Buck Meta Chapter 1 Notes PhL 1.11.15

Meta Notes. I took this approach with Stakgold. While the original raw notes contain lots of detail and added comments, examples, questions and their answers, the Meta notes are at a higher level and try to give a simple and compact summary of the content.

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Chapter 1: Sets and Functions [pp 1-54]

1.1 A short intro.

1.2 The Geometry of E^n Cartesian space.**[2]**

Algebraic refers so the field properties of $E^{\mathbf{n}}$ = n-space. Vector space.
Quadrilateral midpoints always mark a parallelogram.

1.3 Distance and Norms. **[9]** **// Theorem 1**

Distance, **Norms**, **absolute value**, **inner products**.
Theorem 1: **Schwarz Inequality** proved: $\mathbf{p} \bullet \mathbf{q} \leq |\mathbf{p}|\,|\mathbf{q}|$ (involves norms and inner product)
Triangle Inequality: [12] Says that $|\mathbf{p}+\mathbf{q}| \leq |\mathbf{p}| + |\mathbf{q}|$ (proved from Schwarz, hence corollary of)
Straight Line as $\mathbf{p} + \lambda(\mathbf{p}-\mathbf{q}) = \mathbf{p}(1-\lambda) + \lambda\mathbf{q}$, segment **p** to **q** is then $0 \leq \lambda \leq 1$.
Convexity of a set in $E^{\mathbf{n}}$, how related to above line segment idea.

1.4 Sets and Functions. **[15]** **// no theorems**

Contains and **covers** means same, **set** and **collection** means same.
Set basics: union, intersection, notation and so on.
Functions, **domain** of, function as **mapping**, **curves** and **surfaces**.
Sequence as a mapping $n \rightarrow a_{\mathbf{n}}$ or integers to some space.
Trace is range/image of sequence.
Notion of A x B as the **graph** of f: A→B, is called **Cartesian Product**

1.5 Topological ideas**. [24]** **// Theorem 2**

Neighborhood (**ball**) of a point in a set.

Interior point has neighborhood entirely in set.
Exterior point has neighborhood entirely outside set.
Boundary point is neither in or out, boundary denoted ∂S.
Open set has only interior points.
Closed set means all points not in set are exterior points, or ∂S is in S.
Closure of set S is $S \cup \partial S = \overline{S}$ is a closed set, but Bucks don't use that overbar notation.
Complement of an open set is closed and vice versa. (a fact)
Cluster point = **accumulation point** = **limit point**: any ball has ∞ points in S.
Isolated point = can have ball with no other points, not a cluster point
Facts about open and closed sets page 28.
Connected = cannot split S into two non-null disjoint sets with no shared ∂S elements
Countable and non-countable sets [30].
Bounded and unbounded sets (requires a norm for a point in the set, $|\mathbf{p}|$)
Topology on set S = a description of the open subsets of S
U is **open relative to** S means U is intersection of an open set in E^2 with set S, p 32 fig.

Theorem 2. Set S = not connected $\Leftrightarrow$ S = A$\cup$B where A and B are disjoint and are each open relative to S. [page 32]

1.6 Ordering for R ≡ E^1 and grab bag of other topics **[34].** // **Theorems 3→5 BW + HB**

Order relation < in sense that a < b for a,b in E^1.
Notation [a,b] for closed interval, and how **interval** is specified using <.
Symbol ∞ (page 35)
LUB and **GLB** of a set with an order relation <

Theorem 3. The line is a connected set.

A **dense** in B means $\overline{A}$ = B. **Rationals** are dense in the reals.
Rationals called R_0 and form a disconnected set (break or gap at every non-rational number)
Set is **complete** if there are no such gaps.

Nested Set Theorem [37]: for an infinite nested set where all C_i are closed and bounded, there is a point which is in all the sets.

Theorem 4. (**Bolzano-Weierstrass**). Every bounded infinite set S in E^n has at least one cluster point.

Heine-Borel Property: A set has this property if any cover of open sets has a finite subcover. A set with this property is said to be **compact**.

Theorem 5. (**Heine-Borel**). Any closed and bounded set in E^n has the Heine-Borel property (is compact).

1.7 Sequences **[40]** // **Theorems 6→11**

Definition of a **sequence** of points $\{p_n\}$: mapping from I → E^n.

Note that a sequence means an infinite sequence in this book.
Sequence **converges** to p ≡ any neighborhood of p has all points p_n from some N on.
$\{p_n\}$ is **divergent** if there is no p to which it converges (does not mean $p_n \to \infty$)
Theorem 6: A convergent sequence is bounded. This seems pretty obvious and proof is tiny.
Theorem 7: Sum of two convergent sequences converges to the obvious sum point.
Theorem 8: The set $\overline{S}$ consists of points which are all limit points of convergent sequences.
Subsequence definition [44]
Theorem 9. Any bounded sequence has a convergent subsequence. [Example p 41]
A **Convergent (to p) Sequence** has $| p_k - p | <$ any ε for k being large enough.
A **Cauchy Sequence** has $| p_k - p_j | <$ any ε for both k and j being large enough.
Fact: Every convergent sequence is a Cauchy sequence. [p 45, Exercise 11]
Theorem 10. Every Cauchy sequence is a convergent sequence (in E^n).
Theorem 11. If $| p_{j+1} - p_j | < c^j$ where $|c| < 1$, then $\{p_j\}$ converges.
Theorem of Exercise 6: If $\{p_n\}$ has a limit point p, then $\{p_n\}$ has a subsequence which converges to that limit point (perhaps the full sequence jumps around if non-convergent as on page 41).

1.8 Real sequences **[48]** // **Theorems 12,13**

Monotonic = can be up or down [48]
Theorem 12: In R, a bounded monotonic sequence must converge.
Theorem 13: For any real x, there is always an integer n such that $n > x$.
Notion of sequence convergence **from above** or **from below** in R.
Use of **lim sup** $\mathbf{a}_n$ and **lim inf** $\mathbf{a}_n$ to handle sequences that jump around and have therefore multiple cluster points and do not converge. These can exist when **lim** $\mathbf{a}_n$ does not exist. The sup and inf are the largest and smallest limit points of a sequence.
Notion of **rate of convergence** of a sequence.

Note: The subject of series and sequences of functions has not yet been brought up.