Buck Chapter 7 : Differential Geometry and Vector Calculus PhL 6.24.15

My Chapter 6 notes were written starting Jan 11 of 2015, but the Ch 6 *meta* notes were only written on May 16, after a large time gap which involved a Cod trip. So now I am starting into the raw notes for Ch 7 after another 6 week gap! I did a cursory review today if earlier meta notes.

Note added 5/23/16: I read all these notes and am happy with things now to page 410. I still should read the rest of the chapter. It all makes sense, but notation is different from mine.

Note added 6/4/16. Finally finished reading and writing these Chapter 7 notes, it took a whole year!

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Chapter 7: **Differential Geometry and Vector Calculus [pp 366-436]**

Rereading this chapter on 5/21/16 after the 5/19 release of wedge doc!

7.1 Integrals over Curves and Surfaces **[366]**

In Chapter 6 we set up all the machinery for integrals over curves and over areas, but the function being integrated there was always just "1", so we were computing arc length and surface area. Here we insert a function f which is either f(γ) for the curve case, or f(Σ) for the surface case.

As before, the curve γ and the surface Σ should be regarded as mappings!

$\gamma: R \to R^3$ or $\gamma: t \to (x,y,z)$.
$\Sigma: R^2 \to R^3$ or $\Sigma: (u,v) \to (x,y,z)$.

Curve case: In wedge doc I show below (10.10.44) that my temperature integral has the form

$$\int_0^a dx^1\, K(x)\, T(\mathbf{F}(x)) \;\to\; \int_a^b dt\; |\partial_t \gamma|\, f(\boldsymbol{\gamma}(\mathbf{t})) \;. \qquad \text{// Buck 367 (7-1)}$$

which does indeed agree with Buck. So wedge doc "has this covered", and I show arc length as well in the case that T = f = 1,

$$L' = \int_{C'} ds' = \int_0^a dx^1\, K(x) = \text{arc length of the curve C' in x'-space .} \qquad (10.10.43)$$

$$K^2(t) = v'^2_1 + v'^2_2 + v'^2_3 = (\mathbf{v}')^2 \quad \Rightarrow \quad K(t) = |\,\mathbf{v}'\,| = |\,\partial_t \mathbf{x}'\,| = |\,\partial_t \mathbf{F}(t)\,| \,. \qquad (10.10.44)$$

Example 1 [368] Compute center of mass for a uniform half-circle wire.

A curve integral here is $\bar{y} = (1/M) \int_\gamma y\, \rho\, ds = (1/M) \int_\gamma \sin(t)\, [M/\pi]\, dt = \pi^{-1} \int_0^\pi \sin(t) dt = 2/\pi$.

It happens for this example that dt = ds is the arc length. How would you write this using dy as the integration variable? In polars we have r = 0 and ds = rdθ = dθ and x = cosθ and y = sinθ so

$$dy = \cos\theta d\theta = \sqrt{1-y^2}\, d\theta \qquad \text{so } ds = d\theta = dy/\sqrt{1-y^2} \quad \text{so then}$$

$$\bar{y} = (1/M) \int_0^1 y\, [M/\pi]\, dy/\sqrt{1-y^2} = (1/\pi) \int_0^1 (y/\sqrt{1-y^2})dy = \int_0^1 A(y)dy$$

and then it is a special case of a 1-form where the dx coefficient vanishes. They will no doubt soon make the point that you cannot write $\int d\boldsymbol{\ell} \bullet \mathbf{F} = \int_\gamma [\, F_{\mathbf{x}}(x,y)dx + F_{\mathbf{y}}(x,y)dy\,]$ as a single-term 1-form, and thus a so-called "line integral" in the plane is *not* just a "curve integral". I guess they are saving this observation for their next section where this stuff is "applied".

Surface case:

The surface integral p 368 replaces $|\gamma'(t)|$ by $|\mathbf{n}(u,v)|$ and I earlier derived the Jacobian way you write this normal thing as shown in (7.3) as the square root of the sum of squared Jacobians.

This now appears in wedge doc in (10.10.20) where now K = $|\, \mathbf{n}'|$. Also I have the area integral,

$$A' = \int_{S'} dA' = \int_S K(\mathbf{x})\, dx^1 dx^2 . \qquad (10.10.21)$$

Example 2: [369] Compute center of mass for a parabolic surface. Requires computation of $|\mathbf{n}|$ and instead of using u,v they just use x,y directly.

Page 370 concerns interpreting our integrals as **functionals** in the language of Stakgold. They write the line integral functional as (γ,f) . Recall from Stak that a functional maps into R^1. Here our functionals are integrals of a function f, so in a sense they are functions of f, but at all points on a curve of surface. These are linear functionals since integral of f + g is integral of (f+g). I will not digress on this Stak for the moment other to say that for Stak, $(\delta_{\mathbf{y}},\varphi)$ meant $\int \delta(x-y)\varphi(x)dx$ where φ was a test function. Here the function being integrated, called f, plays the role of the test function, and γ or Σ plays the role of δ. So I am fairly comfortable with the notion of $\int_\Sigma f$ being a **surface-functional** (Σ,f) and so on. Similarly Bucks define a **curve-functional** and a **region-functional** for 3D region Ω.

Note added: Only now do I realize that this will become my "second definition" when integrating k-forms. I quote an example in wedge doc, but I should have referred to this Buck section as well. *****. Bucks are "setting us up" for that k-form discussion.

Page 371 talks about right-hand rule for normal **orientation** of boundaries see drawing there. In 1D we have a very trivial orientation idea with the explicit a,b endpoints. For a curve integral, $F(-\gamma) = -F(\gamma)$ based on the idea that you integrate in the reverse direction. Similarly, $\int\int_{-D} = -\int\int_{D}$ (opposite sides of surface have opposite sign area) and same for 3D, and some simple example pictures are given (in 3D, volume is positive for right-handed system, negative for left, same for dV integrated against a function $f(\Omega)$. They come up on page 374 with a sort of hokey idea of notating the two orientations for a 2D surface integral using dydx = - dxdy which again is like their "box notation" and you don't take these to be products of differentials. And similarly for dxdydz.

Note added: Bucks have now *worked in* the idea that dydx = - dxdy without mentioning wedge products. They argue that this is just a "device" for handling "orientation". And similarly for dxdydz.

7.2 Differential Forms **[376]**

I feel I am on the move once again, this is a long-wondered-about topic for me. This is an amazing section.

Fact: In E^n you can have a k-form for k = 0,1,2..n. Examples:

$\omega(0\text{-form}) = f(\mathbf{r})$ a point function

$\omega(1\text{-form}) = A_1(\mathbf{r})dx_1 + A_2(\mathbf{r})dx_2 + ... + A_n(\mathbf{r})dx_n = A_i dx_i$

$\omega(2\text{-form}) = A_{12}(\mathbf{r})dx_1dx_2 + A_{13}(\mathbf{r})dx_1dx_3 + ... + A_{1n}(\mathbf{r})dx_1dx_n$
$+ A_{23}(\mathbf{r})dx_2dx_3 + A_{24}(\mathbf{r})dx_2dx_4 + ... + A_{2n}(\mathbf{r})dx_2dx_n$
$+ .... + A_{(n-1)n}dx_{n-1}dx_n$ n! terms

$= A_{ij}dx_i dx_j$ i,j = 1,2...n // $dx_2dx_2 = 0$ so OK to include

and so on for higher and higher. In working with a k-form in E^n you need to think about both the value of k and the value of n.

Rules: sign rule dydx = -dxdy for 2-forms
sign rule dxdydz = $(-1)^s$ d*d*d* etc (sign of permutation)
dxdx... = 0

Note: Ara below shows how these are really things like $dy \wedge dx = -dx \wedge dy$ with wedge products where the wedged product defines an oriented 2-piped in E^n.

The subject of interest here is integrating a k-form (a "differential form") over a region of dimension k.
For example, integrate a 1-form over a region of dimension 1 which is a curve $\gamma(t)$.
For example, integrate a 2-form over a region of dimension 2 which is a surface $\Sigma(u,v)$.
For example, integrate a 3-form over a region of dimension 3 which is a surface V(u,v,w)

The Bucks treat things one case at a time, so I will do that here in these notes as well.

k = 1 n = 2: the 1-form in E^2

$\omega = A(\mathbf{r})dx + B(\mathbf{r})dy \qquad \mathbf{r} = (x,y)$

Imagine a curve $\gamma: E^1 \rightarrow E^2$ which says $\mathbf{r} = (x,y) = \mathbf{r}(t) = \boldsymbol{\gamma}(t)$ so $x = \gamma_{\mathbf{x}}(t)$ and $y = \gamma_{\mathbf{y}}(t)$ eg. Then you can write $x = \gamma_{\mathbf{x}}(t)$ so $dx = \gamma'_{\mathbf{x}}(t)\,dt$ and $dy = \gamma'_{\mathbf{y}}(t)\,dt$ and so

$$\omega = A(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{x}}(t)\,dt + B(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{y}}(t)\,dt = [\,A(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{x}}(t) + B(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{y}}(t)]\,dt$$

and then here is your integral of interest

$$\int_{\gamma} \omega = \int_{t_1}^{t_2} [\,A(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{x}}(t) + B(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{y}}(t)]\,dt = \int_{\gamma} [\,A(\mathbf{r})dx + B(\mathbf{r})dy\,] \quad \text{// Buck p 376 (7-7)}$$

The last form is easiest to comprehend, the dt form shows how you actually DO the integral. Bucks do three examples for various γ and ω choices. Notice that so far we have no dω, only ω.

Note Added: in wedge doc the above appears as

$$\int_{\varphi} \alpha_{\mathbf{x}} = \int_{\varphi} \mathbf{f}(\mathbf{x}) \bullet \boldsymbol{dx} = \int_{[0,1]} g(t)\,dt$$

$$= \int_0^1 \mathbf{f}(\boldsymbol{\varphi}(t)) \bullet (\partial_t \boldsymbol{\varphi}(t))\,dt = \int_0^1 f_i(\boldsymbol{\varphi}(t))\,[\partial\varphi^i(t)/\partial t]\,dt\,. \qquad (10.12.11)$$

k = 1 n = 3: the 1-form in E^3

$\omega = A(\mathbf{r})dx + B(\mathbf{r})dy + C(\mathbf{r})dz \qquad \mathbf{r} = (x,y,z) = \boldsymbol{\gamma}(t) = (\gamma_{\mathbf{x}}(t),\gamma_{\mathbf{y}}(t),\gamma_{\mathbf{z}}(t))$

$\omega = A(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{x}}(t)\,dt + B(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{y}}(t)\,dt + C(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{z}}(t)\,dt$

$= [\,A(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{x}}(t) + B(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{y}}(t) + C(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{z}}(t)]\,dt$

$$\int_{\gamma} \omega = \int_{t_1}^{t_2} [\,A(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{x}}(t) + B(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{y}}(t) + C(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{z}}(t)\,]\,dt$$

$$= \int_{\gamma} [\,A(\mathbf{r})dx + B(\mathbf{r})dy + C(\mathbf{r})dz]$$

Here we get just one example on page 380 top.

k = 2 n = 2: the 2-form in E^2 [starts mid p 380]

Imagine a surface $\Sigma: E^2 \rightarrow E^2$ which says $\mathbf{r} = (x,y) = \Sigma(u,v)$ so $x = \Sigma_{\mathbf{x}}(u,v)$ and $y = \Sigma_{\mathbf{y}}(u,v)$ eg.

$\omega = A(\mathbf{r})dydz + B(\mathbf{r})dzdx + C(\mathbf{r})dxdy$ $\quad\quad$ $\mathbf{r} = (x,y) = \Sigma(u,v) = (\Sigma_{\mathbf{x}}(u,v),\Sigma_{\mathbf{y}}(u,v))$

Before doing examples we need more info, so we put this case on hold for a little while.

Adding k-forms is very simple, just remember the sign rules for ordering things. Ex 1 p 381.

For **multiplication**, you can multiply an i-form by a j-form to get an i+j form. If i+j > n the result is zero due to the "repeated differential rules" such as dxdx = 0 (no area). Several examples p 382 for doing this multiplication of k-forms. [Ara shows how this follows from distributive and scalar "rules"]

[383] To **differentiate a k-form to get dω** you replace the coefficient functions like A with dA where dA is the usual chain rule differential. You do this for any k-form, though Bucks show three cases on page 383. Examples are provided bottom p 383.

PL Problem 1. Consider

$\omega = Adx + Bdy + Cdz$

$d\omega = dA\,dx + dB\,dy + dC\,dz$ $\quad\quad$ // note that d(dx) = 0, etc (Ara)

where A,B,C are each functions of x,y,z. Then,

$dA = A_1dx + A_2dy + A_3dz$
$dB = B_1dx + B_2dy + B_3dz$
$dC = C_1dx + C_2dy + C_3dz$

Then

$d\omega = (A_1dx + A_2dy + A_3dz)\,dx + (B_1dx + B_2dy + B_3dz)\,dy + (C_1dx + C_2dy + C_3dz)\,dz$

$= A_2dydx + A_3dzdx + B_1dxdy + B_3dzdy + C_1dxdz + C_2dydz$

$= (B_1-A_2)dxdy + (C_2-B_3)dydz + (A_3-C_1)dzdx$

$= (C_2-B_3)dydz + (A_3-C_1)dzdx + (B_1-A_2)dxdy$
x-term y-term z-term

and this is the result shown in Exercise p389 14. I already see a curl connection going on here in some dim manner!

Now with this extra info, we get to **Theorem 1**. The theorem here is that, if you follow the simple rules for using 1-forms in E^2, your result *is consistent with* the known Jacobian rule for changing variables. You just write it out as shown. Bucks are a bit hazy with this Theorem. You could regard it as a proof of the Jacobian rule if you really accept the k-form rules.

I can now add some clarity: Theorem 1 really says this

$$du \wedge dv = \frac{\partial(u,v)}{\partial(x,y)}\, dx \wedge dy$$

which is a statement involving 2-forms. The Jacobian integration rule does not involve 2-forms, it involves products of differentials where dxdy = dydx for example. You can see how Bucks got into trouble here by not showing the wedge symbols.

Recall that on page 301 the Bucks do a huge messy derivation of the E^3 Jacobian rule using their set function ideas. My tensor doc derivation of the Jacobian integration rule is this

• The volume of the N-piped is given by (see Section 5.12 concerning J)

$$V = |\det[\,\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \ldots \mathbf{e}_N]\,| = |\det(S^a{}_b)| = g'^{1/2} = |J| \quad . \tag{8.3.8}$$

I guess we can regard Theorem 1 as a statement about the dxdy type objects which appear in differential forms.

Now let's jump back to where we were before the new facts were added:

k = 2 n = 2: the 2-form in E^2 [mid p 380]

Imagine a surface $\Sigma: E^2 \rightarrow E^2$ which says $\mathbf{r} = (x,y) = \Sigma(u,v)$ so $x = \Sigma_x(u,v)$ and $y = \Sigma_y(u,v)$ eg.

k = 2 n = 3: the 2-form in E^3 [top p 381]

$\omega = A(\mathbf{r})dydz + B(\mathbf{r})dzdx + C(\mathbf{r})dxdy$ $\quad \mathbf{r} = (x,y) = \Sigma(u,v) = (\Sigma_x(u,v),\Sigma_y(u,v))$

We have $x = \Sigma_x(u,v)$ so

$dx = (\partial_u\Sigma_x)du + (\partial_v\Sigma_x)dv = d\Sigma_x$ $\quad d\varphi^i = R^i{}_j\, du^j$ wedge (10.9.17) !
$dy = (\partial_u\Sigma_y)du + (\partial_v\Sigma_y)dv. = d\Sigma_y$
$dz = (\partial_u\Sigma_z)du + (\partial_v\Sigma_z)dv. = d\Sigma_z$

Using the forms on the right, we would say

$$\omega = A(\Sigma(u,v))d\Sigma_y d\Sigma_z + B(\Sigma(u,v))d\Sigma_z d\Sigma_x + C(\Sigma(u,v))\, d\Sigma_x d\Sigma_y$$

and this really is what appears top page 385. They don't go the next step in general but consider

$$dxdy = [(\partial_u\Sigma_x)du + (\partial_v\Sigma_x)dv][(\partial_u\Sigma_y)du + (\partial_v\Sigma_y)dv] = [\,(\partial_u\Sigma_x)(\partial_v\Sigma_y) - (\partial_v\Sigma_x)(\partial_u\Sigma_y)]\, dudv$$

$$= [\,(\partial_u x)(\partial_v y) - (\partial_v x)(\partial_u y)]\,dudv \;=\; \frac{\partial(x,y)}{\partial(u,v)}\,dudv$$

Then I guess we would have

$$\omega = [\,A(\mathbf{r})\frac{\partial(y,z)}{\partial(u,v)} + B(\mathbf{r})\frac{\partial(z,x)}{\partial(u,v)} + C(\mathbf{r})\frac{\partial(x,y)}{\partial(u,v)}\,]dudv$$

$$\int\int_{\Sigma}\omega \;=\; \int\int_{\Sigma}[\,A(\mathbf{r})\frac{\partial(y,z)}{\partial(u,v)} + B(\mathbf{r})\frac{\partial(z,x)}{\partial(u,v)} + C(\mathbf{r})\frac{\partial(x,y)}{\partial(u,v)}\,]dudv$$

but Bucks never write this out in this manner. On page 385 they do give some examples for various ω and Σ choices. [Ara below confirms the above expression]

Note added 5/20/16: BUT, I write the above in wedge doc:. First,

$$\int_{\varphi}\alpha_{\mathbf{x}} \equiv \int_{[0,1]^2}\varphi^*(\alpha_{\mathbf{x}})$$

$$= \text{the integral in t-space of the pullback of } \alpha_{\mathbf{x}} \text{ over the 2-cube } [0,1]^2 \qquad (10.13.3)$$

and then

$$\varphi^*(\alpha_{\mathbf{x}}) \;=\; [\,f_{12}(\boldsymbol{\varphi}(\mathbf{t}))\frac{\partial(\varphi^1,\varphi^2)}{\partial(t^1,t^2)} + f_{13}(\boldsymbol{\varphi}(\mathbf{t}))\frac{\partial(\varphi^1,\varphi^3)}{\partial(t^1,t^2)} + f_{23}(\boldsymbol{\varphi}(\mathbf{t}))\frac{\partial(\varphi^2,\varphi^3)}{\partial(t^1,t^2)}\,]\;dt^1 \wedge dt^2$$

$$= G(\mathbf{t})\;dt^1 \wedge dt^2$$

where

$$G(\mathbf{t}) \equiv [\,f_{12}(\boldsymbol{\varphi}(\mathbf{t}))\frac{\partial(\varphi^1,\varphi^2)}{\partial(t^1,t^2)} + f_{13}(\boldsymbol{\varphi}(\mathbf{t}))\frac{\partial(\varphi^1,\varphi^3)}{\partial(t^1,t^2)} + f_{23}(\boldsymbol{\varphi}(\mathbf{t}))\frac{\partial(\varphi^2,\varphi^3)}{\partial(t^1,t^2)}\,]\,. \qquad (10.13.6)$$

Theorem 2:

(i) Two smoothly equivalent curves have the same 1-form integral
(ii)Two smoothly equivalent surfaces have the same 2-form integral

The proofs are given p 386-387 and make use of the earlier "change of variable" Jacobian integration rule. This just means that these integrals are properties only of the traces of the mappings.

Technically, the 1-form integral is called a **line integral**, while that 2-form one is a **surface integral**. These are special definitions!

Now here is a point that they mention early on: Consider from above this sample,

$$\int_{\gamma} \omega = \int_{t_1}^{t_2} [\, A(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{x}}(t) + B(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{y}}(t)] \; dt = \int_{\gamma} [\, A(\mathbf{r})dx + B(\mathbf{r})dy \,]$$

If you had only the first term, so B = 0, you would have

$$\int_{\gamma} \omega = \int_{t_1}^{t_2} [\, A(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{x}}(t)]dt = \int_{\gamma} A(\mathbf{r})\, ds = \int_{\gamma} A(x,y,z)\, ds$$

This matches the form shown on page 367. We are allowed to choose a curve mapping where the speed is set so ds is arc length. The point here is that A is a function of **r**, not a function of γ, although you evaluate A at points on γ. So in this case our more general "line integral" reduces to a simple "integral along a curve". When both A and B terms are present, perhaps you cannot set both terms to have only arc length at the same time, so you cannot write the first integral as a simple ds integral. But why not split it into two terms? Then perhaps each term is a simple arc length integral.

I am hazy on this point which Bucks make but don't explore in any way. Is it possible to put the general 1-form integrals as a sum of simple curve-functionals? I am pretty sure they will come back to this point after doing some "vector analysis".

Angus Taylor book on differential forms. Page 433. [I am not sure on 12.11.16 which Taylor book this is. I found Taylor and Mann but that seems not right.] They talk here about a 1-form in the plane and call it Mdx+Ndy. They note that *some* such forms are exact differentials in the there exists some function u such that du = Mdx+Ndy. This not mentioned by Bucks (yet). How can you tell just looking at M and N if there is a function such that $u_{\mathbf{x}}$= M and $u_{\mathbf{y}}$ = N? Taylor focuses on this question. I see nothing about dxdy changing sign with order. The 1-form in E^3 then appears on page 471. The same question is asked about whether u exists so du = the form. // so forms get some mention, but not much. No other calculus books on my shelf deal with this topic! Wiki is too advanced for me on this subject right now.

********** **Arapura Notes** **********

Arapuru 38 page PDF (2015). They are also interested in whether the 1-form is exact. Now this author states that there are two parallel universes, diff forms and the usual vector. Stuff. He first shows that

THEOREM 6.1

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Arapuru calls this Green's Theorem. I have notes on this "meanings of Green's Theorem.doc" which shows that you can derive the above just using the "integral of a gradient theorem" in 2D. I also show in my "integral theorems.doc" that this is just a special case of Stokes's Theorem:

2. Stokesian: (for n=2, $\mathbf{A} = f\hat{\mathbf{x}} + g\hat{\mathbf{y}}$, $\nabla \mathbf{x} \mathbf{A} = (\partial_{\mathbf{x}} g - \partial_{\mathbf{y}} f)\, \hat{\mathbf{z}}$, $d\boldsymbol{\ell} \bullet \mathbf{A} = f dx + g dy$, => "Green's Theorem")

$$\int_{\mathbf{S}} d\mathbf{S} \bullet (\nabla \mathbf{x} \mathbf{A}) = \oint d\boldsymbol{\ell} \bullet \mathbf{A} \qquad \textbf{(7)} \qquad \text{Schaum 22.60}$$

OK, fine. Ara is going I think to use this as a case study, so at least I understand the starting point. Ara next defines a **2-vector** as an oriented parallelogram P, very good. Has a plane, and area, an orientation. Then cP means area scaled by c. If the edges vectors are u and v, then Ara writes P = u $\wedge$ v where this is the famous "wedge product". In E^3 you can associate u $\wedge$ v with u x v, but the cross product only has meaning in E^3 whereas u $\wedge$ v has meaning for any E^n: It is a parallelogram spanned by the two vectors. In tensor doc I could talk about $\mathbf{e_1} \wedge \mathbf{e_2}$ in n dimensions, but I had no use for such a thing (but perhaps this would clarify my area stuff).

"A 2-vector is a finite sum of oriented parallelograms."

A 2-form is like a 2-vector but built using forms. On $\mathbb{R}^3$, this would be an expression:

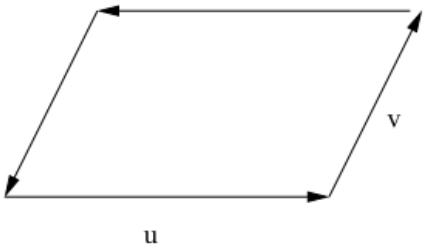
$$F(x,y,z)dx \wedge dy + G(x,y,z)dy \wedge dz + H(x,y,z)dz \wedge dx$$

where F, G and H are functions defined on an open subset of $\mathbb{R}^3$. Any wedge product of two 1-forms can be put in this format. For example, using the above rules, we can see that

$$(3dx + dy) \wedge (dx + 2dy) = 6dx \wedge dy + dy \wedge dx = 5dx \wedge dy$$

So OK, what Bucks write as dxdy, Ara is writing as dx $\wedge$ dy. Here is more from Ara earlier

orientation if $c > 0$ and the opposite orientation if $c < 0$. Given two vectors $\mathbf{u}, \mathbf{v}$ we define the parallelogram $\mathbf{u} \wedge \mathbf{v}$ to be given as follows ($\wedge$ is pronounced "wedge").



Evidently

$$\mathbf{v} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{v}$$

$$\mathbf{u} \wedge \mathbf{u} = 0$$

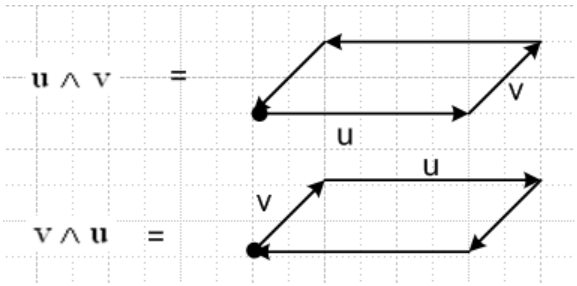
and

$$c(\mathbf{u} \wedge \mathbf{v}) = (c\mathbf{u}) \wedge \mathbf{v} = \mathbf{u} \wedge (c\mathbf{v})$$

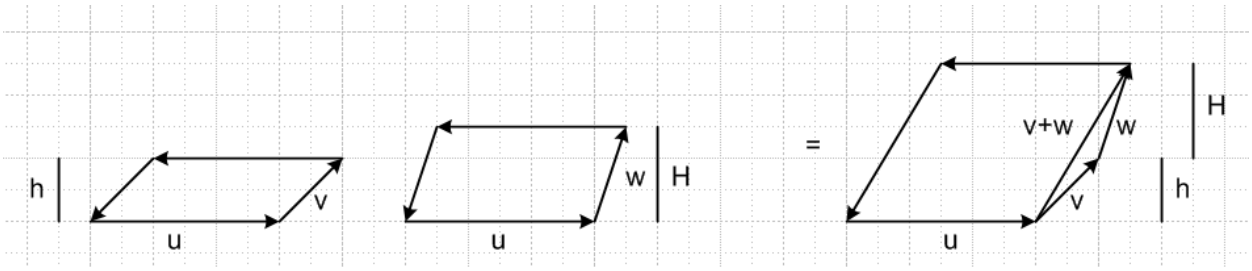
The sum of vectors can be defined geometrically using the parallelogram law. There is a limited version of this for 2-vectors: if two parallelograms share one side, then the remaining sides can be added using the parallelogram law. In terms of algebra, this is just the distributive law:

$$\mathbf{u} \wedge \mathbf{v} + \mathbf{u} \wedge \mathbf{w} = \mathbf{u} \wedge (\mathbf{v} + \mathbf{w})$$

Now where does Ara come up with his "evidently" rule with the minus sign? This is totally non-obvious to me. Consider:



By MY definition, layout out the first vector's tail at the dot, then lay out the second vector's tail at the end of the first vector. The minus sign is because the orientations are different! Now what happens if you wedge u with itself? Area is obviously 0. So I guess a 2-piped with zero area is defined to be a zero vector. Then you get that u ∧ u = 0 which is the Buck idea that dx dx = 0. At least Ara is giving my better motivation! I can see the validity of the c rule above, but how does distributive work?



This seems more of a definition. What does it mean to "add two parallelograms having a common edge"? This is the generalization of the idea of adding two vectors (1-vectors). For a vector space concept, the sum must be itself a parallelogram, and we have achieved that much. The sum of the areas of the two 2-pipeds is indeed the area of the sum one (based on the one half base times height rule). If they do not have a common edge, then we don't care because we are just trying to justify the distributive rule above.

The distributive rule is very important because they you can do things like this (here I leave out the wedge symbol

(adx + bdy)∧(cdx+ddy) = (adx + bdy)∧(cdx) + (adx + bdy)∧(ddy) // distrib

= (adx)∧(cdx) + (bdy)∧(cdx) + (adx)∧(ddy) + (bdy)∧(ddy)

= ac (dx)∧(dx) + bc (dy)∧(dx) + ad (dx)∧(dy) + bd (dy)∧(dy)

= (ad - bc) dx∧dy

So here the functions coefficients are treated like the scalars in the rule above. Wiki has more to say about the idea that this wedge stuff is some kind of "vector space".

Comments: At least I have some better support for the idea that dxdy = -dydx and dxdx = 0. Bucks are using the wedge product, but they don't display the symbol!

Now what about the "correspondence" stuff. Ara says

In order to make the comparison with traditional vector calculus, we note that we can convert vector fields to 2-forms and back

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \leftrightarrow F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy,$$

Earler, we learned how to convert a vector field to a 1-form:

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \leftrightarrow F_1 dx + F_2 dy + F_3 dz$$

So the first line is roughly saying that $\hat{\mathbf{x}} \leftrightarrow$ dy ∧ dz = **d**y x d**z** = dydz $\hat{\mathbf{y}}$x $\hat{\mathbf{z}}$ = dydz $\hat{\mathbf{x}}$ so the correspondence seems to ignore the differential scale factor dydz. I can live with that. The second line is similar where we are saying $\hat{\mathbf{x}} \leftrightarrow$ dx = d**x** = dx $\hat{\mathbf{x}}$, so again we ignore the scaling dx factor.

Ara then says you can move between the two forms above in this manner:

To complete the triangle, we can interchange 1-forms and 2-forms using the so called Hodge star operator.

$$*(F_1 dx + F_2 dy + F_3 dz) = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

$$*(F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy) = F_1 dx + F_2 dy + F_3 dz$$

Obviously one must have ** = 1 for this operator (not quite, there is a phase here 5/23/16). Fine.

Next comes differentiation:

$$d(\alpha + \beta) = d\alpha + d\beta$$
$$d(f\alpha) = (df) \wedge \alpha + f d\alpha$$
$$d(dx) = d(dy) = d(dz) = 0$$

where α and β are 1-forms and f is a function. Recall that

The first line is a normal rule from calculus that moves to this new world.
The second is the product world where f = scalar and α = 1-form. Notice that you have to deal here with dα, where α is some 1-form, but the last line seems to kill this off so dα = 0?? Well no, only true on the primitives. The first two rules seem reasonable, but why would you say d(dx)= 0? Is it just that dx is a primitive fixed thing which has no variation, it is a "constant" in this algebra world?

Ara then uses the above rules to obtain a Buck result

$$d(F dx + G dy + H dz) = (G_x - F_y) dx \wedge dy + (H_y - G_z) dy \wedge dz + (F_z - H_x) dz \wedge dx$$

Note added 5/23/16: See dα in standard form in wedge doc, which is same as the Ara result above:

$$d\alpha = \Sigma_{j_1<j_2} (\partial_{j_1} f_{j_2} - \partial_{j_2} f_{j_1})\, dx^{j_1} \wedge dx^{j_2} \qquad (10.3.24b)$$

$$d\alpha = \Sigma_{i<j} (\partial_i f_j - \partial_j f_i)\, dx^i \wedge dx^j$$

Ara continues

"We also have 3-vectors which correspond to oriented parallelepipeds."

Ara shows that the "d" operator turns n-forms into (n+1)-forms. Then comes

PROPOSITION 9.1 $d^2 = 0$

[1]This takes a bit of work that I'm leaving as an exercise. It's probably easier to work backwards. You'll need the product rule for dot products and the chain rule.

Ara goes on to do surface integral stuff. Here is one of his results

DEFINITION 11.1 *The integral of a 2-form on S is given by*

$$\iint_S F dx\wedge dy + G dy\wedge dz + H dz\wedge dx = \iint_D \left[F\frac{\partial(x,y)}{\partial(u,v)} + G\frac{\partial(y,z)}{\partial(u,v)} + H\frac{\partial(z,x)}{\partial(u,v)} \right] du dv$$

which is a result I actually derived above.

OK, I think that is enough Ara for now. He has filled in some of the missing pieces. I see the Buck approach, they want to keep it simple and not get all involved in algebraic details. Maybe I will reread the above notes with the extra Ara info.

Note added 5/21/16. I remember reading the above Arapura stuff. In wedge doc

$$\mathbf{a} \wedge \mathbf{b} = \det\begin{pmatrix} a^1 & b^1 \\ a^2 & b^2 \end{pmatrix} \mathbf{u}_1\wedge\mathbf{u}_2 \;=\; \det(\mathbf{a},\mathbf{b})\, \mathbf{u}_1\wedge\mathbf{u}_2 \;=\; [\, a^1b^2 \,-\, a^2b^1]\;\; \mathbf{u}_1\wedge\mathbf{u}_2 \;. \tag{4.3.14}$$

I comment on the fact that the coefficient here equals the area of a n-piped spanned by a and b, but I would never claim that **a** ^ **b** "was" that 2-piped. It is a 2-blade in L^2 which of course means it is also a tensor product in V^2. Such objects are not 2-pipeds. But it was a mystery that Arapura somehow was associating these wedge products with little differentials that at least look like they would be reasonable to integrate over. I think I have greatly clarified all this stuff now in wedge doc.

********** end Arapura notes **********

7.3 Vector Analysis **[390]**

The first 10 pages is old hat for me. They do use the terms "bound vector" and "free vector", a concept I always wondered about -- they at least give them names! A passing mention of **quaternions** on page 391 leaves me pretty cold since I never knew what these were. Associated with E^4, discussion is what kinds of multiplication can you do in E^4? We know the inner and outer product in E^3 for example. This is something I could ponder at some point, another "hole" in my house of math.

[400] What on earth is going on here? As for A, you can see how $\hat{\mathbf{x}}$ might be "associated with" dx or the combination dydz = $A_{\mathbf{x}}$ where dx is the one thing missing. In some sense things are vector like here, whereas in B the two items are scalars so a 1 on the right.

Now we just linearly combine the A and B ideas to get the C idea.

Now Bucks consider some k-form products. They first define ν, ω and ω* as shown, and then the products νω and νω* are as shown top of page 401 (which I could verify). [I now know that the * is the Hodge star operator]. Now here comes the big deal: these two products are on the k-form side of the correspondence fence. If you see what corresponds to these expressions on the vector side of that fence, using the correspondence rules A,B,C of p 400, you find that $\nu\omega \leftrightarrow \mathbf{V} \times \mathbf{W}$ and $\nu\omega^* \leftrightarrow \mathbf{V} \bullet \mathbf{W}$. As Bucks point out, the single concept of "multiplication of differential forms" produces both these vector products. There is some kind of homomorphism going on here which they are not talking about, or isomorphism. [Yes, there is something going on here in the algebraic sense, but wiki is still over my head, Ara was helpful].

Note added 6.4.16: I have already added this to wedge doc, and today I added it even more severely, both as (4.3.18b) and as a comment leading to (H.1.26).

They now examine "further correspondences" between the two worlds:

[401]. Consider $df = f_1dx + f_2dy+f_3dz \leftrightarrow f_1\hat{\mathbf{x}} + f_2\hat{\mathbf{y}} + f_3\hat{\mathbf{z}} = \nabla f$ (d applied to a 0-form)

So right out of the box we get d(0-form) corresponding to gradient.

[401] Now consider d(1-form):

$\omega = f_1dx + f_2dy + f_3dz = \mathbf{f} \bullet d\mathbf{r}$ // but *now* f_i is component of **f**, not $\partial_i f$

$d\omega = df_1dx + df_2dy+df_3dz \qquad df_1 = \partial_x f_1dx+ \partial_y f_1dy+ \partial_z f_1dz$

$\Rightarrow d\omega = (\partial_x f_1dx+ \partial_y f_1dy+ \partial_z f_1dz)dx + (\partial_y f_2dy+ \partial_z f_2dz+ \partial_x f_2dx)dy+(\partial_z f_3dz+ \partial_x f_3dx+ \partial_y f_3dy)dz$

$= (\partial_y f_1dy+ \partial_z f_1dz)dx + (\partial_z f_2dz+ \partial_x f_2dx)dy+(\partial_x f_3dx+ \partial_y f_3dy)dz$

$= (\partial_2 f_1dy+ \partial_3 f_1dz)dx + (\partial_3 f_2dz+ \partial_1 f_2dx)dy+(\partial_1 f_3dx+ \partial_2 f_3dy)dz$

$= (\partial_2 f_3 - \partial_3 f_2)\, dydz + \text{cyclic}$

$\leftrightarrow \quad (\partial_2 f_3 - \partial_3 f_2)\hat{\mathbf{1}} + \text{cyclic} = \text{curl}(f) = \nabla \times f$

To summarize this case:

$\omega = \mathbf{f} \bullet d\mathbf{r} = f_1dx + f_2dy + f_3dz \qquad \leftrightarrow \mathbf{f} \qquad$ 1-form

$\Rightarrow$ $d\omega = (\partial_2 f_3 - \partial_3 f_2)dydz + \text{cyclic}$ $\leftrightarrow \nabla \times \mathbf{f}$ 2-form

[401] Now consider d(2-form) :

[Note: this is the first Buck use of the symbol ω* :]

$\omega^* = f_1 dydz + f_2 dzdx + f_3 dxdy$ $\leftrightarrow \mathbf{f}$ 2-form

$d\omega^* = df_1 dydz + df_2 dzdx + df_3 dxdy$

$= (\partial_x f_1 dx + \partial_y f_1 dy + \partial_z f_1 dz)dydz + (\partial_y f_2 dy + \partial_z f_2 dz + \partial_x f_2 dx)dzdx + (\partial_z f_3 dz + \partial_x f_3 dx + \partial_y f_3 dy)dxdy$

$= (\partial_x f_1 dx)dydz + (\partial_y f_2 dy)dzdx + (\partial_z f_3 dz)dxdy$

$= (\partial_x f_1 + \partial_y f_2 + \partial_z f_3)dxdydz$ $\leftrightarrow$ div f $= \nabla \bullet \mathbf{f}$

To summarize this case

$\omega^* = f_1 dydz + f_2 dzdx + f_3 dxdy$ $\leftrightarrow \mathbf{f}$ 2-form

$\Rightarrow$ $d\omega^* = (\text{div f})dxdydz$ $\leftrightarrow \nabla \bullet \mathbf{f}$ 3-form

[402] **Theorem 3**: d(dω) = 0 for any k-form in E^n [Ara claims this as well]

Maybe ε notation will help. Start with <u>ω being a 1-form.</u>

$\omega = f_1 dx + f_2 dy + f_3 dz = f_i dx_i$ // implied sum on i $(\partial_i f)dx^i$ = scalar

$d\omega = (df_i)dx_i$

$df_i = \partial_x f_i dx + \partial_y f_i dy + \partial_z f_i dz = \partial_j f_i dx_j$ // sum only on j

Then

$d\omega = (\partial_j f_i dx_j)dx_i = (\partial_j f_i)dx_j dx_i$ // sum on i and j

$d(d\omega) = [d(\partial_j f_i)]dx_j dx_i = ?$

From above

$df_i = (\partial_k f_i)dx_k$ now left $f_i \rightarrow \partial_j f_i$

$d(\partial_j f_i) = (\partial_k \partial_j f_i)dx_k$

Then we have

$$d(d\omega) = [\ (\partial_k\partial_j f_i)dx_k\ ]dx_j dx_i \ = \ (\partial_k\partial_j f_i)\ dx_k dx_j dx_i$$

But now $(\partial_k\partial_j f_i)$ is sym on k↔j whereas $dx_k dx_j dx_i$ is antisym, so d(dω) = 0 QED.
[I think the above is MY proof only]

Addendum: We showed above that

$$d(d\omega) = \ (\partial_k\partial_j f_i)\ dx_k dx_j dx_i = 0$$

Claim as noted below that you can write

$$dx_k dx_j dx_i \ = \varepsilon_{kji}\, q \qquad\qquad q = (1/N!)\ \varepsilon_{kji}\ dx_k dx_j dx_i = dx_1 dx_2 dx_3 \text{ I guess}$$

Thus the above says

$$(\partial_k\partial_j f_i)\varepsilon_{kji}\ q = 0$$

or

$$q\ (\partial_k \varepsilon_{kji}\partial_j f_i) = 0$$

or

$$\partial_k \varepsilon_{kji}\partial_j f_i = 0$$

or

$$\partial_k\ [\ \text{curl}(\mathbf{f})\ ]_k = 0 \qquad \Leftrightarrow \qquad \text{div curl (f)} = 0. \qquad \text{A}$$

Suppose ω is a 0-form. Then

$$\omega = f \qquad\qquad \text{// in tensor doc, f = scalar}$$
$$d\omega = df = \ (\partial_j f)\ dx_j$$

$$d(d\omega) = \ [d(\partial_j f)]\ dx_j \ = \ [\ (\partial_k\partial_j f)dx_k]\ dx_j \ = 0 \text{ by the same sym/antisym rule}$$

Addendum. Consider the well known fact that curl(gradf) = 0. This says

$$[\text{curl}(\text{gradf})]_i = 0 \qquad \mathbf{V} = \text{gradf} \ = \nabla f \qquad [\text{curl}\ \mathbf{V}]_i = 0$$

$$\varepsilon_{ijk}\ \partial_j V_k = 0 \qquad \Rightarrow \qquad \varepsilon_{ijk}\ \partial_j\partial_k f \ = 0$$

Now according to something I did not too long ago, you can write

$$A_{ij} \ = \varepsilon_{ijk} Q_k$$

Where was this? Maybe in revamped tensor doc? Where did I carefully store my conclusions of that discussion? Just above (D.4) in tensor doc. OK then, write

$$dx^k dx^j = \varepsilon^{jki} q_i$$

Apply ε_{jkm} to both sides

$\varepsilon_{jkm}dx^k dx^j = (\varepsilon^{jki}\varepsilon_{jkm})q_i = 2!\, q_m$ $\qquad$ $q_m = (1/2)\varepsilon_{jkm}dx^k dx^j$

But just leave it as is so we can say (noting that $q_i \neq 0$)

$0 = d(d\omega) = (\partial_k\partial_j f)dx^k dx^j = (\partial_k\partial_j f)\,\varepsilon^{jki}q_i = q_i\,\varepsilon^{ijk}\,(\partial_j\partial_k f)$

$= q_i\,[\,\nabla \times \nabla f]_i = q \bullet [\,\nabla \times \nabla f\,]$ B

Somehow you argue that q is not perp to $\nabla \times \nabla f$ and you then conclude that $\nabla \times \nabla f = 0$.
So I guess I buy the claim that $d(d\omega) = 0$ for a 0-form tells you that $\nabla \times \nabla f = 0$.

Suppose ω is a 2-form. Then dω will be a 3-form (label functions as shown)

$\omega = f_{23}dydz + f_{31}dzdx + f_{12}dxdy = f_{23}dx_2dx_3 + f_{31}dx_3dx_1 + f_{12}dx_1dx_2 = f_{ij}dx_i dx_j$

Notice that the notation $f_{ij}dx_i dx_j$ works for E^n and not just E^3. The diagonal terms are formally present in the double sum, but they are in fact zero, so OK to include them. So $f_{ij}dx_i dx_j$ has n^2 terms, but only n(n-1) of these terms are non-zero and n are zero, the diagonal terms. Now

$\omega = f_{ij}dx_i dx_j$ // in tensor doc this would be $f_{ij}dx^i dx^j$ = scalar!

$d\omega = [d(f_{ij})]dx_i dx_j$

$d(f_{ij}) = (\partial_m f_{ij})\,dx_m$

$d\omega = (\partial_m f_{ij})dx_m dx_i dx_j$

$d(d\omega) = [d(\partial_m f_{ij})]dx_m dx_i dx_j$

$d(\partial_m f_{ij}) = (\partial_n\partial_m f_{ij})\,dx_n$

$d(d\omega) = (\partial_n\partial_m f_{ij})dx_n dx_m dx_i dx_j$ = 0 by symmetry

Suppose ω is a k-form. Then dω will be a (k+1)-form (label functions as shown) . This works the same as the 2-form above but you add lots of dots:

$\omega = f_{ij\ldots}\, dx_i dx_j \ldots\ldots$ // there are k dx factors and k summation indices and n^k terms
// of which many are zero.

$d\omega = [d(f_{ij}\ldots)]dx_i dx_j \ldots.$

$d(f_{ij}\ldots.) = (\partial_M f_{ij\ldots})\,dx_M$

$d\omega = (\partial_M f_{ij}...)dx_M dx_i dx_j...$

$d(d\omega) = [d(\partial_M f_{ij})]dx_M dx_i dx_j.....$

$d(\partial_M f_{ij}...) = (\partial_N \partial_M f_{ij}...)\, dx_N$

$d(d\omega) = (\partial_N \partial_M f_{ij}...)dx_N dx_M dx_i dx_j....$ = 0 by symmetry

I am happy with Theorem 3 and these facts:

d(dω) = 0 for 0-form	↔	curl grad f = 0	see B above
d(dω) = 0 for 1-form	↔	div curl (**f**) = 0	see A above

One wonders how to interpret the result for a 2-form?

Added 6.4.16, I have added all the above stuff as an Exercise starting with H.5.12.

Digression on the Multi-Index, the Ordered Multi-Index, and Counting Terms

This idea was somewhere in Stak and also appears on p 19 of Sjamaar's pdf. Write the above as

$\omega = f_{ij}... dx_i dx_j..... = f_I\, dx_I$ I = i,j,k..... I = multi-index

I think I did this informally in my Lagrange Appendix on determinants. Now, since each dx_i can be one of n choices in E^n, and since you can have k dx_i factors for a k-form, you have $\mathbf{n}^k$ **ways** to write out the dx_i product so you would say there at most n^k possible terms. But due to the minus sign rule, lots of these dx_I groups are either the same or 0. So how many unique terms are there?

Suppose the indices are i_1, i_2 .... i_k for the term $dx_{i1}dx_{i2}....dx_{ik}$. If you only want to count unique terms, you must restrict your count so that $1 \le i_1 < i_2 < i_3 .....< i_k \le n$. How many ways can you select a set of i_r so this is true? Think of the integers 1,2,3....n as n empty boxes. Think of the indices as k balls. You want to put the balls into the boxes, but at most one ball can go into any box. The number of ways to put the k index balls into the n boxes is the same as the number of ways to simply select k boxes from a group of n boxes, and that number is (n,k). For each selection of k boxes, you drop in your indices in increasing order left to right. The result (n,k) is confirmed by Sjamaar page 19.

For example, suppose n = 3 and k = 2. There are then (3,2) = 3 2-forms: dxdy, dxdz,dydz.

So you can rewrite the situation above keeping only unique terms by saying

$\omega = f_{ij}... dx_i dx_j..... = f_I\, dx_I$ I = i,j,k..... I = ordered multi-index

[402] **Theorem 4.** Consider this integral of a *general* 1-form situation.

$$\int_{\gamma} \omega = \int_{\gamma} F_i dx_i = \int_{\gamma} \mathbf{F} \bullet d\mathbf{r} \qquad \text{sum on i}$$

Define

$$F_T = \mathbf{F} \bullet d\mathbf{r}/dt = \mathbf{F} \bullet \mathbf{v} = \mathbf{F} \bullet \boldsymbol{\gamma}'(s)$$

Assume arc length ds is the same as the integration parameter, so $|\boldsymbol{\gamma}'(s)| = 1$. This means that $\boldsymbol{\gamma}'(s)$ is a unit vector, which let us call $\hat{\mathbf{v}}$. Then $F_T = \mathbf{F} \bullet \hat{\mathbf{v}}$ and we see that F_T is the projection of **F** along the curve trace, which means it is the component of F *tangent* to the curve trace, and hence it is called F_T for Tangent. Then

$$\boldsymbol{\gamma}'(s) = d\gamma/ds = d\mathbf{r}/ds = (\partial_s x, \partial_s y, \partial_s z) = \partial_s x\, \mathbf{i} + \partial_s y\, \mathbf{j} + \partial_s z\, \mathbf{k}$$

Then

$$F_T = \mathbf{F} \bullet (\partial_s x\, \mathbf{i} + \partial_s y\, \mathbf{j} + \partial_s z\, \mathbf{k}) = F_x\, \partial_s x + F_y\, \partial_s y + F_z\, \partial_s z = \mathbf{F} \bullet (\partial_s \mathbf{r}) = \mathbf{F} \bullet \hat{\mathbf{v}}$$

Now

$$\int_{\gamma} \omega = \int_{\gamma} F_i dx_i = \int_{\gamma} \mathbf{F} \bullet d\mathbf{r} = \int_0^1 ds\ \mathbf{F} \bullet (\partial_s \mathbf{r}) = \int_0^1 ds\, F_T$$

This says there exists a function (F_T) such that the line integral can in fact be written as an ordinary curve integral of the type considered in Section 7.1.

Theorem 5. Consider this integral of a 2-form situation. We seem to simply rederive a result found earlier, but which they did not state but I did state. So here is the Bucks' derivation of my fact

$$\int\int_{\Sigma} \omega = \int\int_{\Sigma} [\, A(\mathbf{r})\frac{\partial(y,z)}{\partial(u,v)} + B(\mathbf{r})\frac{\partial(z,x)}{\partial(u,v)} + C(\mathbf{r})\frac{\partial(x,y)}{\partial(u,v)}\,]dudv$$

$$= \int\int_{\Sigma} [\, A(\mathbf{r})dydz + B(\mathbf{r})dzdx + C(\mathbf{r})dxdy] = \int\int_{\Sigma} F_N\, dA$$

The scalar function F_N here is the *normal* component of **F** at each surface point. Again I think this is the simple area integral functional talked about in Section 7.1. So there exists a function F_N which makes this work, and it is shown as p 403 C. It is noted that

$\int\int_{\Sigma} F_N\, dA$ = mass flow across the surface Σ of $\mathbf{F} = \rho\mathbf{v}$ in fluid mechanics

$\int_{\gamma} F_T\, ds$ = measure of how well a fluid flows along path γ (circulation if closed path!)

This was a pretty hard section for me to chew down. // I have done some back and fill and wrote a doc about respeeding a curve.

Note added 5/23/16. I think the last items above are well covered in wedge doc!

7.4 Theorems of Green, Stokes and Gauss **[405]**

Here comes the big payoff of doing all the prep work above. These three very complicated theorems have absolutely trivial statements in the language of differential forms. But first, let's write them as I normally do. First, here is the general Stokes's Theorem which applies to any bounded surface (could be non-planar)

$$\int_S d\mathbf{S} \bullet (\nabla \times \mathbf{A}) = \oint d\boldsymbol{\ell} \bullet \mathbf{A} \qquad \textbf{(7)} \qquad \text{Schaum 22.60}$$

If the surface S lies in the xy plane, so does the boundary curve, and then we get "Green's Theorem":

$$\int_S dxdy\, (\partial_x A_y - \partial_y A_x) = \oint d\mathbf{s} \bullet \mathbf{A} = \int_{\partial S} [\, A_x dx + A_y dy]$$

The divergence theorem is this

$$\int_V dV\, \nabla \bullet \mathbf{A} = \int_S d\mathbf{S} \bullet \mathbf{A} = \int_S dS\, \hat{\mathbf{n}} \bullet \mathbf{A}$$

Now no k-forms appear in these three statements except for Green's Theorem where there is no real distinction between dx as a differential of x, and dx as a differential form. So let's restate these three theorems in the Buck order:

Green's Theorem: E^2 $\quad \int_S dxdy\, (\partial_x A_y - \partial_y A_x) = \oint d\mathbf{s} \bullet \mathbf{A} = \int_{\partial S} [\, A_x dx + A_y dy]$

Stokes's Theorem: E^n $\quad \int_S d\mathbf{S} \bullet (\nabla \times \mathbf{A}) = \oint d\mathbf{s} \bullet \mathbf{A}$

Gauss's Theorem: E^n $\quad \int_V dV\, \nabla \bullet \mathbf{A} = \int_S d\mathbf{S} \bullet \mathbf{A} = \int_S dS\, \hat{\mathbf{n}} \bullet \mathbf{A}$

Bucks are going to study these in the order shown.

[406] **Theorem 6:** Green's Theorem (see just above) with convex domain restriction.

$$\int_{\partial D} \omega \equiv \int_{\partial D} [\, A_x dx + A_y dy] = \int\int_D dxdy\, (\partial_x A_y - \partial_y A_x) = \int\int_D d\omega \qquad (*)$$

The domain is shown page 407 which I suppose is a general case with the restriction given. Notice the function g(x) on the top as γ_1 and f(x) on the bottom in the figure as γ_2. The proof first treats just the $A_x dx$ term which they call Adx. I am happy with p 407 E which consider an intermediate result for the case that $A_y = B = 0$. So they have now shown that $\int_{\partial D} \omega = E$ when B=0. Now work on RHS (*):

Using the same ω shown above, they compute dω and find, for this one Adx term, that

$d\omega = d(A_x dx + A_y dy) = d(A_x)dx + d(A_y)dy = [(A_x)_1 dx + (A_x)_2 dy]\, dx + [(A_y)_1 dx + (A_y)_2 dy]\, dy$

$= (A_x)_2 dy dx + (A_y)_1 dx dy = [\,(A_y)_1 - (A_x)_2]\, dx dy$

$= - (A_x)_2 dx dy$ // if we set $A_y = B = 0$ for the moment.

Pause: With the drawing on page 407 and related text, Bucks have shown directly that $\int_{\delta D}\omega = \int_D d\omega$ which is Stokes' theorem for the special case that ω is a 1-form and D has the special shape shown in the figure, so they have proven Stokes in a simple case, their purpose here. They do this proof by looking at the two terms in ω = Adx + Bdy separately and Stokes works for each term.

Digression:

Now what this really says is

$d\omega = - (A_x)_2\, dx \wedge dy$

but somehow Bucks treat dx ∧ dy as if it were an area differential dxdy. So something is missing here in my book. They are completely silent of course since their notation makes no distinction (!!). I need someone to talk about integration of a 2-form!

Look at **Arapura** page 16 which I quote

DEFINITION 11.1 *The integral of a 2-form on S is given by*

$$\iint_S F dx\wedge dy + G dy\wedge dz + H dz\wedge dx = \iint_D \left[F\frac{\partial(x,y)}{\partial(u,v)} + G\frac{\partial(y,z)}{\partial(u,v)} + H\frac{\partial(z,x)}{\partial(u,v)}\right] du dv$$

In practice, the integral of a 2-form can be calculated by first converting it to the form $f(u,v)du \wedge dv$, and then evaluating $\int\int_D f(u,v)\, dudv$.

Bucks have also developed this thing. In our case we have only the first term so it says

$$\int\int_S F\, dx \wedge dy = \int\int_D F \frac{\partial(x,y)}{\partial(u,v)} du dv = \int\int_S F\, dx\, dy$$

Here S is a surface in x,y,z space, and D is that same surface in u,v space. Now if we simply make the definitions u = x and y = v (since our surface S is in a plane!!), then the you get right form above, so THAT is how this little mystery is resolved!

So we then have (resuming notes on Buck here)

$$\int\int_D d\omega = \int\int_D [-\ (A_x)_2\ dx \wedge dy] = -\int\int_D (A_x)_2 dxdy$$

$$= \int_a^b dx \int_{f(x)}^{g(x)} dy\ (A_x)_2 \quad = \text{page 408 C.}$$

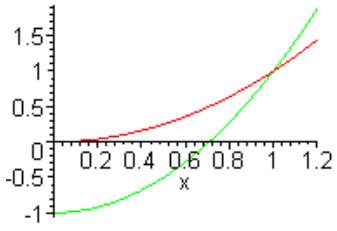
But since page 408 C = p 407 E, they have shown that

$$\int_{\partial D} \omega = \int\int_D d\omega$$ for a convex region as shown page 407, for the A_x term

One can repeat the derivation for the A_y term and then you have proven Theorem 6 QED.
This seems a perfectly reasonable proof to me, with my little added item.

Comment: This is where I first encountered the mystery that dx ∧ dy → dxdy calculus.

Bucks next point out that the method of the page 407 figure does not require convex, it requires that you be able to describe the area by four bounding functions as on page. They then give a little example where you do have four bounding functions but region is not convex



// non-convex region D example

Such a boundable-by-four-functions region is called a **standard region**. The fancy with-hole page 409 picture is partitioned so as to be a union of these standard regions. They do not worry about proving when this is or is not possible. In the union process, the area integrals add and only the boundary curves are left, so they have thus generalized Theorem 6 to such a fancy region.

[408 bottom] **What happens to differential forms under a transformation**?

Digression: explanation of Buck's pullback notation **********

Bucks present a treatment of this subject without talking about pullbacks. Now that I have read about that somewhat in Sjamaar, I am going to add some notes interspersed into the following earlier notes.

1-forms under transformations

First consider only the world of 1-forms, where

$\omega \equiv A_1 dx_1 + A_2 dx_2 = \mathbf{A(x)} \bullet d\mathbf{x}$ $\quad\quad$ $\mathbf{x} = (x,y)$ $\quad\quad$ $\mathbf{x}' = (u,v)$

I could "transform" this thing from x-space to x'-space in the sense of tensor doc, assuming that the thing A is a tensorial vector.

$\omega' \equiv A'_1 dx'_1 + A'_2 dx'_2 = \mathbf{A'(x')} \bullet d\mathbf{x'}$ $\quad\quad$ // using tensor doc notation

But this is NOT what Bucks mean by ω*. Here is what they mean:

$\omega = \mathbf{A(x)} \bullet d\mathbf{x}$

$\omega^* \equiv \mathbf{A(x(x'))} \bullet d\mathbf{x}(\mathbf{x}') = \mathbf{A^*(x')} \bullet d\mathbf{x}$ $\quad\quad$ (***)

They just want you to EXPRESS ω in terms of **x**' instead of **x**. Since the functional form changes when you do this with **A**, they write the new functional form as **A***, so that is then the meaning of **A***.

Translate:

$\omega = \mathbf{A(x')} \bullet dx'$

$F^*(\omega) = \mathbf{A}(F(\mathbf{x})) \bullet F^*(dx') = \mathbf{A}(F(\mathbf{x})) \bullet [R\, dx] = \mathbf{A}(F(\mathbf{x})) \bullet dF$ wedge doc (10.8.16) , (10.8.20)

Sjamaar. For him, the transformation is from y-space on the right (where ω = α lives) to x-space on the left (where φ*α lives). In my notation above, I instead have x-space on the right, and I have x'-space on the left. The Sjamaar transformation might then be $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{x}')$ [instead of $\mathbf{y} = \varphi(\mathbf{x})$]. The pullback would then be $\varphi^*\alpha = \varphi^*\omega = \varphi^*(\mathbf{A}) \bullet \varphi^*(d\mathbf{x}) = \mathbf{A}(\boldsymbol{\varphi}(\mathbf{x}')) \bullet d\boldsymbol{\varphi}$. This is my (****) where I write $\mathbf{x} = \mathbf{x}(\mathbf{x}')$

Conclusion: **Bucks' notation is that ω* = φ*ω for an implied transformation φ !** On page 409 A they have a transformation $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{u})$ [so **u** is space on left, **x** space on right] for example. In this case, for a 0-form (a function) you make the transformation as $f^*(\mathbf{u}) = f(\varphi(\mathbf{u}))$ which Sja would call $\varphi^* f = f(\varphi(\mathbf{u}))$. Good! For a 1-form Bucks say $\omega = \Sigma_i A_i(\mathbf{x}) dx_i$. Sja would say

$\varphi^*\omega = \Sigma_i\, \varphi^*(A_i)\, \varphi^*(dx_i) = \Sigma_i\, A^*_i(\mathbf{u})\, d\varphi_i = \Sigma_i\, A^*_i(\mathbf{u})\, (\partial\varphi_i/\partial u_j)\, du_j$

and this is exactly what you see happening in p 409 C,D,E. Bucks call φ*ω by name ω*, so the transformation name is then suppressed. So I have made contact. Here is more evidence on page 410

Bucks 410: $(d\omega)^* = d(\omega^*)$

Sjamaar p39: $\varphi^*(d\alpha) = d(\varphi^*\alpha)$

So you now see how Bucks are translating things: φ*(dα) is written as just (dα)*.

So finally I have an understanding of the Buck's notation ω* ! It is just Sjamaar's φ*ω with the transformation name implicit, and the * moved to the right.

Now I may finally have an interpretation of Buck (7.31)A:

$$\int_{\gamma} \omega^* = \int_{\gamma^*} \omega \qquad \omega = \text{a 1-form}$$

How recall Sjamaar:

$$\int_{[a,b]} c^*\alpha = \int_c \alpha$$

In Buck notation this would read

$$\int_{[a,b]} \omega^* = \int_c \omega$$

Now in the Buck discussion, γ is a curve which bounds a region in u-space (space on the left), so it is like the interval [a,b] in the Sjamaar discussion. The Buck curve γ* is the forward mapping of this γ curve into the space on the right, and this is what Sja calls c. So here is the path from Sja to Bucks:

$$\int_{[a,b]} c^*\alpha = \int_c \alpha \qquad \text{// Sjamaar pullback of a curve}$$

$$\int_{[a,b]} \omega^* = \int_c \omega \qquad \text{// Buck notation}$$

$$\int_{\gamma} \omega^* = \int_{\gamma^*} \omega \qquad \text{// Buck curve names}$$

So at least things are getting better. True, Sjamaar has not yet mapped two arbitrary curves, but I could do that myself, or he will be doing it soon.

<u>0-forms under transformations</u>

$\omega = f(\mathbf{x})$

$\omega^* = f(\mathbf{x}(\mathbf{x}')) = f^*(\mathbf{x}')$ p 411 A

Again, you just EXPRESS ω in terms of the x' coordinates and keep track of the functional form with f*.

[still inside digression here]

Example #1. Compute ω* for an arbitrary 1-form ω: [done on page 409]

$\omega = \mathbf{A(x)} \bullet d\mathbf{x} = A(x,y)dx + B(x,y)dy$ page 409 B1

$\omega^* = \mathbf{A^*(x')} \bullet d\mathbf{x} = A^*(u,v)\, dx + B^*(u,v)\, dy$ page 409 C where $dx = d\varphi$ and $dy = d\psi$

Now we can still call upon tensor doc in this sense:

$$\begin{array}{llll} d\mathbf{x}' = R(\mathbf{x})\, d\mathbf{x} & R_{ik}(\mathbf{x}) \equiv (\partial x'_i/\partial x_k) & R = S^{-1} & // \; dx'_i = R_{ij}\, dx_j \\ d\mathbf{x} = S(\mathbf{x}')\, d\mathbf{x}' & S_{ik}(\mathbf{x}') \equiv (\partial x_i/\partial x'_k) & S = R^{-1} & // \; dx_i = S_{ij}\, dx'_j \,. \end{array} \qquad (2.1.6)$$

So then [see p 409 A]

$S_{11} = \partial_u x = \varphi_1 \qquad S_{21} = \partial_u y = \psi_1 \qquad S_{12} = \partial_v x = \varphi_2 \qquad S_{22} = \partial_v y = \psi_2$

Then we can write

$d\mathbf{x} = S d\mathbf{x}' \qquad dx_1 = S_{11} dx'_1 + S_{12}\, dx'_2 = \varphi_1 dx'_1 + \varphi_2\, dx'_2 = \varphi_1 du + \varphi_2\, dv = dx$

Then you get

$\omega^* = \mathbf{A^*(x')} \bullet d\mathbf{x} = A^*(u,v)\, dx + B^*(u,v)\, dy$

$= A^*(u,v)\, [\varphi_1 du + \varphi_2\, dv] + B^*(u,v)\, [\psi_1 du + \psi_2\, dv]$ page 409 D

$= [A^*\, \varphi_1 + B^*\, \psi_1]\, du + [A^*\, \varphi_2 + B^*\, \psi_2]\, dv$ page 409 E

Example #2. Compute ω* for an arbitrary 0-form ω: [done on page 411]

$\omega = f(\mathbf{x})$

$\omega^* = f(\mathbf{x}(\mathbf{x}')) = f^*(\mathbf{x}')$ p 411 A

Example #3. Compute dω and (dω)* for the above Example #2 (each of these is a 1-form)

$d\omega = df(x) = \partial_i f\, dx_i = f_i\, dx_i$

$(d\omega)^* = f_i(\mathbf{x}(\mathbf{x}'))\, dx_i(\mathbf{x}') = f_i\, dx_i = \mathbf{f} \bullet d\mathbf{x} = f_i\, [S dx']_i$

$= f_i\, S_{ij} dx'_j \,.$

Now assume specifically that the transformation of interest is $(u,v) \rightarrow (x,y,z)$ so $E^2 \rightarrow E^2$ as stated in page 411 F. Then we have $S_{ik} = \partial x_i/\partial x'_k$ so,

$S_{11} = \partial_u x = \varphi_1$ $\quad S_{21} = \partial_u y = \psi_1$ $\quad S_{31} = \partial_u z = \theta_1$

$S_{12} = \partial_v x = \varphi_2$ $\quad S_{22} = \partial_v y = \psi_2$ $\quad S_{32} = \partial_v z = \theta_2$

and then

$(d\omega)^* = f_1[S_{11}dx'_1 + S_{12}dx'_2] + f_2[S_{21}dx'_1 + S_{22}dx'_2] + f_3[S_{31}dx'_1 + S_{32}dx'_2]$

$= f_1[\varphi_1 dx'_1 + \varphi_2 dx'_2] + f_2[\psi_1 dx'_1 + \psi_2 dx'_2] + f_3[\theta_1 dx'_1 + \theta_2 dx'_2]$

$= f_1[\varphi_1 du + \varphi_2 dv] + f_2[\psi_1 du + \psi_2 dv] + f_3[\theta_1 du + \theta_2 dv]$ page 411 D1

$= [f_1\varphi_1 + f_2\psi_1 + f_3\theta_1]\, du + [f_1\varphi_2 + f_2\psi_2 + f_3\theta_2]\, dv$ page 411 D2

Example #4: Show that $(d\omega)^* = d(\omega^*)$ for any 0-form.

$\omega = f(\mathbf{x})$

$\omega^* = f(\mathbf{x}(\mathbf{x}')) = f^*(\mathbf{x}')$

$d(\omega^*) = df(\mathbf{x}(\mathbf{x}')) = f_i\, \partial x_i/\partial x'_j\, dx'_j = f_i\, S_{ij}\, dx'_j$

But in Example #3 we shows that

$(d\omega)^* = f_i\, S_{ij} dx'_j$

and thus we have $d(\omega^*) = (d\omega)^*$ for any 0-form ω.

2-forms under transformations

$\omega = A_{ij}(\mathbf{x})\, dx_i dx_j$

$\omega^* = A_{ij}(\mathbf{x}(\mathbf{x}'))\, dx_i dx_j = A^*_{ij}(\mathbf{x}')\, dx_i dx_j$

Example #5. Show that $(d\omega)^* = d(\omega^*)$ for ω = any 1-form.

Start with

$\omega = \mathbf{A(x)} \bullet d\mathbf{x} = A_i dx_i$

$d\omega = dA_i\, dx_i = \partial_j A_i\, dx_j\, dx_i = (A_i)_j\, dx_j dx_i$ // second sub = derivative

$(d\omega)^* = (A_i)_j(\mathbf{x}(\mathbf{x}'))\, dx_j dx_i = (A_i)_j\, dx_j dx_i$ // dependences not shown

Meanwhile,

$\omega^* = \mathbf{A(x(x'))} \bullet d\mathbf{x} = A_i(\mathbf{x}(\mathbf{x}'))\, dx_i$

$d(\omega^*) = dA_i(\mathbf{x}(\mathbf{x}'))\, dx_i = (A_i)_j\, dx_j\, dx_i$

Therefore for any 1-form we have that $(d\omega)^* = d(\omega^*)$.

Comment: since the * operator simply changes the way you EXPRESS a result, the result does not seem too surprising to me.

k-forms under transformations

$\omega = A_{ijk..}(\mathbf{x})\, dx_i dx_j dx_k ...$

$\omega^* = A_{ijk..}(\mathbf{x(x')})\, dx_i dx_j dx_k ...$

Example #6. Show that $(d\omega)^* = d(\omega^*)$ for ω = any k-form for $k \geq 3$:

$\omega = A_{ijk..}(\mathbf{x})\, dx_i dx_j dx_k ...$

$d\omega = dA_{ijk..}(\mathbf{x})\, dx_i dx_j dx_k ... = (A_{ijk..})_I\, dx_I\, dx_i dx_j dx_k ...$

Meanwhile,

$\omega^* = A_{ijk..}(\mathbf{x(x')})\, dx_i dx_j dx_k ...$

$d(\omega^*) = [dA_{ijk..}(\mathbf{x(x')})]\, dx_i dx_j dx_k ... = (A_{ijk..})_I dx_I dx_i dx_j dx_k ...$

Therefore for any k-form with $k \geq 3$ we have that $(d\omega)^* = d(\omega^*)$.

End Digression **********

We now examine the claims of (7-31) using the ω^* definitions given above

Dealing with (7-31A) on page 410

Buck Chapter 7 suddenly grinds to a full stop for me with this equation. I have been stuck for 2-3 days at this point and cannot move forward. [this is an original red note on date of original reading]

They claim the equation "follows at once" from Section 6.1, but I see no connection whatsoever between the subject here and that Section. I am unclear as to the *meaning* of integrals of 1-form ω.

Here are the two objects which Bucks claim are obviously equal in their (7-31A)

(1) $\int_{\gamma^*} \omega = \int_{\gamma^*} \mathbf{A(x)} \bullet d\mathbf{x}$ $\qquad \mathbf{x} = \boldsymbol{\varphi}(\mathbf{u})$

(2) $\int_Y [\omega^*] = \int_Y [\mathbf{A}(\mathbf{x}(\mathbf{x}')) \bullet d\mathbf{x}(\mathbf{x}')] = \int_Y [\mathbf{A}(\boldsymbol{\varphi}(\mathbf{u})) \bullet d\boldsymbol{\varphi}]$

Here I am injecting my conjecture (***) above that ω* means you just express ω in terms of the x' [**u**] variables. That conjecture has "worked" for all the examples I did above. [ah, but ω* means two things, you have to worry about the function coefficients AND action on the differential part]

$\alpha = \Sigma_i f_i(\mathbf{x})\, dx_i \qquad \varphi^*\alpha = \Sigma_i f_i(\varphi(\mathbf{u}))\, d\varphi_i = \Sigma_{ij} f_i(\varphi(\mathbf{u}))\, (\partial\varphi_i/\partial u_j) du_j \qquad \mathbf{x} = \boldsymbol{\varphi}(\mathbf{u})$

$\omega = \Sigma_i f_i(\mathbf{x})\, dx_i \qquad \omega^* = \Sigma_i f_i(\varphi(\mathbf{u}))\, d\varphi_i = \Sigma_{ij} f_i(\varphi(\mathbf{u}))\, (\partial\varphi_i/\partial u_j) du_j \qquad \mathbf{x} = \boldsymbol{\varphi}(\mathbf{u})$

[But I guess my original expression in (2) above is OK, except that dx is not a function.]

Let's start with trying to understand the simpler item (1). In my Section 7.2 notes above I have this claim

$$\int_Y \omega = \int_{t_1}^{t_2} [\, A(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{x}}(t) + B(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{y}}(t)]\, dt = \int_Y [\, A(\mathbf{r})dx + B(\mathbf{r})dy\,]$$

I think I could then apply this to item (1) replacing γ→γ* to get

$$\int_{Y^*} \omega = \int_{t_1}^{t_2} [\, A(\boldsymbol{\gamma}^*(t))\gamma^{*\prime}_{\mathbf{x}}(t) + B(\boldsymbol{\gamma}^*(t))\gamma^{*\prime}_{\mathbf{y}}(t)]\, dt$$

So if you hand me a curve γ*(t), I know what everything means in the above expression and so I can in theory compute the integral. So I understand "the meaning" of item (1), at least in E^2 as written.

I am at a complete loss right now for how to compute integral (2). Offhand, it seems I can just delete the x' sub-arguments and then I get (2) = $\int_Y \mathbf{A}(\mathbf{x}) \bullet d\mathbf{x}$, but this is then not the same as (1) as they are claiming.

Fact: **I need another source** which talks in more detail about (7-31)

(1) I cannot understand Bucks' claim as to why it is true

(2) I am unable to derive it myself

****

Notes below cannot be trusted since I am blocked above.

Theorem PL3: Next, consider

$$\int\int_D (d\omega)^* = \int\int_D [\, (\partial'_j \mathbf{A}'(\mathbf{x}'(\mathbf{x}))\, dx'_j(\mathbf{x})] \bullet d\mathbf{x} \qquad \text{// one of my "two interp ways" above}$$

$$= \int\int_{D'} [\, (\partial'_j \mathbf{A}'(\mathbf{x}'))\, dx'_j] \bullet d\mathbf{x}' \qquad \text{// run } dx'_j dx'_k \text{ over D' instead of D}$$

$$= \int\int_{D'} [\,(\partial_j \mathbf{A}(\mathbf{x}))\, dx_j] \bullet d\mathbf{x} \qquad \text{// dot product is scalar}$$

$$= \int\int_{D'} (d\omega) \qquad \text{// my interp above}$$

$$= \int\int_{D^*} d\omega \qquad \text{// their name for D'}$$

The above is then my proof of (7-31) B. But there are some subtleties here! Consider

$\mathbf{V}'(\mathbf{x}') = (\partial'_j \mathbf{A}'(\mathbf{x}'))\, dx'_j$

$V'_k(x') = (\partial'_j A'_k(\mathbf{x}'))\, dx'_j = (\partial'_j A'_k) dx'^j$ = component k of a vector in x'-space

So our integrand is

$[\,(\partial'_j \mathbf{A}'(\mathbf{x}'))\, dx'_j] \bullet d\mathbf{x}' = V'_k(x')\, dx'^k$ = a true scalar for any transformation!

$= \mathbf{V}'(\mathbf{x}') \bullet d\mathbf{x}'$

$= \mathbf{V}(\mathbf{x}) \bullet d\mathbf{x}$ // since a true scalar

This occurs in the first proof as well where we have

$[\mathbf{A}'(\mathbf{x}') \bullet d\mathbf{x}'\,] = A'_k\, dx'^k$

but in order to really see this, you have to use covariant notation with up and down indices. I can see why Buck's don't want to get into that right here. Does Arapura address this stuff? The answer is NO. I would have to find some other source. I just found a pretty good source that talks about more advanced things like pullbacks and I think that does relate to the above, but let's not wander off right now!!

Status: I will go with my little derivations of page 410 A and B (7-31). These were hugely non-obvious and Bucks are a little dishonest in their presentation I would say. But let's now move on. The proof of Theorem 7 is now clear:

Resume notes on Buck

Theorem 7: If Green's Theorem (in the plane) holds in x-space, it also holds in x'-space.

Note that this is for an arbitrary transformation they call T, and I call **F**. Perhaps this would have some implication for curvilinear coordinates. So far we don't have much motivation for why Theorem 7 is important.

[411] **Theorem 8:** Proof that $(d\omega)^* = d(\omega^*)$ // see translation table below, * is pullback

This is a 3-page proof !!! But I think I did my own proof above as Theorem PL1. I wonder if there is any connection between our proofs? Mine is probably no good. My proof was only for 1-forms!

0-forms.

$\omega = f(\mathbf{x})$ $\quad d\mathbf{x} = Sd\mathbf{x}'$ contravariant

$\omega^* = f'(\mathbf{x}') = f(\mathbf{x}(\mathbf{x}'))$ // seems only possibility // same as page 411 A

$d[f(\mathbf{x}(\mathbf{x}'))] = \partial_i f \; dx_i = \partial'_i f' \; dx'_i$ // your choice, both ways are right

$d(\omega^*) = d[f(\mathbf{x}(\mathbf{x}'))] = \partial_i f \; dx_i = \nabla f \bullet d\mathbf{x} = \nabla f \bullet [Sd\mathbf{x}']$ // probably what B says

Let's try to explicitly confirm that the above agrees with p 411 B: the following based on p 411 F:

$S_{11} = \partial_u x = \varphi_1$ $\quad S_{21} = \partial_u y = \psi_1$ $\quad S_{31} = \partial_u z = \theta_1$
$S_{12} = \partial_v x = \varphi_2$ $\quad S_{22} = \partial_v y = \psi_2$ $\quad S_{32} = \partial_v z = \theta_2$

Then

$$d(\omega^*) = \nabla f \bullet [Sd\mathbf{x}'] = \Sigma_{i=1}^{3} \partial_i f \, \Sigma_{j=1}^{2} S_{ij} \, dx'_j = \Sigma_{i=1}^{3} f_i \, \Sigma_{j=1}^{2} S_{ij} \, dx'_j$$

$$= f_1 [S_{11}dx'_1 + S_{12}dx'_2] + f_2 [S_{21}dx'_1 + S_{22}dx'_2] + f_3 [S_{31}dx'_1 + S_{32}dx'_2]$$

$$= [f_1 S_{11} + f_2 S_{21} + f_3 S_{31}] \, dx'_1 + [f_1 S_{12} + f_2 S_{22} + f_3 S_{32}] \, dx'_2$$

$$= [f_1\varphi_1 + f_2\psi_1 + f_3\theta_1] \, du + [f_1\varphi_2 + f_2\psi_2 + f_3\theta_2] \, dv$$

Digression 5/23/16.

The Bucks have made a valiant effort on Stoke's related stuff, but it is just impossible for me to get happy with their limited notation where I can never see what is a wedge product and what is not.

Reading Buck, I got completely hung up on page 410 (7-31). But now I understand that each of these is the same statement that you evaluate a form integral by doing its pullback integral. $F^*(\omega) = [\omega \; \mathcal{R}]$. Bucks don't say $F^*(\omega)$, they say ω^* . This is very confusing because on page 400 they used ω^* to represent the Hodge thing! Maybe that is what messed me up:

	me	Bucks	
Hodge	$*\alpha$	ω^*	p 400 bottom
pullback	$F^*(\alpha)$	ω^*	p 410 7-31

In the latter case, Bucks are using blank-space and *-space the way I use x'-space and x-space.

$$\int_{S'} \alpha_{\mathbf{x}'} \equiv \int_S F^*(\alpha_{\mathbf{x}'}) = \int_S \beta_{\mathbf{x}} \qquad \text{ME} \qquad (10.11.2)$$

$$\beta_{\mathbf{x}} \equiv F^*(\alpha_{\mathbf{x}'}) = \Sigma'_J\, g_J(\mathbf{x})\, dx^J \quad \text{where} \quad g_J(\mathbf{x}) = \Sigma'_I\, f_I(\mathbf{F}(\mathbf{x}))\, \det(R^I{}_J)\,. \qquad (10.11.1)$$

$$\int_{Y^*} \omega = \int_Y \omega^* \qquad \text{BUCKS}$$

Again I have x'-space on the left and x-space on the right.
Bucks have * space on the left and no-* space on the right.

But they use * as both the space label and the pullback operator symbol, VERY confusing to a reader.

My approach bases everything on the underlying transformation, and a pullback is tied to such a transformation. I think that is the point I try to get across in my pullback chapter. Bucks do not mention the word pullback, they are writing in 1956 and 1965.

Probably later books since 1965 take more my approach, but maybe my emphasis is unique in the literature, but I am not really interested in that role. I am interested in clean exposition to myself of this topic, and that is what wedge doc is all about. (Bernie: What this campaign is about is ....).

To their credit also, the Bucks do hit on all the key ideas like $d^2 = 0$ and F* and d commutating (7-30). They are just doing what I am doing in Chapter 10 but I have more notational power.

Let's look now at some other Buck equations

Buck	me
7.31 pullbacks	noted above
$(d\omega)^* = d(\omega^*)$	$F^*(d\alpha) = dF^*(\alpha)$ (10.7.22)
$(\alpha\beta)^* = \alpha^*\beta^*$	$F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta)$ (10.7.21)
$d(\alpha\beta) = (d\alpha)\beta + \ldots$	$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$ (10.3.27)

Bucks page 414-421 are doing the various Stokes' Theorem cases, curl, div and all that. It appears that I did not read these pages and won't now because that stuff is old hat to me.

I should read the rest of this chapter starting on 410 more or less where I hit the wall. All in good time.

Resume Buck notes on 5/24/16 starting at black square page 413 (end of long Theorem 8 proof) I note in passing that on page 413 before the black square that the Bucks have stated two of my and Sjamaar's general theorems, but in Buck's special notation. I have not studied this long proof because I have my own proof in wedge doc which seems very simple.

Theorem 9: (R^2) [pullback of a 2-form] This theorem states p 414 A. I show that this is just

$$\alpha_{\mathbf{x}} = f_{12}(x)\, dx^1 \wedge dx^2$$

$$\varphi^*(\alpha_{\mathbf{x}}) = f_{12}(\boldsymbol{\varphi}(\mathbf{t}))\, \frac{\partial(\varphi^1, \varphi^2)}{\partial(t^1, t^2)}\, dt^1 \wedge dt^2 \qquad \text{wedge doc 10.13.5}$$

with appropriate integrals added. The area case is then shown in B, all is well. They are pulling back a 2-form here. I treat this in wedge doc in Section 10.13 (integration of 2-forms) where I state results for R^3:

$$\alpha_{\mathbf{x}} = \Sigma_{1\le i_1<i_2\le 3} f_{i_1 i_2}(\mathbf{x})\, {}_{\mathbf{x}}\lambda^{\mathbf{I}} = \Sigma_{1\le i_1<i_2\le 3} f_{i_1 i_2}(\mathbf{x})\, dx^{i_1} \wedge dx^{i_2}$$

$$\varphi^*(\alpha_{\mathbf{x}}) = \Sigma_{1\le i_1<i_2\le 3} f_{i_1 i_2}(\boldsymbol{\varphi}(\mathbf{t}))\, \frac{\partial(\varphi^{i_1}, \varphi^{i_2})}{\partial(t^1, t^2)}\, dt^1 \wedge dt^2$$

If you restrict this to R^2 there is only the one terms shown above. My statement of the integral is

$$\int_{\varphi} \alpha_{\mathbf{x}} \equiv \int_{[0,1]^2} \varphi^*(\alpha) = \int_{[0,1]^2} G(\mathbf{t})\, dt^1 \wedge dt^2 \,. \qquad (10.13.7)$$

where I use the 2-cube instead of the general region they use.

Theorem 10: (R^3) [Stokes' theorem for a 1-form : traditional Stokes' theorem with curl.] This theorem states p 414 C. I treat this in Section H.5 on the curl and I show there that

$$\alpha = \Sigma_j F_j\, dx^j = \mathbf{F} \bullet d\boldsymbol{x} \qquad \text{// 1-form} \qquad (H.5.1)$$

$$d\alpha = \Sigma_{i<j} (\partial_i F_j - \partial_j F_i)\, dx^i \wedge dx^j \qquad \text{// (10.3.24b), } d\alpha \text{ written in standard form} \qquad (H.5.2)$$

$$\int_{\mathbf{M}} d\alpha = \int_{\partial \mathbf{M}} \alpha$$

$$\int_{\mathbf{M}} \Sigma_{i<j} (\partial_i F_j - \partial_j F_i)\, dx^i \wedge dx^j = \int_{\partial \mathbf{M}} \mathbf{F} \bullet d\boldsymbol{x} \qquad (H.5.5)$$

The last line H.5.5 is in fact Bucks' theorem 10 page 414 C. Bucks prove Theorem 10 by using the generalized Stokes with pullback then a push forward, I am content without their details.

[415] They comment that it is OK for surface to have **holes**, as long as you include their boundaries in your Stokes analysis, just like the outer boundary. Nice picture.

[416] Comment on **Mobius** that boundaries don't cancel right so things don't work. Non-orientable. Picture bottom of page 416.

Theorem 11: [Stokes' theorem for a 2-form : the divergence theorem] In Appendix H.4 I have

$$\alpha = F_i\, dx^i = \mathbf{F} \bullet d\boldsymbol{x}$$

$$*\alpha = F_i\, (*dx^i) = \mathbf{F} \bullet *d\boldsymbol{x} = \mathbf{F} \bullet d\boldsymbol{A} \qquad \text{// this is Bucks' } \omega \text{ in page 416 A}$$

$$\int_{\mathbf{M}} d(*\alpha) = \int_{\partial \mathbf{M}} (*\alpha)$$

$$\int_{\mathbf{M}} \operatorname{div} \mathbf{F}\ dV = \int_{\partial \mathbf{M}} \mathbf{F} \bullet dA\ . \qquad \text{(H.4.6)}$$

So I am applying Stokes' general theorem to ω = *α.

The Bucks prove Theorem 11 in part by using a cube and doing just the first term ω = Adydz. On page 417 they first do the volume integral and then they do the surface integral, and they show that the two are equal, so for the cube and for this one term the divergence theorem is proved.

Comment: Today is 5/24/16 and I am now reading starting at page 418 for the very first time!

[418-421] Here Bucks write the divergence theorem and Stoke's traditional theorem in vector notation, and then proceed to give the usual physical interpretations of these theorems. They say all the right things using mass flow for the divergence theorem in R^3 and then they talk about fluid curl with an attempted drawing on page 420.

Comment: They are mixing diffop vector analysis with differential forms analysis here, it is a bit confusing. For me, the notions of the divergence theorem and Stokes' theorem exist totally devoid of differential forms. It just seems confusing to mix them together as they do, but at least this gives them a place in the book to present these two theorems. The chapter 7 title is "differential geometry and vector calculus". It is true that Stokes' theorem generalizes these R^3 theorems to n dimensions and to manifolds and all that stuff. I will grant that this section 7.4 on the Stokes family of theorems got me motivated to write wedge doc.

Section 7.5 Independence of path, forms that are exact and closed. original reading 5/24/16

1-form ω is exact means ω = df where f is a function.

Theorem 13: [423] If ω is exact, then a line integral is *path independent* since it depends only on the endpoints. I have this in wedge doc somewhere. And if ω is exact, closed line integral is zero. (Corollary) This requires that there be no holes so simply connected or convex will do I suspect.

Theorem 14: [424] Exact ⇒ Closed *and* exactness can be stated in R^2 using Cauchy Riemann (maybe...) it looks like with some similar statement for R^3.

Theorem 15: [425] Closed ⇒ Exact in a spherical region! Bucks only showing this for ω = 1-form. They do it by starting with ω and actually constructing a function f such that ω = df. This function f is a certain integral from the origin to a point using three segments as in drawing page 426. I guess you need the spherical region so you can get to any point in this manner from the origin.

Corollary: [426] If you consider a local open region of a point and put a ball around that point, then Closed ⇒ Exact in that little ball.

The **Angle Form Example** [426] where dω = 0 so closed: Let D be an annular ring around the origin. Line integral around a path in the ring of the angle form ω gives 2π. Thus we don't have closed line integral vanish, thus ω is not exact, though it is locally exact. The Bucks will attribute this to the fact that

the domain of this annular ring has a hole in it, a slightly different view than my Poincare. Also the region is not convex.

Theorem 16: [427] If ω integral is path independent in an open connected set, then ω is exact in that set. This allows for a non-convex set. If there were a hole, you would not have path independence of the integral, so this theorem then does not apply. This theorem is arrow α in fig on page 428.

Definition. Imagine a surface with no hole and pick any closed loop on this surface. That loop is the entire boundary of a new smaller orientable surface. The surface is then **simply connected**. If surface has a hole, then pick a loop around the hole. That loop is NOT the boundary of a surface, it is the boundary of two different surfaces, the inside one and the outside one. So not simply connected.

Alternate: every closed curve can be contracted to a point means simply connected. Otherwise the surface is **multiply connected**.

Theorem 17: [427] If ω is exact in a simply connected region, then it is closed. Angle form has that point at the origin. This is a little like the star thing, contractible to a point.

Corollary [429] Local exactness implies exactness if region is simply connected.

Bucks then show that if ω = df is exact, you an compute f by integrating over any path you want. They give an example ω and compute the integral three ways and get the same f, picture top page 430.

Suddenly they change topics to PDE's written in terms of forms. So for example ω = Adx +Bdy = 0 is a "differential equation" in this language. If you can find a function g such that (gω) = df, meaning then that gω is exact, g is an "integrating function", and you can then integrate to find f. If you then look at the equation d(gω) = 0, that will be a PDE for the function g where A and B are givens. In 2D claim is that if functions are reasonable, such a g always exists, so that PDE always has a solution.

What about in 3D with **Pfaffian** equation ω = 0 (1-form) as shown bottom page 430? In this case the integrating factor g might not exist. Bucks then come up with a necessary condition that must be met by the coefficients A,B,C so that a factor g exists. It is p 431 A. So ends this little digression into PDE theory and is relation to forms being exact.

holding here to work on car pre-Torrey. // That was 5/24/16 and resuming here now on 6/4/16

Theorem 18: [431] When can you represent a field **F** as **F** = ∇φ (**potential** φ) ? One thing you know is that this can only be true for an irrotational field curl **F** = 0 because curl ∇φ = 0. If curl **F** ≠ 0, then representation **F** = ∇φ would be impossible since that forces curl **F** = 0. You can find φ by doing line integrals of **F** to various points **x** from some fixed point **a**, but domain must have no holes so paths are unique. You can identify curl **F** = 0 with F being **conservative**. Examples are irrotational flow velocity **V** and electrostatic field **E**. The thing -φ is the **work done**. The usual stuff.

Definition: Exactness for a 2-form σ: σ = dω (earlier Bucks did 1-form).

Theorem 19 [432] . Integral of *exact* 2-form σ over surface tied down at boundary is same for any such surface, it does not depend on the surface shape. This is the flux business. Integral over closed surface gives 0. This is similar to exact 1-form having same integral over any path tied down at the endpoints.

Theorem 20 [432]. Given the 2-form shown, you compute dσ as shown, and then if σ = dω is exact, then σ is also closed then dσ = 0, and this produces the little equation shown.

Theorem 21 [433] If σ is closed on a convex region, then it is exact in that region. This is the Poincare direction.

Theorem 22 [434] If div **F** = 0, then you can represent **F** as **F** = curl **V** . Such an **F** is solenoidal. The magnetic field is an example, and **B** = curl **A** implies that div **B** = 0.

And so on 6.4.16 my long voyage through Buck Chapter 7 comes to an end! It took me a full YEAR to get through this chapter, with all my various digressions and wedge doc and so on.