0 in T,
we can find $\delta > 0$ such that $|f(t,p) - f(t_0,p)|$ when $|t-t_0| < \delta$.

I would describe this as "uniform continuity over T which is also uniform over P".
This latter is a strange new use of the word "uniform". The statement involves being uniform in two different spaces at the same time, E and T. The "continuity" aspect only involves space T.
One could also say we have (uniform convergence)^{p} of $f(t_n,p) \to f(t_0,p)$ as $t_n \to t_0$.
The sequence involved is a sequence of *points* $\{t_n\}$ or $\{ f(t_n,p) \}$; only function is f.
Notice that we still have the E^m norm, not the max norm. We still have only one function.
So even this double upgrade thing has nothing to do with (uniform convergence)^{f} of a set of functions $f_n \to f$ in the Hilbert Space of the max norm.

Now can I connect what Bucks say on page 204 with one of the "upgrades" shown above? I first thought they were doing the complicated upgrade #3, but now I think it is upgrade #2 where we have ordinary continuity at the point t_0 and where we are uniform in the bystander space p in E.

Wrong hence blue font: On page 204 the Bucks use the above double-upgrade concept, which involves to different underlying sets E and T. They want to say that the result is uniform convergence of $f(t_n,p) \to f(t_0,p)$ since it is uniform in both the T and E sense. I will call this (uniform convergence)^{p2} to indicate that we are still using the point norm for $f(t_n,p) \to f(t_0,p)$, but that we are uniform in 2 spaces, T and E.

So this really is a brand new definition of the term "uniform convergence".

Comment: Earlier I was thinking of combining the real variables t and p into a single parameter $\mathbf{x} = (t,p)$ which lies in E^2 and trying to use the notion of uniform convergence in E^2 as a combined way to handle the "uniformity" of both variables. I now think this is not useful because the variables are really on a different footing. We have uniform *continuity* only with respect to t, and p has its own uniform sense which is different. Still I suppose you could say we have things being uniform over E^2 and more generally over $E^k \times E^m$.

(c) Resume at the start of Section 4.4

Let's look again at Upgrade #2 above:

2. For any $\varepsilon > 0$ and for *all* p in E and for a specific t_0 in T,
we can find $\delta(t_0) > 0$ such that $|f(t,p) - f(t_0,p)| < \varepsilon$ when $|t-t_0| < \delta$.

I would describe this as "continuity at a point t_0 in T which is uniform over E".
I would also say that $f(t_n,p)$ (converges uniformly)^{p} over E to $f(t_0,p)$ as $t_n \to t_0$.
Notice that there is only one function here, it is called f.

Restate this:

For any $\varepsilon > 0$ and for a for specific t_0 (in some space T in E^n)
one can find $\delta > 0$ such that
$| f(t,p) - f(t_0,p) |_{Em} < \varepsilon$ when $|t-t_0|_{En} < \delta$, where δ works for all p in E.

In the Buck definition of "uniform convergence with a parameter", instead of saying one can find a δ where $|t-t_0| < \delta$ (a neighborhood ball), they say one can find a neighborhood N around t_0. For me, the distinction is very minor, though N is allowed to be more general in shape than a ball. I can restate the above ε,δ definition of "uniform convergence with a parameter" using their N :

For any $\varepsilon > 0$ and for a for specific t_0 (in some space T in E^n)
one can find a neighborhood N around t_0 such that
$| f(t,p) - f(t_0,p) |_{Em} < \varepsilon$ when t lies in N, where N works for all p in E.

Notice that in the first definition, since $\delta > 0$, we never have $t = t_0$ in the above ε,δ process. So perhaps that is why they exclude the point $t = t_0$ in their specification of N for the second definition.

Example: Point p is now called x, it is the bystander and lies in some interval E. Consider

$f(t,x) = \sin(xt)/ [t(1+x^2)]$ and $t_0 = 0$ is the specific point of interest in T

$f(t_0,x) = f(0,x) = \lim_{t\to 0} \{\sin(xt)/ [t(1+x^2)]\} = x/(1+x^2) \equiv F(x)$

Our goal is to prove that the limit converges uniformly for all real x.

In order to show that $f(t,x) = \sin(xt)/ [t(1+x^2)]$ converges uniformly for all real x to $x/(1+x^2)$, we start this way. We are supposed to meet this requirement for any selected $\varepsilon > 0$,

$$| f(t,x) - F(x)| < \varepsilon$$

which can be written

$$\frac{|x|}{1+x^2} \left| \frac{\sin(xt)}{xt} - 1 \right| < \varepsilon \qquad *$$

We now arbitrarily assume that $|x| \le R$ where R is some positive number. I agree that one can certainly find a β such that

$$\left| \frac{\sin(xt)}{xt} - 1 \right| < \varepsilon_1$$

when $|xt| < \beta$. And I agree one can then define $\delta > 0$ by the relation $\delta = \beta/R$ where $|x| \le R$. Meanwhile, note that

$$|x| \le R \le R(1+x^2) \quad \Rightarrow \quad \frac{|x|}{1+x^2} \le R \qquad **$$

So at this point I agree that one can say, using * and **, that

$$\frac{|x|}{1+x^2} \left| \frac{\sin(xt)}{xt} - 1 \right| \le R\varepsilon_1 \equiv \varepsilon \qquad \text{when } |xt| < \beta = \delta R$$

Suppose we know that $|x| \le R$ and $|t| < \delta$. Then $|xt| = |x|\,|t| < R\delta$.
That is to say, if $a < b$ and $c < d$, then $ac < bd$. So choosing $|t| < \delta$ for this particular δ, we obtain the result that $|f(t,x) - F(x)| < \varepsilon$ for all $|x| \le R$. Thus, we have uniform convergence on $E = [-R,R]$.

Note that I disagree with the Buck presentation in that I use ε_1 above and then define $\varepsilon = R\varepsilon_1$. For any desired $\varepsilon > 0$ this gives some required $\varepsilon_1 > 0$ as long as $R > 0$. So we end up with

$$|f(t,x) - F(x)| < \varepsilon \qquad \text{when} \quad |x| \le R \text{ and } |t| < \delta \qquad \text{and } \delta = \beta/R \text{ and } \beta \text{ exists to get } \varepsilon_1 \text{ etc.}$$

Can R really be an arbitrary positive number here? I don't see why not, it is just a selected but finite parameter.

Continue onto 2nd part. I agree that $\left| \frac{\sin(xt)}{xt} \right| \le 1$ for any x and any t, since this is the sinc function. The triangle thing tells us that $\left| \frac{\sin(xt)}{xt} - 1 \right| \le \left| \frac{\sin(xt)}{xt} \right| + |-1|$ and then this is $\le 1 + 2 = 2$, so we get that $\left| \frac{\sin(xt)}{xt} - 1 \right| \le 2$. Then we have this new result,

$$|f(t,x) - F(x)| = \frac{|x|}{1+x^2} \left| \frac{\sin(xt)}{xt} - 1 \right| \le \frac{2|x|}{1+x^2} \quad \text{as claimed in C.}$$

This is driven small as x becomes large, so imagine there is some number R_0 such that
$\frac{2|x|}{1+x^2} < \varepsilon$ when $|x| \ge R_0$. Then the above line becomes

$$|f(t,x) - F(x)| < \varepsilon \quad \text{when } |x| \ge R_0 \text{ and for any t.} \qquad \text{// 2nd part}$$

and therefore, by restricting in t, we certainly also know that

$$|f(t,x) - F(x)| < \varepsilon \quad \text{when } |x| \ge R_0 \text{ and } |t| < \delta \text{ .} \qquad \text{// 2nd part, t restricted}$$

Now for our concluding third part : select $R = R_0$ in the first part, so we then have from the first part.

$$|f(t,x) - F(x)| < \varepsilon \quad \text{when } |x| \le R_0 \text{ and } |t| < \delta \text{ .} \qquad \text{// 1st part}$$

These last are the two inequalities which we combine to get

$| f(t,x) - F(x)| < \varepsilon$ when $|x|$ = anything and $|t| < \delta$

Thus, we have (finally) demonstrated that the limit of f(t,x) shown in p 204 F converges uniformly for all x on the real axis. QED for this Example.

Preliminary for next section p 205. Recall from above this definition of uniform convergence with parameter,

For any $\varepsilon > 0$ and for a for specific t_0 (in some space T in E^n)
one can find $\delta > 0$ such that
$| f(t,p) - f(t_0,p) |_{Em} < \varepsilon$ when $|t-t_0|_{En} < \delta$, where δ works for all p in E.

Now change notation replacing t by r, and t_0 by ∞, use E^1, replace f by g, and remove the sequence subscripts:

For any $\varepsilon > 0$ and for $r_0 = \infty$ (in E^1)
one can find $\delta > 0$ such that
$| g(r,p) - g(\infty,p) |_{E1} < \varepsilon$ when $|r-\infty|_{E1} < \delta$, where δ works for all p in E.

This doesn't really work because $|r-\infty|_{E1}$ is hazy, although the general idea is correct. For the special case that $t_0 = \infty$, let's try this alternate definition of "uniform continuity with a parameter p" :

For any $\varepsilon > 0$ and for $r_0 = \infty$ (in E^1)
one can find $R > 0$ such that
$| g(r,p) - g(\infty,p) |_{E1} < \varepsilon$ when $r > R$, where R works for all p in E.

This is similar to our definition of uniform convergence for a sequence (from above)

Def: [182A] A *sequence* $\{f_n(x)\}$ **converges uniformly** to F(x) on domain D if it converges for all x in D "with the same N". That is, there is an N such that $|| f_n(x) - F(x) || < \varepsilon$ for n> N *and* for all x in D. The point is that the same large integer N works for all x in D, you don't need N(x).

which I can rewrite in our current notation (uniform convergence of a sequence)

For any $\varepsilon > 0$ and for $n_0 = \infty$ (in E^1)
one can find $N > 0$ such that
$| g(n,p) - g(\infty,p) |_{E1} < \varepsilon$ when $n > N$, where N works for all p in E.

Now, if g(n,p) represents the partial sum of a series, $g(n,p) \equiv \Sigma_{k=1}^{n} a_n(p)$, then you can see that
$| g(n,p) - g(\infty,p) |_{E1} < \varepsilon$ says $| tail(n,p)|_{E1} < \varepsilon$ which in turn says $| \Sigma_{k=n}^{\infty} a_n(p) | < \varepsilon$ for $n > N$.

Now go back to the continuum case (n discrete, r continuous)

For any $\varepsilon > 0$ and for $r_0 = \infty$ (in E^1)

one can find R > 0 such that
| g(r,p) - g(∞,p) |$_{E1}$ < ε when r > R, where R works for all p in E.

Here if g(r,p) represents the finite "partial integral" to r, g(r,p) ≡ $\int_c^r$ du f(p,u) , then you can see that

| g(r,p) - g(∞,p) |$_{E1}$ < ε says | tail(r,p)|$_{E1}$ < ε which in turns says | $\int_r^\infty$ du f(p,u) | < ε for r > R.

So, here we have the notion of the uniform convergence of an integral with upper endpoint → ∞. It is completely analogous to the notion of the uniform convergence of a series, and we could have dealt with the definition on page 205 without even bothering with the opening subject of "uniform convergence with a parameter". But we have sort of fitted this second definition into the framework of the more general first definition. Note that p can be regarded as "the parameter" for either series or integral or general case.

Bucks claim upcoming theorems are very similar to what we have already seen before, proofs are almost the same.

Theorem 26 [206] (**Comparison Test**).

If:

- f(p,u) is continuous in p over E and I guess in u on [c,∞)
- |f(p,u)| ≤ f_1(u) for all p in E and for u in [c,∞)
- $\int_c^\infty$ du f_1(u) converges

Then:

$\int_c^\infty$ du f(p,u) "converges uniformly" for all p in E to some F(p)

I suppose we have then a finite upper bound Q on our integral of interest, value for all p in E:

$$\int_c^\infty \text{du } f(p,u) \le \int_c^\infty \text{du } |f(p,u)| \le \int_c^\infty \text{du } f_1(u) = Q$$

and therefore the integral cannot diverge.

Theorem 27 [206] (**Follow On**). Assume the premises shown above so F(p) ≡ $\int_c^\infty$ du f(p,u) converges uniformly on E. Then F(p) is continuous at p.

It is not claimed that F(p) is *uniformly* continuous on E. A simple proof is outlined. Notice that this is the first time in this section we have ever talked about continuity wrt to the bystander parameter p.

Corollary [206]. Assume F(p) ≡ $\int_c^\infty$ du f(p,u) converges uniformly on E. The above theorem then shows that $\lim_{p \to p_0} F(p) = F(p_0)$, which is the meaning of F(p) is continuous. This says that

$$\lim_{p\to p_0}\left[\int_c^\infty du\, f(p,u)\right] = \int_c^\infty du\, \{\lim_{p\to p_0}[f(p,u)]\} = \int_c^\infty du\, f(p_0,u)$$

and that in turn says you are allowed to move the limit through the integration *in this case*! As a sort of aside to this main point, the continuity of F(p) says that as $p_n \to p_0$ we have $F(p_n) \to F(p_0)$ [recall that convergence is preserved by continuity]. This is true for any sequence p_n , and for any sequence p_n it seems pretty obvious that p_0 is a "cluster point". Perhaps they are using this phrase to protect against the possible case that the p domain E has some discrete points and if p_0 is such a point, then you cannot smoothly approach it with a sequence $p_n \to p_0$ and p_0 in that case is not a cluster point.

Example 1: Uses p = x in E^1 and $f(x,u) = x^2/[1 + x^2u^2]$ and c = 1. Using the comparison test above, it is easily shown that $\int_1^\infty du\, f(x,u)$ converges uniformly for all x , so $[-\infty,\infty] \equiv E$. Now take the point p_0 to be $x = \infty$ which lies in E. Then the Corollary above says

$$\lim_{x\to\infty}\left[\int_1^\infty du\, f(x,u)\right] = \int_1^\infty du\, f(\infty,u) = \int_1^\infty du/u^2 = 1.$$

This same result is shown to obtain by just "doing the integral" naively and using l'Hospital.

Comment: I could imagine my doing the above naively. You really need to make sure that the integral converges for all x near $x = \infty$ before you do the above limit/integral order interchange.

Theorem 28 [207]
If :

- f is continuous in u for $[c,\infty)$ and in x for [a,b]
- $\int_c^\infty du\, f(x,u)$ converges uniformly for x in [a,b]

then $$\int_c^\infty du\left[\int_a^b dx\, f(x,u)\right] = \int_a^b dx\left[\int_c^\infty du\, f(x,u)\right]$$

This says that under these conditions, you can interchange order of integration, even though one of the integrals is "improper". Normally the function f(x,u) will always be continuous in my dealings, so it is really that second bullet item that is important. I scanned the proof, seems all OK.

Example 1 [207]. This is a counterexample to show that you can get into trouble with order interchange if you don't have uniform convergence. In this example [a,b] = [0,1] and c = 0. It is shown that in one ordering you get double integral = 0, while in the other order you get 1. The function being integrated is not very strange, but we do have that ∞ endpoint indicating user take caution.

Look at the single integral in (4-7). Is this F(x) integral uniformly convergent for x in [0,1] ? The integral is given by $\lim_{R\to\infty}[xR^2e^{-xR}]$. Suppose we try the value x = 1/R, which is in this interval. We that have $\lim_{R\to\infty}[Re^{-1}] = \infty$, so for x = 1/R we get no convergence, hence not uniform convergence on [0,1]. What this really says is that things fail as $x \to 0$ where as $R\to\infty$ the curves look like page 182 and the max point has no bound as $R\to\infty$. $R\to\infty$ is like $n\to\infty$ in that drawing.

I wonder if I have ever made this error? I think Jim was always interested in cases like this, but I could never find one for him to ponder.

The next theorem considers when you can take ∂_x through an integral, another order interchange, again when we have our improper infinite endpoint.

Theorem 29 : [208]
If:

- $\int_c^\infty$ du f(x,u) converges uniformly for x in [a,b]
- f(x,u) and ∂_xf(x,u) are both continuous in [a,b] for x, and in [c,∞) for u
- $\int_c^\infty$ du ∂_xf(x,u) converges uniformly for x in [a,b]

Then:

$$\partial_x \left[\int_c^\infty du\ f(x,u) \right] = \int_c^\infty du\ [\partial_x f(x,u)] \qquad \text{for any x in [a,b]}$$

In Section 3.2 we had this same theorem as Theorem 10 [120] except there the upper endpoint was the finite value d rather than ∞. Proof really is very similar.

We now conclude this section with four interesting examples, each one following a different thread through our theorems. The Bucks really are "example people" and I like that a lot.

Example 1 [209]. We start with (4-8) which says $1/x = \int_0^\infty e^{-xu}\ du$ which we know is valid uniformly in x for x > 0. Apply ∂_x and assume can go through to get $1/x^2 = \int_0^\infty u\ e^{-xu}\ du$. Again, this is OK uniformly for x > 0. We therefore have established the first and third bullets above for x in [δ,∞] which is our [a,b] in this example, where δ is any δ > 0. Everything is continuous in u and x. We conclude from Theorem 29 that we can move ∂_x through the integral for any x in [δ,∞] which is basically x > 0.

Now we can recur this example by trying to apply ∂_x to our $1/x^2$ result. If we do this n times, meaning we apply ∂_x^n to the 1/x result, we get result (4.10). As the powers u^k build up, the ∂_x^k integrals are always uniformly convergent on [δ,∞] since these powers don't really change anything in the argument. So you can keep applying ∂_x as many times as you want.

Example 2 [209] . Strange integral (4-11) is considered. First step is to write it as a double integral. We then apply Theorem 28 (where one of the integrals has upper endpoint ∞) . Since $f(x,u) = e^{-xu}$ we know that the du integration is UC for x in [δ,∞] because we just did that above. The order interchange is then justified, and the integral is trivially found to be equal to log(2). So this example perhaps suggests a way to evaluate a nasty single-variable integral if you can convert it do a double integral and then it is easy to do. This rings a distant bell with Sneddon and his Abel transforms.

Example 3 [210] We are back to integral with a parameter, $F(u) = \int_0^\infty dx\, e^{-xu} \sin(x)/x$. We would like to take the $u \to 0$ limit right through the integral to conclude that $F(0) = \int_0^\infty dx\ \text{sinc}(x) = \pi/2$. My first reaction is that this looks like Corollary p 206. You would have to know that the F(u) integral is uniformly continuous for a set including u = 0. This seems manifestly obvious, so what is the point of this Exercise? Well I guess the point is that we might not know that $\int_0^\infty dx\, \sin(x)/x$ is a convergent integral. The comparison test tells us that it is less than $\int_0^\infty dx\ 1/x$ but this diverges, which is why they say "conditionally convergent" on the first line of p 210. So I guess that is the point, we want to show this thing converges.

To that end, we apply ∂_u to both sides of F(u) = integral to obtain (4-14). We can see that this equation F'(u) = integral only converges for u > 0 which is to say, u in $[\delta,\infty]$.

Having applied, ∂_u , we next integrate (4-14) over u to get F(u) = C - arctan(u). But now to evaluate the constant C, we have to assume Fact A from which we conclude that $F(\infty) = 0$ which says $C = \pi/2$. That would tell us that $F(u) = \pi/2 - \arctan(u)$ and therefore $F(0) = \pi/2$. However, in order to associate this F(0) with $\int_0^\infty dx\, \sin(x)/x$, we have to assume Fact B.

So as the Bucks say, we have two "gaps" in our work here, Fact A and Fact B.

By doing the parts integration on bottom of page 210, Bucks show that the integral appearing in Facts A and B is in fact uniformly continuous for u in $[0,\infty]$. They show here that the tail vanishes as $R \to \infty$. Therefore, since Fact A involves $u = \infty$ and Fact B involves u = 0, both facts are justified by Corollary p 206 concerning pushing a limit through in integral to ∞ since you have shown uniform convergence.

So this example is a fancy one since it requires us to first do ∂_u F and then to integrate that over u.

Note: We treated the integral $\int_1^\infty dx\, \sin(x)/x$ in Section 3.4 and did parts there at least to show that it converges, but it is some Si function thing. I guess Bucks have not treated $\int_0^\infty dx\, \sin(x)/x$ prior to this Example 3.

Example 4 [211]. Here we want to evaluate another strange integral, reminiscent of Example 2, but different. By defining F(u) as in (4-16), we note that F(0) seems to be $F(0) = \int_0^\infty dx\, \exp(-x^2) = \sqrt{\pi}/2$, while at the same time F(1) seems to be the goal integral of (4-15). But again, in both cases we are assuming we can take a $\lim_{u \to u0}$ through the integrals.

As in the previous example, they want to apply ∂_u to (4-16) to get (4-17). To justify this, they have to show uniform convergence for some range of u. For F(u), we have this UC for $u \geq 0$ (a fact which they later prove on p 212). But to justify the ∂_u action we have to also show that the (4-17) integral is also UC. They do this with a dense little argument I do not follow the details of, but they conclude that we have UC of this thing again for $[\delta,\infty]$.

In the previous example, after doing ∂_u we did $\int du$, but that method does not fly here because the integral is a mess. But by slight of hand just changing variables, they are amazingly able to show that

F'(u) = -2F(u). This simple ODE says F(u) = C e^{-2u} for some constant C, still all for u > 0. But since F(u)'s integral is OK for u ≥ 0, we can write F(0) = C = $\sqrt{\pi}$ /2 so then F(u) = ($\sqrt{\pi}$ /2) e^{-2u} , and so we have evaluated the integral in (4-16). Our original strange integral is F(1) = ($\sqrt{\pi}$ /2) /e^2 and this is the final conclusion on page 212 and that is the end of this Section 4.4

If I were more serious about this delicate subject, I would want to do all the exercises. But this is my very first time "through Buck" (in the modern era), and I don't want to take the time to do that, since basically I have a finite amount of time left in all that I might do. I have no idea how rigorous my pass through this section was (red pencil underlining) or when that pass was. I suspect I was just skimming for something I was doing perhaps 10 years ago, or maybe in the Stakgold push era.

4.5 The Gamma Function **[213]**

An idea presented here is the definition of a function in terms of an integral of another function. The first example is log(x) as integral of 1/x. Second example is arctan(x) as an integral of 1/(1+x^2), and how given that integral, you can derive various properties of the artctan function. But the big example is the Gamma function integral shown in (4-20). Cleverly, Bucks already computed this integral for integer x and showed it gage the n! result as top page 214. This example is going to continue now for 4 pages.

Theorem 30 [214] Γ(x+1) = xΓ(x) for any x> 0.

This familiar fact is proved from the integral definition (4-20) using parts integration.

By changing integration variables in (4-20) you can write down lots of other forms of the integral definition of Γ(x) and four such alternatives are given on mid p 214.

Theorem 31 [214] **Half integer formulas**. We know Γ(1/2) just doing the integral, so Γ(n/2) can be done using Theorem 30's little formula. They then talk about "extending" the definition of Γ(x) to negative half integer values in the obvious manner. They don't mention the fact that really the parts integration provides an analytic continuation into a new convergence region, perhaps they have that in their appendix and I have seen it perhaps in Ahlfors or elsewhere. It then exposes the poles on the left. In this section the term pole does not appear since they have not talked at all about complex variables.

Theorem 32. [216] **Stirling's** Formula for Γ(x).

The proof is a long one using a certain approximation method of breaking an integral into 3 terms and showing that only the middle one matters for large x. Two Lemmas are required along the way. I did some of it, but really it is not critical, nothing magic is done here.

Theorem 33. [216] **Beta** B(p,q) defined as a certain integral is shown equal to Γ(p)Γ(q)/Γ(p+q).

The proof is pretty easy. It starts by writing Γ(p)Γ(q) as a double integral as shown in A. This is converted to polar coordinates, and further shuffling gives the desired result as shown p 219 B.

End of Chapter 4 !!!! Next task is to write the Meta notes!