Buck Meta Chapter 3 : Integration PhL 1.12.15

This meta doc at 15 pages is half the size of the 31 page raw notes as of 1/14/15.
The book Chapter 3 is 61 pages long and is brimming chock full of theorems (19) and examples.

3.1 The Definite Riemann Integral [97] // theorems 1→4 1
3.2 Evaluation of Definite Integrals: a grab bag of methods [107] // theorems 5→10 3
3.3 Taylor's Theorem [124] // theorems PL1,PL2 and 11 5
3.4 Improper Integrals [131] // theorems 12→17 6
1D Improper Integrals [134] 7
2D Improper Integrals [144] 11
3.5 Set Functions [151] // theorems 18,19 12

3.1 The Definite Riemann Integral [97] // theorems 1→4

On pages 97-99 the Bucks assemble a veritable battalion of measuring tools to deal with Riemann integration. An arbitrary area (domain D) is pictured on page 98 with a grid and regions 1,2,3. Here we can just admire the list of symbols used in this **general-rectangle assault** on the subject:

D,R = bounded domain D, enclosed by rectangle R
Γ = **boundary** of D
f = a function define over D
N = any **grid** of rectangles which "cover" R and this D within R (grid need not be evenly spaced)
N' = a **refinement** of N = a grid with more "lines" than grid N
R_{ij} = grid rectangles partitioned into 3 groups as Fig 3-1 shows. Group 2 are the straddlers.
d(N) = max of **diameters** of all the R_{ij} in the grid N
$\overline{S}(N,D)$ = the area of 1+2 (outside) has GLB = S
$\underline{S}(N,D)$ = the area of 1 (inside) has LUB = s
imagine considering ALL possible covering sets $\{R_{ij}\}$, with arbitrarily small rectangles
$\overline{A}(D)$ = GLB of area over all possible sets $\{R_{ij}\}$ = S
$\underline{A}(D)$ = LUB of area over all possible sets $\{R_{ij}\}$ = s
$A(\Gamma) = \overline{A}(D) - \underline{A}(D)$ = area of boundary, which we know will be 0
p_{ij} = arbitrary point within R_{ij}
$\{p_{ij}\}$ = set where we pick some p_{ij} in each R_{ij}
M_{ij} = max value of f over all R_{ij}
m_{ij} = min value of f over all R_{ij}

With these symbols, we then write the **Riemann integral** of f(x,y) as the limit of the **Riemann sum**:

$$S(N, f, \{p_{ij}\}) \equiv \Sigma_{ij}\, f(p_{ij})\, A(R_{ij}) \rightarrow \iint_R f = \text{the "integral"} \quad \text{as } d(N) \rightarrow 0$$

The initial two theorems and all related work assume temporarily that D = R which is the bounding outer rectangle.

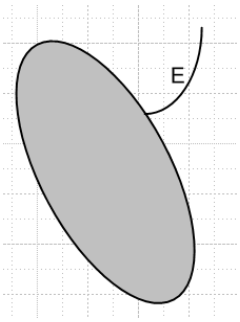
Theorem 1: [100] If f is continuous, then the above integral limit exists. Note integral is over R.

The lengthy proof involves three Lemmas and wraps up on page 102.

Theorem 2: [102] Even if f is not continuous at certain points which comprise a set E of zero area within R, the limit still exists, but now you have to assume f is bounded on R. So restate:

Theorem 2: If f is bounded on R and is continuous on R except for E, $\int\int_R f$ **exists**.

Another long proof. I picture a typical such zero-area region as perhaps a whisker sticking out (the ellipse should still be a rectangle R at this point).



Theorem 3: [103] If D is a bounded set having an area (some sets don't), and
if f is bounded on D, and
if f is continuous on the entire interior of D
then $\int\int_D f$ **exists and** is independent of the enclosing R.

The "proof" here is just to cookie-cut R to D by setting f = 0 outside D within R.

Bucks comment that all this stuff works for D in E^n for any n. For n = 1 they restate things, and in particular Theorem 2:

Theorem 2': (1D) [104] If f is bounded on [a,b] and is continuous on [a,b] except for certain points which make up our set E, $\int_a^b f(x)\,dx$ **exists**.

An example of an allowable f would be one that is piecewise continuous, but Bucks do not mention this phrase.

Theorem 4: Collection of **facts**:

(i) $\int_D (f+g) = \int_D f + \int_D g$

(ii) $\int_D (Cf) = C\int_D f$

(iii) $f(D) \geq 0 \Rightarrow \int_D f \geq 0$

(iv) $\int_D f$ exists $\Rightarrow \int_D |f|$ exists and $|(\int_D f)| \le \int_D |f|$ (like $|a+b| \le |a|+|b|$ triangle inequality)

(v) $\int_{D1} f + \int_{D2} f = \int_{D1 \cup D2} f$ or in 1D : $\int_a^b f + \int_b^c f = \int_a^c f$

Point of Confusion: With all the tools and arbitrary D, picture on page 98 makes sense and we have three classes of small rectangles as shown. But on page 99 they set D = R for a while. This means there are only rectangles of class 1, so then having $\overline{S}(N,D)$ = the area of 1+2 (outside) makes no sense. I think what they mean is that they temporarily take D = R' where R' is some rectangular D that lies inside the outer rectangle R and which does not align with the grid so there are still classes 1,2 and 3 of small rectangles. This confusion does I think greatly weaken their discussion. And the no-area set E is also a bit confusing. Luckily, I know what they mean here. Maybe things are better in 3rd edition. I am sure others have noticed this problem.

3.2 Evaluation of Definite Integrals: a grab bag of methods [107] // theorems 5→10

Bucks compute a certain 2D integral (**Example 1** p 108) by actually laying down a uniform **grid** and doing it just as the previous theory says: compute the sum then take n→∞. The 1D Example 2 integral yields a sum we don't know how to take the n→∞ limit of, but with a non-uniform grid we get a sum we can do. They are trying to justify the generality of their theory method which allows for non-uniform grids (most books would not do this I suspect).

Theorem 5: If f is continuous on your 1D closed interval, the **antiderivative** F exists.

Theorem 6. You can always **add a constant** to get a new antiderivative.

Theorem 7. This constant cancels out if you do a definite 1D integral. That is, $\int_a^b f = F(b) - F(a)$.

We are warned that the dx in an integral is not really a differential, it is a notation. If you take it strictly as a differential, you get into **sign problems with Jacobians**. dx is really the area of a box in 1D and it has to always be positive, just as a rectangle area R_{ij} in 2D is positive. Suggest a "box notation" for dx.

Change of integration variables.

Example 1' shows how you can get a total trash result if you use non-continuous f(x) on interval.
Example 2' shows the issue with the sign of dx and du.

So much for 1D integrals, the Bucks return to **2D integrals.**

The idea that you can do one of them first and then do the other is subject of Theorem 8 (below), at least for integral over a simple aligned rectangle. Just proving this obvious theorem using their various "power tools" takes a full 2 pages. The "mean value theorem for integrals" is used, and I have added comments on this above below the other mean value theorems.

Page 113 has a technical issue that the inner integral might not exist for certain values of the outer variable, but this might still be OK in terms of doing the double integral. This is subject of **Lebesgue** integration which avoids such problems, and we are given **Munroe's 1953 book** as a reference.

Theorem 8. [111] (**Either Order**) If f is continuous on a closed rectangle R in 2D, then you can say

$$\int_R f \;=\; [\, \int_a^b \int_c^d dx\, dy \,]\, f(x,y) \;=\; \int_a^b dx\, [\; \int_c^d dy\, f(x,y)\,] \;=\; \text{could do in other order}$$

Theorem 9. [114] Generalization of Theorem 8 to situation where f is continuous *except* on some zero-area subset E like my whisker thing earlier. As long as the whisker is not exactly vertical or horizontal, you can compute the integral as shown in Theorem 8 above.

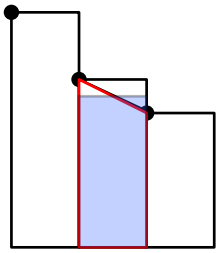
The reason you can see is that a vertical whisker will have some vertical "area" for its single integration dy which is done first, then this gums up the works. Another long proof.

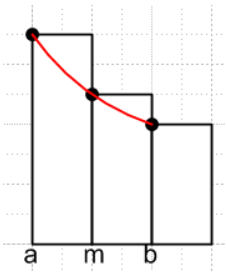
Corollary: For non-rectangular D and continuous f, you can do the integral this way

$$\int_D f \;=\; \int_a^b dx\, [\; \int_{c(x)}^{d(x)} dy\, f(x,y)\,]$$

if you know the **equations of the boundaries**. Two **examples** are given, including the complicated one having the picture on page 116.

Next topic p 117 is approximating an integral using the **trapezoidal rule** or **Simpson's rule**. I looked up the latter to verify the little equation and drew some pictures :





I guess for a full integral you would add up the Simpson rectangles a pair at a time like the pair shown above on the right. It turns out that Simpson's rule gives the exact integral for f(x) cubic or less, even though Simpson uses a quadratic fit! I explain how this works in the raw notes.

The next section shows some **examples of doing 2D integrals in either order** to get same result.

Theorem 10. [120] If f and $\partial_x f$ are both defined and continuous on x in [a,b] and y in [c,d], then

$$\{\partial_x [\; \int_c^d dy\, f(x,y)\,]\}\,(x) \;\equiv\; \partial_x F(x) \;= F'(x) \;= \{\; \int_c^d dy\, [\partial_x f(x,y)]\,\}\,(x) \qquad \text{for all x in [a,b]}$$

Here (x) just shows that something is a function of x. This says you can "**differentiate under the integral sign**". Two examples (Ex1 and Ex 2 p 121) are given of computing F'(x).

Last topic is how to compute such an F'(x) if **endpoints are functions of x**. You get extra terms:

$$F(x) = \int_{c(x)}^{d(x)} dy\ f(x,y)$$

$$F'(x) = \{ \int_{c(x)}^{d(x)} dy\ [\partial_x f(x,y)] \}\ (x) + d'(x)\ f(x,d(x)) - c'(x)\ f(x,c(x)) \qquad \text{// p 122}$$

3.3 Taylor's Theorem [124] // theorems PL1,PL2 and 11

Class C^n means $f^{(n)}$ exists and is continuous, and $C' = C^1$ and $C'' = C^2$ and some f(x) can be C^∞.

Theorem PL1: if f is continuous on [a,b], then

$g = \int f$ is continuous on [a,b]

$g = \int f$ is differentiable on (a,b) // see raw notes

Theorem PL3: If f(x) is C^m, it is also C^{m-1}, C^{m-2}....C^1, C^0. // shown in raw notes

A **Taylor Series** at a point x_o seeks to match a function f(x) by a polynomial $P(x; n,x_0)$ of degree n such that all derivatives up through order n match.

Theorem 11 [126] gives an exact **formula for the error** as an integral:

$$R_n(x; x_0) \equiv f(x) - P(x; n,x_0) = (1/n!) \int_{x_0}^{x} dx'\ f^{(n+1)}(x')\ (x'-x_0)^n$$

In the raw notes I derive this fact, parsing what the text says. Since the integral is a bit inconvenient, using the mean value theorem you can come up with another error statement :

Corollary 1. [126] $R_n(x; x_0) \equiv f(x) - P(x; n,x_0) = [\ f^{(n+1)}(x_1) / (n+1)!]\ (x-x_0)^{n+1}$

Here x_0 is the expansion point, x the variable, and x_1 is "**some point**" (unknown) in (x_0,x). You might be able to bound the error by letting x_1 (called τ) vary over the interval. I derive this corollary in raw notes.

Corollary 2. [127] (**Taylor's Theorem**) This just writes things out as a restatement of the above:

$$f(x) = P(x; n,x_0) + R_n(x; x_0)$$

$$= \Sigma_{m=0}^{n} (1/m!)\ f^{(m)}(x_0)\ (x-x_0)^m + f^{(n+1)}(x_1)\ (x-x_0)^{n+1} / (n+1)! \qquad \text{for some } x_1 \text{ in } (x_0,x)$$

truncated Taylor — remainder part

For n = 1 this says $f(x) = f(x_0) + f'(x_1)(x-x_0)$ which is the "mean value theorem of diff calculus" which we encountered as Theorem 21 in section 2.7.

Example 1 is $f(x) = e^x$ with $x_0 = 0$. Remainder is then $e^{x1}\, x^{n+1} / (n+1)!$ But on [-1,1] $|x|^{n+1} \le 1$ so we find that $|\text{error}| \le e^{x1} / (n+1)!$ But worst case $e^{x1} = e$, so result can be written $|\text{error}| \le e / (n+1)!$ This is an example of the "you might be able" quoted above. We of course get $e^x = 1 + x + x^2/2! + ...$

Def: [128] f(x) is **analytic** at x_0 if it is C^∞ in some open interval around x_0 AND $R_n \to 0$ on interval.

In other words, you can use an ***infinite*** **Taylor series** for an **analytic function**. The need for the last condition $R_n \to 0$ is demonstrated by the example $f(x) = \exp(-1/x^2)$ which is in fact NOT analytic at x=0. Bucks do not use the phrase **essential singularity**. Here remainder is $R_n = \exp(-1/x^2)$ for any n.

Ex 2: [129] They comment that the Taylor series **may *not* be the most accurate** polyn for a given f(x).

Example 4: One way to integrate a function is to **approximate** the integrand as a Taylor series. Here the function $\exp(\sqrt{x}\,)$ is expanded in a Taylor series and the integral done by approximation in this way.

Question: Why was the Taylor expansion presented in this chapter on "integration"? It seems perhaps a little out of place. In 3rd Ed, the integration chapter is preceded by a differentiation chapter and the Taylor series is put there. Amazon has the TOC for 3rd Ed.

3.4 Improper Integrals [131] // theorems 12→17

An **improper integral** $\int_D f$ is one where **either D or f is unbounded, or both.** The **first case** is treated first, and initially we just use f = 1 in 2D, so we are talking integrals which are areas.

Bucks apply their rectangle method of Section 3.1 and are able to obtain a monotone increasing sequence for area $A(D_n)$ where $D_n = D \cup R_n$. You must define an "**expanding set of nested rectangles**" R_n which cover more and more of E^2. If the sequence converges $A(D_n) \to A$, then A is the value of the improper integral. You really only have to show that the sequence $A(D_n)$ is bounded since it is monotone.

For **Example 1** $f(x) = (1+x^2)^{-1}$ over $(-\infty,\infty)$ it is D that is unbounded. A simple set of R_n results in $A(D_n) = 2 \tan^{-1}(n) \to 2(\pi/2) = \pi$ and so the integral is π. [131]

For **Example 2** of the page 132 figure the improper area of interest is defined by $|x^2 - y^2| \le 1$. This is not simply the area under a function f(x), but breaking the problem into 8 identical pieces, the 1/8th area becomes the area between two functions as shown. Again, an expanding set of R_n is constructed, but this time it is shown that $A(D_n) \ge 4+4\log n$, so $A = \infty$, and this area is undefined. Again, D is unbounded.

Example 3 changes this last to $|x^4 - y^4| \le 1$ with similar picture, and now $A(D_n) \le 8$ so A exists, but we don't know what it is.

Theorem 12 [133]. The claim is that the integral found by the "expanding rectangles method" is the same **no matter what set of rectangles you select**, as long as the set forms an expanding cover of the plane.

I review the proof in the raw notes, and it involves the Heine-Borel theorem.

1D Improper Integrals [134]

On p 133 at this point Bucks change from 2D to 1D and allow some $f \neq 1$ as integrand. Once $f \neq 1$, we are in the realm of "improper integrals".

As an injected topic, Bucks discuss their **Cauchy Principal Value** integral

$$CPV = \lim_{r\to\infty} \int_{-r}^{r} dx\ (1+x)/(1+x^2)$$

First, notice the idea that the domain D is unbounded while f(x) is bounded here. If you break this integral into two obvious pieces at x = 0, the $x/(1+x^2)$ "odd" terms diverge on $(0,\infty)$, but in the CPV integral this divergence does not occur since even integral of an odd function. Just saying.

At bottom page 135 we do first example of the **second case**: the domain is bounded but the function f is unbounded over the domain. The example is $\int_0^1 x^{-1/2}\, dx$ and f is unbounded at x = 0. The expanding set of rectangles R_n are not shown in Fig 3-8 but I add them as dashed line. The intersection $D_n = D \cup R_n$ is the gray area where you have to correct the figure by making the top part white. The integral of $1/\sqrt{x}$ is the area under the curve, so on p 136 they compute the area as $A(D_n) \to 2$ as a simple "area under curve" dy dx integration, and so our example improper integral is evaluated. They generalize this method by defining $f_n(x) = \min [\ f(x),n]$ and then you write $A(D_n) = \int f_n(x)$ and imitate the example. They claim this is not the best way to proceed. I call this the **slice-off method**.

On page 137 they forget the rectangle stuff and just express the desired integral as a limit of the finite integral $\int_r^1 x^{-1/2}\, dx$ as $r\to 0+$ and then this serves as a definition of $\int_0^1 x^{-1/2}\, dx$. I suppose you could do this method as well in terms of an expanding set of rectangles, but they don't talk about that.

We now review our two methods.

There are two ways to take the limit for an integral of the type shown in Fig 3-8 where the integrand is unbounded at one end, which we will assume is the "a" end.

1. $A(D) = \lim_{n\to\infty} \int_a^b dx\ f_n(x)$ $\qquad$ $f_n(x)$ = f(x) sliced off at y = n
2. $A'(D) = \lim_{r\to a} \int_r^b dx\ f(x)$

In the first method, we first integrate the "sliced off function $f_n(x)$" over the full interval [a,b] at each fixed n and get some A_n, then we take $A = \lim_{n\to\infty} A_n$. In the second method we integrate the full function only on [r,b] to get some A_r and then take $A' = \lim_{r\to a} A_r$.

Theorem 13: [137] If f is continuous and $f \geq 0$, then A = A', both ways give same result.

The result applies if the integral is finite or infinite. The requirement $f \geq 0$ allows one to maintain the area concept without dealing with "negative areas" and of course this fact is used in the proof. We then get a large set of examples:

Example 1: $\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$ using Method 2, result is $\pi/2$.

Example 2: $\int_0^1 \frac{1}{\sqrt{x}\,(1-x)}\, dx$ using Method 2, diverges at the x=1 end.

Example 3: $\int_0^\infty e^{-x}/\sqrt{x}\, dx$. This has an f = unbounded issue at x = 0, and a D unbounded issue at $x = \infty$. They break it into $\int_0^\infty = \int_0^1 + \int_1^\infty$ in order to separate the two issues, but they stop at that point since we can see that both integrals will be finite, and in fact Maple says the result is $\sqrt{\pi}$.

Example 4: $\int_0^1 dx/\sqrt{x}$ using $x = u^2 = \varphi(u)$ (which is class C' on (0,1)). Here $dx = 2u\, du$ so the integral becomes $dx/\sqrt{x} = 2u\, du / u = 2du$ with same (0,1) endpoints so magically, after changing variables, the issue at x = 0 simply goes away!

Examples 5,6: Two other examples where changing variables makes the issue "go away". In both examples, the endpoint ∞ becomes finite after changing variables from x to θ or x to u.

Warning:

1. $\int_a^c dx\, f(x) = \int_a^b dx\, f(x) + \int_b^c dx\, f(x)$ // this is always OK
2. $\int_a^c dx\, [\, f(x) + g(x)\,] = \int_a^c dx\, f(x) + \int_a^c dx\, g(x)$ // can have a problem

Example 7 demonstrates line 2 above where the LHS integral converges but both RHS don't.

Theorem 14 [140] (**Comparison Test**) If $0 \leq f \leq g$ on [a,b) then $\int_a^b dx\, f(x) \leq \int_a^b dx\, g(x)$.

This is only useful if the g integral converges, otherwise you learn nothing from this test. Short proof.

Corollary. [140] (**Ratio Test**) If $f \geq 0$ and $g \geq 0$ on [a,b) and if lim f/g = L as $x \to b$ where $0 < L < \infty$, then the two integrals $\int_a^b dx\, f(x)$ and $\int_a^b dx\, g(x)$ both converge or both diverge.

This is then another convergence test to put in your **toolbox**. I parsed the short proof.

Theorem 15. [141] $\int_a^b |f|$ converges $\Rightarrow \int_a^b f$ converges .

A trivial tiny proof is given which I buy. Cannot just invoke Thm 14 with g = |f| since no f≥0.

Theorem 15A (combination of 14 and 15, and several examples of use follow) :

$\int f \le \int |f|$ and $|f| < |g|$ $\Rightarrow$ $\int f \le \int |g|$ so that $\int |g|$ converges $\Rightarrow$ $\int f$ converges

Example 1: $f = \sin(x)/x^2$, $g = 1/x^2$, $|f| < |g|$ so $\int dx/x^2$ converge at $x=\infty \Rightarrow \int dx\, f$ converges.

Example 2: $f = \cos(1/x)/\sqrt{x}$, $g = 1/\sqrt{x}$, $|f| < |g|$ so $\int dx/\sqrt{x}$ converge at $x=0 \Rightarrow \int dx\, f$ converges.

Example 3: $f = \sin(x)/x^{3/2}$. Break into two integrals to isolate issues:

$f = \sin(x)/x^{3/2}$ on $[0,1]$ see that $x \to 0$ has $f = x^{-1/2}$ converges, no issue

$f = \sin(x)/x^{3/2}$ on $[1,\infty]$, $g = 1/x^{3/2}$, $|f| < |g|$ so $\int dx\, x^{-3/2}$ conv at $x=\infty$ so $\int dx\, f$ converges.

Definitions: [141]

$\int f$ = **absolutely convergent** if $\int |f|$ is convergent. [so then we know $\int f$ also converges]

$\int f$ = **conditionally convergent** if $\int f$ converges but $\int |f|$ diverges.

Example 4: $f = \sin(x)/x$ on $(1,\infty)$ not resolved by Theorems above, need better test. Here Bucks use **integration by parts** [142] to show in line A that $\int f$ converges at $x = \infty$ (so dicing fixes log div).

Example 5: $f = |\sin(x)/x|$ on $(1,\infty)$. Bucks show that this one diverges (slicer is disabled!).

Therefore $f = \sin(x)/x$ on $(1,\infty)$ is conditionally convergent.

Example 6: $f = \cos(x)/\log(x)$ on $[2,\infty)$. Cannot just apply Thm 15A since $\int 1/\log$ diverges. So again do the parts integration, then show that the second term on line B converges due to 15A. The parts integration done the right way "softens" problems.

Theorem 16 [143] (**Dirichlet Test for Product of Two Functions**). This is a fairly complicated theorem to state, and it relates to the product of two functions f(x)g(x) for an integral with upper endpoint ∞. You assume the 4 items:

0. f, g and g' are continuous on $[c,\infty)$
1. $g(x) \to 0$ as $x \to \infty$.

2. $\int g'$ is absolutely convergent (which just means $\int |g'|$ is convergent)

3. Define $F(r) \equiv \int_c^r f(x)dx$ and assume F(r) is bounded on $[c,\infty)$

THEN $\int fg$ converges. I parsed the simple proof.

Note added: I guess $F(\tau)$ could bounce around so it does not converge, but as long as it is bounded, the g factor going to 0 causes the integral of fg to converge.

Theorem 16A (Corollary 1) [143] Replace items 1+2 with "g(x) decreases monotonically to 0", the conclusion is still true.

I show how Thm 16A is similar to the wiki Dirichlet Test for sequences.

Suppose f = sin(x) or f = cos(x). Then f is continuous in item 0 and F(r) is bounded in item 3. So we then get

Corollary 2. [144] If g and g' are continuous and g(x) decreases monotonically to 0 on $[c,\infty)$, then Thm 16A shows that $\int_c^\infty \sin(x)\, g(x)$ and $\int_c^\infty \cos(x)\, g(x)$ both converge.

Note that "g and g' are continuous" is the same as saying "g is Class C^1". Saying the latter says that if g exists, then g' exists and g' is continuous, and we know from Theorem PL3 above that g is also Class C^0 which means g itself is continuous as well.

Comment added: I sometimes use $g(x) = e^{-\varepsilon x}$ as a "converger function" and then take $\varepsilon \to 0$, but this idea is not discussed by Bucks yet. I think ε = complex variable etc etc.

My example from raw notes: $\int_c^\infty dx \cos(x)\, 1/x^\alpha$ = convergent for $\alpha > 0$. For example, $\alpha = .0001$ says that the $1/x^\alpha$ factor decays VERY slowly, and the integral without sin(x) would diverge since this is much worse than the log case $\alpha = 1$ that diverges. The slicing of the sin(x) makes things convergent. It turns out that this integral is $\Gamma(1-\alpha)\sin(\pi\alpha/2)$ for $0 < \alpha < 1$ so I have an explicit value for all α in this range. Probably can get result for $\alpha \geq 1$ by doing parts integration (my guess).

Example 1. $\int_0^\infty \cos(x^2)\, dx$ transforms to $(1/2)\int_0^\infty \cos(u)\, du/\sqrt{u}$. We can see that at low end have $1\sqrt{u}$ which converges there. At high end it is my example above with $\alpha = 1/2$ so converges there as well. This example in x shows that an integral can converge even though the integrand does not $\to 0$.

Example 2. $\int_0^\infty \sin(1/x)\, dx/x$ transforms to $\int_1^\infty \sin(u)\, du/u$, again OK at high end since we studied this situation in Example 4 on page 141.

2D Improper Integrals [144]

Bottom half p 144. For really the first time in this subsection 3.4 on Improper Integrals, Bucks consider a domain D in E^2 instead of E^1.

Page 146 states a "Definition" of the meaning of the convergence of $\int_D f$. It allows D to be infinite, but need f continuous and *positive* on D (but f could be unbounded). Now instead of $D_n = D \cup R_n$ with rectangles R_n, you can use any more general "expanding sequence of closed sets" D_n which converges to D. It then says the double integral converges if the sequence $\int_{Dn} f \rightarrow$ a finite value c. I wonder if this is guaranteed to be the same value c for any choice of the sets D_n? (answer coming soon)

Example 1 [145] $f(x,y) = xy \exp(-x^2-y^2)$ integrated over the entire plane. They pick the obvious set of rectangles, get the integral for R_n (which luckily they can do) and then as $n\rightarrow\infty$ integral is 1/2. This example has **D unbounded and f bounded**.

Example 2. [145] $f(x,y) = y/\sqrt{x}$ over unit square, so **D bounded and f unbounded**. Think of 3D plot where you define f_n with the infinite top shaved off. Recall this was one of two methods earlier for f(x).

Doing this f_n sawed off method in detail then taking $n\rightarrow\infty$ they find integral = 1. They break the integral over the unit square into three pieces integrals 1,2,3 shown p 145A and marked in the drawing on page 146. Region 1 has a flat top at f = n coming up out of the plane of paper.

Then on top p 146 they redo this example using the alternative $r\rightarrow$ method at the left edge of the unit square, and again find integral = 1. This case is also shown in the same p 146 figure, and as $r\rightarrow 0$ the left dotted edge moves to the left edge of the unit square.

Both methods give the same result 1 for the integral. We don't know if all rectangle choices would give this same result, only the two choices tried here. Perhaps Thm 17 below says yes since $f(x,y) = y/\sqrt{x} = |f(x,y)|$ is continuous *within* the unit square.

Example 1 [146-7] . We take $f = (x^2+y^2)^{-\lambda}$ and D = unit disk. We take the D_n to be an annulus of inner radius r, for r>0 we avoid the problem at r = 0. It is easy to compute $A(D_n) = A(r)$, and we find that the answer is integral $= \pi(1-\lambda)$ as long as $\lambda < 1$ **using the r→0 limit**. Otherwise the integral diverges.

Up to now, f has been non-negative on the domain D, but now we allow general f, and we get right into some trouble.

Example 2. [147] We take $f = \sin(r^2)$ and D = first quadrant. So f happens to be bounded, but the domain is infinite. Obviously f takes both positive and negative values and this is going to make trouble:

In their first gambit, they take the R_n squares with obvious corner at (n,n). The double integral of f is easy to do and the result is $\pi/4$ for this integral.

In the second approach, the D_n are a set of quarter disks of radius n. Using polars, what we get this time is that $A(D_n) = (\pi/4)[1 - \cos(n^2)]$ which does not converge! So this **different choice of the D_n** gives a completely different result !!! We are then led to

Theorem 17. If f is continuous on D and if $\int_D |f|$ converges, then $\int_D f$ converges and has the same value for all choices of the D_n.

In Example 2 above, then, it must be that $\int_D |\sin(r^2)|$ does not converge, causing $\int_D \sin(r^2)$ to give different results for different D_n sets.

Conclusion regarding 2D integrals. Here are the theorems:

$\int_D f$ exists $\Leftrightarrow$ must get same convergent result for any D_n // bottom p 148

$\int_D f$ exists $\Leftrightarrow$ $\int_D |f|$ exists // top p 149.

In the 1D world you had in contrast

$\int_D f$ exists $\Leftarrow$ $\int_D |f|$ exists "conditional convergence"

This conditional convergence situation therefore **does not apply** to the 2D integration world. Nothing is said about 3D or other dimensionality. [In 2D, $\int_D f$ exists $\Rightarrow$ $\int_D |f|$ exists , but in 1D it does.]

Example 1. [149] The 1D integral $\int_0^\infty dx\, e^{-x^2}$ (Gaussian). Maple says the indefinite integral involves the error function, fine. There are two things to be said for this example:

(a) You can show that the tail above R is less than e^{-R} so if you want the integral accurate to .001, you could first pick R to make the tail less than .0005. You are left with the finite (0,R) integral which you could then do for example by Simpson's Rule to accuracy of .0005. Then add and the result is accurate to the desired .001. So Bucks are talking about an **approximation** method here, an interest of theirs. [For the student, approximation became less interesting with the advent of calculators doing 10 places.]

(b) In this particular example, there is a **trick** for doing the integral exactly. You write it squared, then combine into a 2D integral where D = first quadrant. Integrand is positive, so any method works. Bucks go to polar coords and use an R method to get the $R\to\infty$ result. The trick here is that dA brings in a factor of r, so then the dr integral is doable as a simple indefinite integral so integral is $\sqrt{\pi}/2$.

Note: Bucks did not mention any 3D integrals, but give one in the last exercise.

3.5 Set Functions [151] // theorems 18,19

A "**set function**" is what it says, a function whose argument is a set S, not a point p. So F(S). Earlier we dealt only with "**point functions**" like f(p). The Bucks come up with a space of sets called $\mathcal{A}$ and any set S lies in this set of sets, or collection of sets. As a **mapping**, a set function looks like F: $\mathcal{A} \to$ R . An

immediate example helps. Perhaps $\mathcal{A}$ is the collection of sets which form a volume in 3D space. A possible S might be three points plus a smooth cloud of points. I think of such a set as a "mask" or "matte". The set S has some "volume" they call A(S). A possible set function might then be M(S) which is the mass associated with set S.

The main point of this chapter is to define a set function in a certain formal and restricted way, and then show that it must be possible to write $F(S) = \int_S \varphi$ and $F'(S;p) = \varphi(p)$, where φ is a point function.

Example: $M(S) = \int_S \rho(p)\, dV$ with $M'(S;p) = \rho(p)$, and also $A(S) = \int_S dV$.

They restrict right away to set functions which are "**finitely additive**". The property stated looks a little like the linearity property of a linear point function: $f(p_1) + f(p_2) = f(p_1 + p_2)$ where p_1 and p_2 are disjoint points in space. The property says instead $F(S_1) + F(S_2) = F(S_1 \cup S_2)$ where S_1 and S_2 are disjoint sets. To me, this says F(S) is describing some "**extensive property**" like mass or charge, not like temperature.

In order to obtain the "**fundamental theorem of calculus**" in the set-function context (a function is the integral of its anti-derivative and so on), Bucks must first define "**continuity at point p_0**" for F(S). [But they don't use the word continuity.] They do this on p 152 by requiring that $| F(S) - L | < \varepsilon$ when $\delta <$ diam(S). This last means that there is a metric (distance between two points) on the space $\mathcal{A}$ and perhaps a natural norm which is this diam thing. The set S is like a vector in a vector space perhaps, (just making this up). In E^n we certainly know what diam(S) means. Maybe there is a norm(S) = diam(S) and no metric $|S_1 - S_2|$, though the latter could be max of $|p_1 - p_2|$ for the two sets. Bucks do not discuss such details. In any event, the idea is that you can imagine that each set S in $\mathcal{A}$ has some diam(S) and you can have a sequence of sets S_i which have less and less diameter until you finally arrive at a limiting set which I guess must contain a single point p_0 of E^n. They mention no such sequence. L is F(S) evaluated for the set S which contains the single point p_0, so you could write $L = F(\{p_0\})$ and then $| F(S) - F(\{p_0\})| < \varepsilon$. Their real goal here is define the meaning of the limit $\lim_{S \to p_0} F(S)$ and that is all they really do.

Immediately they define **uniform continuity** over a region D of space containing many points. This seems to be in direct analogy to our normally defined uniform continuity. Now they require

$| F(S) - F(\{p\})| < \varepsilon$ for all p in D when diam(S) $< \delta$ [δ independent of p]

and they denote $F(\{p\})$ by g(p). As usual, for any $\varepsilon > 0$ you can find a δ (independent of p) such that $| \ | < \varepsilon$ for all points p. Compare the above to regular set continuity at a point p_0

$| F(S) - F(\{p_0\})| < \varepsilon$ when diam(S) $< \delta(p_0)$

Definition of a Set Function Derivative: [153] $F'(S; p) \equiv \lim_{S \to p} [F(S)/A(S)]$ exists for all p in D.

If this limit exists, then F(S) is **differentiable** on the set D. As usual, this limit means there is some limiting object L(p) such that $| F(S)/A(S) - L(p) | < \varepsilon$ when diam(S) $< \delta(p)$. If you can find a δ which works for all p in D, then F(S) is **uniformly differentiable** on D. L(p) is going to be our $\varphi(p)$ point function when the dust settles.

For a 1D normal f'(x) we have A(S) = Δx where Δx is volume of an interval S and F(S) = ΔF is the amount of "stuff" in this volume, and we take $\Delta x \to 0$. In 3D we are used to $\nabla f(\mathbf{r})$ as one form of derivative, but the scalar set-function-derivative is a very different animal. The set derivative always generates some kind of scalar density function which of course is their goal.

Theorem 18. [153]

Let: D = open set in E^n whose boundary has "no area".
φ = a point function continuous and bounded on D
$F(S) \equiv \int_S \varphi$

Then: F(S) is uniformly differentiable on any rectangle R in D
and the set-function derivative is in fact F'(S;p) = φ(p).

So this theorem says that, given any reasonable φ, you can construct a set function F(S) which has this φ as its derivative. The Bucks are restricting things to a 2D region called D by the word "rectangle". They give a proof which seems not too fancy. Then we come to the Main Act:

Theorem 19. [154] (**Fundamental Theorem of Calculus for Set Functions**).

Let: F(S) = defined for at least all rectangles in R^n (probably restricted to n = 2)
F(S) = uniformly differentiable on some closed rectangle E in R^n
where the derivative is F'(S) = φ(p) for p in E
F(S) = finitely additive as defined above

Then: φ(p) is continuous on E, *and* one can write $F(R) = \int_R \varphi$ for any rectangle R inside E.

This theorem says that if you start off with some finitely additive and uniformly differentiable F(S) having derivative F'(S) = φ(p), then you can write F(S) as the integral of φ(p). The theorem does not allow for a general region D, but is limited to a rectangle E. The proof is a full page.

Corollary [155]. If you add to the assumptions of Theorem 19 $S_1 \subset S_2 \Rightarrow F(S_1) \subset F(S_2)$, which is the notion of "**monotone**" for set functions [where $\leq$ is replaced by $\subset$], *then* the above theorem can be extended to apply not just to a rectangle R in E but to any set S in E.

Bucks claim they will use some of this set function theory for handling change of variables in integration in Chapter 6. The general subject is called "**measure theory**" and involves something called the **Radon-Nikodym Theorem** (1930). A "set function" is really within the "theory of **Jordan content** and finitely additive measures". You can see that an integration volume dV is a "measure" and changing variables is going to deal with this object. Think of the invariant Haar measure dg on a group, etc. Bucks give a reference to Halmos's 1950 Measure Theory book. The RN theorem and set functions are addressed on page 137 of my own Riesz Sz.-Nagy book on Functional Analysis. Though this book does not seem to mention "Jordan content", it is very strong on Lebesgue measure; here from wiki:

In mathematics, the **Peano–Jordan measure** (also known as the **Jordan content**) is an extension of the notion of size (length, area, volume) to shapes more complicated than, for example, a triangle, disk, or parallelepiped.

It turns out that for a set to have Jordan measure it should be well-behaved in a certain restrictive sense. For this reason, it is now more common to work with the Lebesgue measure, which is an extension of the Jordan measure to a larger class of sets. Historically speaking, the Jordan measure came first, towards the end of the nineteenth century.