Buck Meta Chapter 2: Continuity PhL 1.11.15

Note: "continuity" is a property that *functions* can have, it is not a property of sets or of sequences. Ch 2 is a long (42 p) and rather detailed chapter with 24 numbered theorems (not all are proved and I added some new theorems). Most of the theorems begin with "assume f is continuous". Probably the theory of non-continuous functions is much less "rich" in what can be said.

This meta doc at 7 pages is half the size of the 17 page raw notes as of 1/11/15.
The book Chapter 2 is 42 pages long and is brimming chock full of theorems (24) and examples.

Buck Meta Chapter 2: Continuity 1
2.1 is a one-paragraph intro..... 1
2.2 The Basics [55] // theorems 1 → 5 1
2.3 Approximation and Uniform Continuity [65] // theorems 6 and 7 2
2.4 Properties of Continuous Functions [72] // theorems 8 → 13 3
2.5 Limits of functions [76] // theorems 14→16 4
2.6 Discontinuities [82] // theorems 17,18..... 4
2.7 Mean value theorems and L'Hospital [89] // theorems 19 → 24 and A,B,C 5

2.1 is a one-paragraph intro.

2.2 The Basics **[55]** **// theorems 1 → 5**

Definition p 56 of **"f(p) continuous at a point p_0"** : For any given ε> 0, you can find a δ small enough so any p in the δ-ball around p_0 meets the requirement | f(p) - f(p_0) | < ε. Shrinking the δ ball in the domain shrinks the image of the δ ball in the range. If you have continuity at all points p_o in some domain region D, then you have **"f(p) continuous on D"**. Note that the required small δ will in general be a function of the location of point p_0, as shown in Fig 2.1 p 56.

Nothing is said at this point about "uniform" continuity.

Theorem 1: Continuity of f(p) implies that, if $p_n \to p$ in the domain, then $f(p_n) \to f(p)$ in the range. One says then that "**continuity preserves convergence**".

This is the first time in the book that the notions of "function" and "sequence" are combined. Above we are suddenly talking about a "function of a sequence of points", $f(p_n)$. In effect the function produces a new and possibly convergent sequence in the range space $f_n \to f$ where $f_n = f(p_n)$. The magic ingredient of "continuity" says that $p_n \to p \Rightarrow f_n \to f$. Again, functions and their continuity are being connected with certain sequences in the domain and range of the functions.

Theorem 2 (converse of above). If for *all* sequences $p_n \to p$ you have $f(p_n) \to f(p)$, then f continuous.

Theorem 3. [60] [Any V open in range $\Rightarrow$ $f^{-1}(V)$ open in domain] $\Leftrightarrow$ f is continuous on domain

This says that if f is continuous, the pre-image of any open set is an open set (and vice versa). Notice that this theorem does not say f maps open sets into open sets. It is the inverse mapping that does this. If you wanted the former, you would need the function f^{-1} to be continuous. There might be a theorem somewhere that says if f is continuous and if f^{-1} exists, then f^{-1} is continuous, but Bucks have not really dealt with the notion of an inverse function at this point.

Theorem 4. The set of real-valued *continuous* functions forms an **algebra**, which means that the set is closed under addition, multiplication, linear combination with constants, and division as long as the denominator function is non-zero.

Corollary. Since 1 and x and y are continuous functions, so are all multivariable polynomials like $f = 2 + 3x^3 + 2xy$. Thus, all **polynomials** $f: E^n \rightarrow R$ are continuous functions over all of E^n.

Theorem 5: Continuity is preserved under functional concatenation, as in f(g(p))

2.3 Approximation and Uniform Continuity [65] // theorems 6 and 7

Definition: F(x) is a **uniform ε approximation** to f(x) if $| f(x) - F(x) | < \varepsilon$ for all x in domain of interest. Example shown page 66 figure. Note that F might not be differentiable everywhere.

Weierstrass Approx Theorem. [66] On a closed interval [a,b] you can uniform-ε-approximate any continuous f by a polynomial (which, as noted above, is also continuous). [not proved by Bucks]

Uniform Continuity. [67] Idea is that you require $| f(p)-f(q) | < \varepsilon$ not just for one particular q in the domain (which would be regular continuity at point q), but you require $| f(p)-f(q) | < \varepsilon$ for all p and q in the domain. You then have continuity uniformly over the domain. The idea is that given any ε, the same δ has to work for all q's in D.

On (open) interval $0 < x < 1$, $f(x) = 1/x$ is not uniformly continuous, but $f(f) = x^{1/2}$ is. I have a lot of raw notes decoding the Buck comments on these two examples on their pages 68 and 69.

Theorem 6. [69] If f is continuous on D, and if D is closed and bounded ("compact"), then f is uniformly continuous on D.

Note that the above requires f to be continuous on the boundary points of D since D is closed.

$f(x) = 1/x$: This function is NOT continuous at x = 0, so the theorem says nothing -- I guess the function could or could not be uniformly continuous on D. Bucks showed it is NOT uniformly continuous.

$f(x) = \sqrt{x}$: This function IS continuous at x = 0 and thus on D = [0,1]. The theorem then applies and we conclude that $\sqrt{x}$ is uniformly continuous on the interval.

$f(x) = x^p$: As long as $p \geq 0$, f(x) will be continuous at x = 0, and x^p will be uniformly continuous. I suspect that for any $p < 0$ it will not be uniformly continuous, but we have only shown that for p = -1, and the Theorem 6 has nothing to say on this question.

Theorem 7. For f continuous on closed and bounded [a, b], it is possible to do the uniform ε approx by a polygonal line (a piecewise linear function).

You could therefore do a polygonal line uniform ε fit for $\sqrt{x}$ on [0,1] but not for 1/x since the latter is unbounded at the 0 endpoint.

2.4 Properties of Continuous Functions [72] // theorems 8 → 13

We open with two theorems:

Theorem 8: If f continuous and f(p)>0, then ∃ B = ball(p) ϶ f(B)>0.

Theorem 9: If f continuous and S = [all p in D ϶ f(S)>c], then S is open (relative to D)
If f continuous and S = [all p in D ϶ f(S)=c], then S is closed (relative to D)

Earlier we talked about a set S of points perhaps having an upper bound, or being bounded, meaning that for all p in S, |p| < M. We now have a new use of the word bounded in relation to a function:

Def: **Function f is bounded on S** if |f(S)| < M. S is in domain, bound is in the range space.

Theorem 10: f continuous on closed and bounded (compact) S in $E^n \Rightarrow$ f is bounded on S .

Example: f(x) = 1/x and S = [0,1] = compact. Since f(x) is not continuous at x = 0, we cannot conclude that f(S) is bounded in the range. And in fact, 1/x is unbounded at x = 0.

Example: f(x) = $\sqrt{x}$ and S = [0,1] = compact. Since f(x) is continuous at x = 0 from above, we can conclude that f(S) is bounded in the range, and in fact $\sqrt{x}$ is bounded in [0,1].

Theorem 10+11: Real f continuous on closed and bounded (compact) S in $E^n \Rightarrow$

- f is bounded on S, meaning $m \le f(S) \le M$
- there is some point s in S where f(s) = the maximum value f takes on S
- there is some point s in S where f(s) = the minimum value f takes on S

Theorem 12. f continuous on S which is compact in $E^n \Leftrightarrow$ **graph** of f is compact

Theorem 13. Assume f is continuous on S which is ***connected***. Assume f(x) < c < f(y) for some points x and y in S. Then there must be some point z in S such that f(z) = c.

Comment: There has been no theorem stating that a continuous function must map open sets to open sets or closed sets to closed sets. We know that the pre-image of an open set is open if f is continuous, but that is a different matter. That is, the inverse map of a continuous function maps open sets to open sets. Wiki discusses special maps that do open to open, or closed to closed, or even both at once (like the identity mapping). These are called "open maps" or "closed maps". This is a subject Bucks do not get into.

2.5 Limits of functions [76] // theorems 14→16

Earlier we talked about the limit of a sequence a_n as $n \to \infty$. One can regard this as the limit of a function a(n) defined on the integers, but now we are really dealing with f(x) where x is in E^n.

[77] Def 4 **"limit of a function "** gives the usual ε δ definition of $\lim_{x \to b} f(x) = L$. In this definition, we have $x \to b$ and $f(x) \to f(b)$, so you would say $\lim_{x \to b} f(x) = f(b)$ and you find δ so $| f(x) - f(b) | < \varepsilon$. This is reminiscent of $| p_n - p | < \varepsilon$ for a sequence for $n > N$ and corresponding $\lim_{n \to \infty} p_n = p$.

Theorem 14 says that limits commute with addition, multiplication and division: For example,

$$\lim_{x \to b} [f(x)\, g(x)] = [\lim_{x \to b} f(x)] [\lim_{x \to b} g(x)]$$

You can define a **Cauchy limit of a function** in this way:

$$\lim_{x \to b,\ x' \to b} | f(x) - f(x') | = 0$$

which is analogous to the Cauchy limit of a sequence from Chapter 1 page 45,

$$\lim_{n \to \infty,\ m \to \infty} |p_n - p_m| = 0$$

Theorem 15: $[\lim_{x \to b,\ x' \to b} | f(x) - f(x') | = 0$ (Cauchy)$] \Rightarrow \lim_{x \to b} f(x)$ exists

The idea here is that both $|x-b| < \delta$ and $|x'-b| < \delta$ to get $| f(x) - f(x') | < \varepsilon$.

Earlier we dealt with a monotonic sequence, one can also have a **monotonic function** f(x).

Theorem 16: If f(x) is bounded and monotonic on (a,b) then both end limits exist with appropriate arrow.

Note that f(x) could in theory have jumps inside the interval and thus not be continuous. I think the monotone is required to rule out possible oscillation at one of the endpoints, like sin(1/x). Such a function would be bounded, but not monotonic. Bucks do not prove this theorem, reader does in an exercise.

2.6 Discontinuities [82] // theorems 17,18

Bucks extend the discussion to domain being E^2 or E^n. The new feature is that if $p \to p_0$ in the domain D, there are an infinite number of paths in D for p to make its approach. For E^1 you had only left or right. They refer to each possible approach path as a set of points S.

Theorem 17. Assume f(p) is defined in an E^n ball around p_0 though perhaps not exactly at p_0. Let S be a path of approach $p \to p_0$. If $f(p) \to$ the same limit L for all sets S, then $\lim_{p \to p_0} f(p) = L$. That is to say, if all approach paths give the same limit, then the limit exists.

Bucks give three examples for f(x,y). For $xy/\sqrt{x^2+y^2}$ they first try axis approaches and then 45 degree ones, and limit seems stable. They then formally prove limit exists at (x,y) = (0,0) and is 0. However, for $xy / (x^2+y^2)$ they find that axis and 45 degree approaches give different results (0 and 1/2), so this function has no limit at (0,0). The last example $xy^2/(x^2+y^4)$ gives the same limit 0 for all linear approaches, but Bucks find a curved approach that gives a different value, so limit does not exist. It would certainly take a long time to check all approaches!

Bucks note that the "double limit" considered here is much different from doing sequential limits. For example:

$$\lim_{(x,y)\to(0,0)} [\, xy^2/(x^2+y^4) \,] = \text{does not exist}$$

whereas

$$\lim_{y\to 0} \{ \lim_{x\to 0} [\, xy^2/(x^2+y^4) \,]\} = \lim_{y\to 0} \{0\} = 0$$
$$\lim_{x\to 0} \{ \lim_{y\to 0} [\, xy^2/(x^2+y^4) \,]\} = \lim_{y\to 0} \{0\} = 0$$

The example x^x is really undefined at x = 0, but the limit from above is 1 and you can just add that point and thereby "remove" the problem at x = 0. A removable discontinuity at x = 0.

Theorem 18: If f is uniformly continuous on bounded D, then you can add ∂D to the domain such that f is then continuous on $\overline{D}$. You have thus *extended* f(D) to f($\overline{D}$) by adding the limit points. Goes both ways:

f uniformly continuous on bounded D $\Leftrightarrow$ you can continuously extend f(D) to f($\overline{D}$)

The proof is long and I did not do it. It must be that the LHS implies first of all that E^n limits *exist* at all boundary points, so at least you can talk about f(∂D) and thus about f($\overline{D}$).

In 1D this says that if f(x) is uniformly continuous and bounded on (a,b), if there are problems at the endpoints, you can fix them up and at worst they are removable discontinuities. For example, given any function f(x) bounded and uniformly continuous on (0,1), you could add the limit points [they must exist by Thm 18] and then make f(x) = 0 outside the interval, and you have thus extended f to (-∞,∞).

Tietze's Extension Theorem (to be treated later) generalizes this from E^1 to E^n and seems to remove the requirement that f be uniformly continuous on D, just continuous will do (along with bounded).

2.7 Mean value theorems and L'Hospital [89] // theorems 19 → 24 and A,B,C

Theorem 19. Consider f: R→R. If there is a local extremum at some point x_0 which lies inside the domain, then $f'(x_0) = 0$. If the extremum is at the end of a domain, then not true.

Theorem 20. (**Rolle's Theorem**) If f is continuous on [a,b] and differentiable on (a,b), and if f(a) = f(b), then there must be a point in (a,b) where the curve f(x) has zero slope.

Note that one might find zero slope at one or both endpoints, but in addition there will be at least one point in the interior (a,b) where the slope is zero as well.

Theorem 21. (**Mean Value Theorem of Differential Calculus**) If f is continuous on [a,b] and differentiable on (a,b), then there will exist a point c in (a,b) such that $f(b) = f(a) + (b-a)f'(c)$.

This says that the slope at c matches the mean (or secant) slope. I give a simple graphical interpretation in the raw notes. If hor axis is time and vert axis distance, then this says that for a linear trip there must be at least one time instant during the trip where the speed matches the average speed.

Theorem 22. (**General Mean Value Theorem**) If f,g are continuous on [a,b] and differentiable on (a,b), then there will exist a point c in (a,b) such that $[f(b)-f(a)]\, g'(c) = [g(b)-g(a)]\, f'(c)$.

Again I give a geometric interpretation in 2D. If argument is time, and if x = f and y = g, then you can regard f and g as a parametric description of a smooth path in the plane. The theorem says there will exist some time instant where vehicle direction matches the average trip (secant) direction.

Theorem 23. Says $\sqrt{u^2+v^2} \approx u + v/2u$ with error $\leq v^2/4u^3$.

Normally you would show this using the Taylor expansion, but here they show it using the Mean Value Theorem, just to give an example of how this theorem might be used. I don't know if they will ever actually use Theorem 23 anywhere.

Theorem 24 (**L'Hospital's Rule**) If f,g are differentiable on [a,b) with $g'(x) \neq 0$, then if at the endpoint b you find that f/g gives a 0/0 or ∞/∞ situation (indeterminate), and if $\lim f'/g' = L$, then $\lim f/g = L$. Moreover, this theorem is valid even if $b = \infty$ and if $L = \infty$.

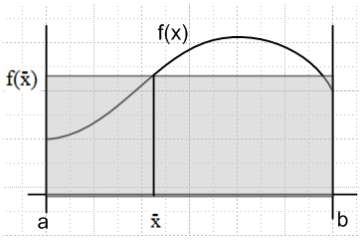
Bucks first ponder $f/g = [1-\cos(x^2)] / x^4$ at x = 0 which is 0/0 case. Then they do x log x which is treated as logx / (1/x) which is ∞/∞ case. Here they find $f'/g' = 0$ so $x\log x \to 0$. As a spinoff result, since one has $x^x = \exp(x\log x)$, we find that $x^x \to \exp(0) = 1$ as $x \to 0$.

They then go into their very long 2 page proof which I did not study. But nice to know where a proof lies if I someday want to prove this theorem.

On my own, I gathered up a few other "value theorems" that are not mentioned in this chapter.

Theorem A. (**Intermediate Value Theorem**) If f is continuous on [a,b], then f(x) must take on every value in the range between f(a) and f(b) somewhere in the interval.

Theorem B. (**First Mean Value Theorem for Integrals**) . If f is continuous on [a,b], then there will exist a point c in [a,b] such that $\int_a^b f(x)\, dx = f(c) \int_a^b dx = f(c)[b-a]$. Here is a picture where $\bar{x} = c$:



The area under a continuous curve is equal to that of a rectangle whose height is the value of the curve at some point in the interval. I give a proof in the raw notes. f(c) is the average value.

Theorem C. (**Second Mean Value Theorem for Integrals**) If f,g are continuous on [a,b] and g≥0, then there will exist a point c in [a,b] such that

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$$ // f(c) is the g-weighted average value

This has a more general form over a domain in E^n (appears in Ex 8 page 106)

$$\int_D f(\mathbf{x})g(\mathbf{x})d^n x = f(\mathbf{c}) \int_D g(\mathbf{x})d^n x$$

D = open and connected in E^n

f,g = continuous and bounded
g = positive definite