Buck Meta Chapter 4: Convergence PhL 1.19.15

This meta doc at 15 pages is one third the size of the 41 page raw notes as of 1/20/15.
The book Chapter 4 is 63 pages long and is brimming chock full of theorems (33) and examples.

Overview and the many meanings of "convergence".

In **Section 4.1** convergence refers to the convergence of **infinite series** of constants, and of the convergence of the sequence of partial sums associated such a series. Double series $\Sigma_{ij}a_{ij}$ are also considered.

In **Section 4.2** convergence refers to the convergence of a series of functions and to the sequences of partial sums associated with such a series of functions. In one sense (pointwise convergence), the functional series converges for some particular argument x with norm that of E^m where $f:E^n \to E^m$. In another sense **uniform convergence**, the functional series converges over a set of x values D with the norm that of $\mathcal{L}_{max}(D)$. I refer to this latter convergence as (uniform convergence)^f. Both these functional convergence notions involve many functions usually called $f_n(x)$.

In **Section 4.3** Bucks apply the uniform convergence ideas to infinite function series $\Sigma_n f_n(x)$ which have the particular functions $f_n(x) = a_n x^n$. These are called **power series**.

In **Section 4.4** Bucks introduce another form of convergence I call (uniform convergence)^p. It relates to a single function of two variables f(t,p) which function is continuous at t_0 in t in a manner that is uniformly convergent for all p in set E as $t \to t_0$. [p = point] An example is the function $f(t,x) = \sin(xt)/[t(1+x^2)]$ which is continuous in t at $t_0 = 0$, and as $t \to t_0 = 0$ this function exhibits (uniform convergence)^p to $F(x) = x/(1+x^2)$ for all real values of x. The variable p or x I call a bystander variable and in one application this is the parameter in an "**improper integral with a parameter**". The main results of this section are a set of what I call "order interchange theorems" which involve limits and operators like ∂_x and the integration operator I. These theorems always require that one or more functions exhibit (uniform convergence)^p.

For integrals, we want to take a limit of the parameter through the integral, or we may want to interchange the order of two integrals where one is improper. In the differential case, we want to differentiate the integral containing the parameter with respect to that parameter, and we would like to move the derivative operator through the integration and act directly on the integrand.

Section 4.5 deals with a special function **Γ(x)** *defined as* an improper integral with parameter.

Chapter 4: Convergence [pp 158-220] 1
4.1 Infinite Series [158] // theorems 1 → 14 1
4.2 Uniform Convergence [180] // theorems 15 → 21 5
4.3 Power Series [196] // theorems 22→ 25 8
4.4 Improper Integrals with a Parameter [204] // theorems 26→29 9
4.5 The Gamma Function [213] // theorems 30→33 14

Chapter 4: Convergence **[pp 158-220]**

4.1 Infinite Series **[158] // theorems 1 → 14**

Bucks develop many theorems here, most are tests for convergence of a series. Subjects include series rearrangements, alternating series, partial sums, comparison of sums with corresponding integrals. The final section has theorems about double sums, including some about the Cauchy product (zigzag order) of two regular series. The word series always means infinite series. Lots of examples in text and raw notes, but they are all omitted here.

An **infinite series** involves terms $\{a_n\}$ and partial sums $\{A_n\}$, both of which are sequences. These two sequences together compose the "infinite series" object.

[160] If the $\{A_n\}$ partial sum sequence converges, then one says "the series converges". So we may say,

$$A_n \to A \quad \Leftrightarrow \quad \text{series } [\ \{a_n\},\{A_n\}\ ] \quad \text{"converges"} \quad \text{// else "diverges"}$$

Note that diverges does not imply $A_n \to \infty$ since you could have a jumping situation where A_n is finite (bounded), but won't stabilize. It is possible for two series to diverge but their sum converges, and an example is given.

Cauchy Product of two series**:** Consider these two power series (from end of this chapter),

$$(\Sigma_{n=0}^{\infty} a_n x^n)\ (\Sigma_{m=0}^{\infty} b_m x^m) \equiv [\ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...]\ [\ b_0 + b_1 x + b_2 x^2 + b_3 x^3 + ...]$$

If you rearrange this series by increasing powers of x ($c_n x^n$) then the coefficients c_n of the powers form the series called the Cauchy Product of the series Σa_n and Σb_n .

Theorem 1: [161] (**Term Limit Test**) If series Σa_n converges, then $a_n \to 0$. Proof is totally trivial using the A_n. Contrapositive gives a very easy test for series convergence!
[Note: not true for analogous $\int_0^{\infty} \cos(x^2)\,dx = (1/4)\sqrt{2\pi}$, p 144 since integrand $\not\to 0$ as $x \to \infty$]

Theorem 2: [161] (**Comparison Test**). If $a_n \leq b_n$ and things non-negative, then Σb_n conv $\Rightarrow \Sigma a_n$ conv.

Theorem is analogous to Comparison Test for integrals Theorem 14 p 140.

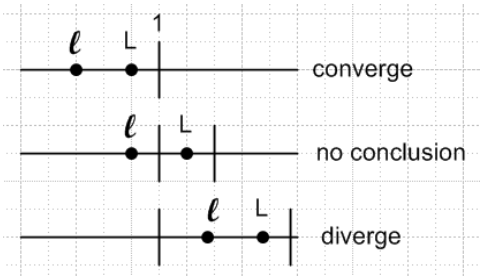
Corollary: [161] (**Two Series Ratio Test**) If $(a_n/b_n) \to L$ (positive and finite) with both a_n and b_n non-negative, then when you consider Σa_n and Σb_n, either both converge or both diverge.

Analogous to Ratio Test for integrals p 140.

Theorem 3. [161] (**Ratio Comparison Test**) Again we have both a_n and b_n non-negative and $a_{n+1}/a_n \leq b_{n+1}/b_n$ for large n. This says that the A ratio is smaller than the B ratio in the limit. Theorem says: Σb_n conv $\Rightarrow$ Σa_n conv. Proof is very simple.

Theorem 4: [161] Consider $\Sigma_{n=0}^{\infty} x^n$. For $|x| < 1$, sum is $1/(1-x)$, otherwise series diverges (real x). Proof shown is very simple.

Theorem 5. [161] (**Bounce Ratio Test**) Assume $a_n > 0$ and allow that the ratio a_{n+1}/a_n might bounce around for large n, so this ratio itself might not have a limit, but assume the ratio is bounded so lim sup ratio = L and lim inf ratio = ℓ . That is to say, the ratio is at least bouncing around inside $\ell \le a_{n+1}/a_n \le L$. Obviously we have $\ell \le L$. Theorem says that $L < 1 \Rightarrow \Sigma a_n$ converges and $\ell > 1 \Rightarrow \Sigma a_n$ diverges, otherwise no conclusion.



Corollary: [162] (**No-Bounce Ratio Test**) Now assume ratio a_{n+1}/a_n does not bounce around but has a simple limit r. Then in the previous theorem, we have L and ℓ both equal r. Then $r < 1 \Rightarrow$ converge, $r > 1 \Rightarrow$ diverge, and $r = 1 \Rightarrow$ no conclusion.

Theorem 6 [162] (**The Root Test**). Again $a_n \ge 0$ and assume lim sup $(a_n)^{1/n} = r$. Then same conclusion as last theorem: $r < 1 \Rightarrow \Sigma a_n$ converges and $r > 1 \Rightarrow \Sigma a_n$ diverges.

Theorem 7 [163] (**Sum/Integral Compare Test**) The sum 1 to ∞ of $f_n = f(n)$ and the integral of f(x) from 1 to ∞ both converge or both diverge if f is continuous and monotone decreasing (to 0, meaning $f \ge 0$) on $[1,\infty)$.

Corollary. [163] Series $\Sigma_1^{\infty} (1/n^p)$ and $\Sigma_1^{\infty} (1/n) / (\ln n)^p$ converge $p > 1$ else diverge.

Theorem 8 [p 163] (**Raabe's Test**). Assume $a_n > 0$ as usual, look at ratio $r_n \equiv a_{n+1}/a_n$. Suppose $r_n \le 1 - p/n$ for $p > 1$ and large n. Then series converges.

Theorem 9: [164] (**Abs Value Test**) If $\Sigma |a_n|$ converges, so does Σa_n.

Fact: It is possible for $\Sigma|a_n|$ to diverge while Σa_n converges. In this case, Σa_n is **conditionally convergent**. If $\Sigma|a_n|$ converges you have **absolute convergence**. This is the same idea as with integrals on page 141.

The following two theorems involve **products of series** :

Theorem 10 [165]: (very obscure) Here $a_n \to 0$, $\Sigma|a_{n+1} - a_n|$ converges, B_n is bounded. Then the product sequence $\Sigma a_n b_n$ converges. Says nothing about whether Σa_n or Σb_n converges. [compare Ch 3 Thm 16]

Theorem 10A = Corollary 1: (very obscure) If monotone $a_n \to 0$ and B_n is bounded, then the product sequence $\Sigma a_n b_n$ converges. Says nothing about whether Σa_n or Σb_n converges. [compare Ch 3 Thm 16A]

Corollary 2 [166] (**Leibniz Alternating Series Test**) : If a_n monotone $\to 0$, then $\Sigma (-1)^{n+1} a_n$ converges.

Notice that this does not seem to have a simple integral analog. But it's a bit like $\int_c^\infty \sin(x)\, g(x)$ where the sign alternation is provided by sin(x), see Ch 3 Corollary 2 [144].

Theorem (Exercise 12 on p 179): For a convergent alternating series, if you stop at some term, the |error| is ≤ the |next term|. Also, the approach to the limit alternates between above and below.
I have not proven this, but it gets used later on page 187. Proof seems would be very easy.

Theorem 11: [169] Only if a series is absolutely convergent are you allowed to **rearrange** the series without changing the sum.

Theorem A1 [170] Assume f is continuous, positive, and *decreasing* on [1,∞]. Then the difference between a sum and an integral from 1 to n is bounded on both sides this way:

$$f(n) \quad \leq \quad [\, \Sigma_{k=1}^{n} f(k) - \int_1^n f(x)dx] \quad \leq \quad f(1).$$

If we express $\Sigma - \int$ as an error C_n, then $\Sigma = \int + C_n$ where $f(n) \leq C_n \leq f(1)$

Theorem A2. $f(1) \leq [\, \Sigma_{k=1}^{n} f(k) - \int_1^n f(x)dx] \leq f(n)$ if f(n) *increasing*

Theorem 12: The quantity $[\Sigma - \int - f(n)/2]$ is bounded above and below provided that f is positive *and* has negative curvature in the entire range 1 to ∞ (see p 171 figure). What this says is

$$a \leq [\Sigma_1^n f(k) - \int_1^\infty f(x)dx - f(n)/2] \leq b \qquad a = f(1) - f(2)/2 \qquad b = f(1)/2$$

This really shows that $\Sigma_1^n f(k) - f(n)/2$ is a good fit to the integral.

Double Series Stuff p 172

The **convergence definition** they choose on page 172 is Cauchy like in that the abs value shown has to → 0 as shown. In the general Cauchy you would have separate N and M, but here it is N and N as shown. They are keeping the matrix square NxN as they increase N to infinity.

Theorem A: [173 top] $\Sigma_{ij} |a_{ij}|$ converges $\Rightarrow \Sigma_{ij} a_{ij}$ converges [like Theorem 9 p 164]
If the abs sum converges, so does the other sum, series is **absolutely convergent**.

Question: Look at Ch 3 meta below Thm 17. I would guess that the double sum is more like the double integral and so either both double sums above converge or both diverge. Is this true?

Theorem B: [173] $\Sigma_{ij}\, a_{ij}$ is absolutely convergent $\Leftrightarrow$ you can rearrange arbitrarily and maintain the sum. This is like Theorem 11 p 169.

Corollary C: [173] If $\Sigma_{ij}\, a_{ij}$ is absolutely convergent, you can rearrange as the zig-zag ordering which is in fact the Cauchy Product sequence as shown above. This theorem involves one double series.

Theorem 13. If Σa_j and Σb_j are each absolutely convergent to A and B, then the zig zag ordered double sum Σc_n converges to AB. This zig-zag ordering is in fact the Cauchy product defined earlier.

Theorem 13A: [173] $\Sigma a_j \rightarrow$ A and $\Sigma|a_j| \rightarrow$ A' (abs conv) => $\Sigma_{zz} a_i b_j \rightarrow$ AB.
and $\Sigma b_i \rightarrow$ B but $\Sigma|b_j|$ diverges

13A is a little stronger than 13. It is called Theorem of Mertens and is in the Knopp book.

Theorem 13B: [174] If both Σa_j and Σb_j are conditionally convergent, then Σc_n may diverge.

Theorem 14. [174] If $a_{ij} \geq 0$, then columns-first sum converges $\Leftrightarrow$ rows-first sum converges

The topic of double series is suddenly ended and here Bucks consider 5 examples of single series which contain a parameter.

4.2 Uniform Convergence **[180]** // **theorems 15 → 21**

Suddenly the topic is changing from the study of convergence of 1D and 2D series of *constants*, to the study of the convergence of 1D series of *functions* of one variable x. Again I suppress Examples.

Since "uniform" has always been a problem for me, I copy all the background from raw notes here:

Note: **Uniform *continuity*** for a function f(x) was treated in Section 2.3. Recall that you need a δ that works for *all* x in the domain D. Here we are talking uniform *convergence* of a sequence of functions or of a series whose terms are functions of x.

Def: [180A] A *series* $\Sigma f_n(x)$ **converges pointwise** to f(x) on domain D if it converges for each x in D. You might find that δ depends on x, and there is no specific δ that works for all x in D.

Def: [180B] A *sequence* $\{f_n(x)\}$ **converges pointwise** to f(x) on domain D if it converges for each x in D. That is to say, for any given x in D, you can find N such that $\| f_n(x) - F(x) \| < \varepsilon$ for $n > N$. Again, you might find that N depends on x, so you really have N(x).

Def: [182A] A *sequence* $\{f_n(x)\}$ **converges uniformly** to F(x) on domain D if it converges for all x in D "with the same N". That is, there is an N such that $\| f_n(x) - F(x) \|_D < \varepsilon$ for $n > N$ *and* for all x in D. The point is that the same large integer N works for all x in D, you don't need N(x).

Important point: Here $\| f_n(x) - F(x) \|_D$ is the max norm, sometimes called the infinity norm. I have much more on this later. Earlier definitions just have a normal E^m geometric norm.

Def: [183A] A *series* $\Sigma f_n(x)$ **converges uniformly** iff the *sequence* of partial sums $F_n(x)$ converges uniformly to F(x) as per the above definition.

Def: [183B] A *series* $\Sigma f_n(x)$ **converges uniformly** iff the tail of the series converges uniformly to 0. Can write this last as $\| \Sigma_{k=n}^{\infty} f_n(x) \|_D \to 0$. This is stated as if it were an alternate definition.

Notice how the series convergence leans on the convergence of the partial sum sequence.

Lots of examples are given. A good one is a series whose terms are $f_n(x) = n^2 x\, e^{-nx}$ on [0,1]. This function has a peak at x = 1/n where $f_n(1/n) = n/e$. Thus, as $n \to \infty$, $\| f_n(x) \|_D \to \infty$ since this is the max norm and the max is going to ∞. Thus, not uniformly convergent on [0,1]. It is pointwise convergent since if you first *pick* some x, you can find N so n>N meets your ε rule (due to the e^{-nx}). This is explained on page 182. It is always a slippery subject.

Definition and Theorem (no number, top page 183)

series is uniformly convergent ⇔ partial sum sequence is uniformly convergent
series is uniformly convergent ⇔ series is pointwise convergent and $| \text{tail}_n |_E \to 0$

def: [183] **Cauchy Property for sequence**:

For any ε>0, you can find N such that $| f_n - f_m |_E < \varepsilon$ for n,m both > N .

Theorem 15 (and precursor p 183)

Sequence f_n has Cauchy property ⇔ sequence f_n is uniformly convergent.

Corollary [p 184] :

series is uniformly convergent ⇐ $\| \text{tail}_{n,m} \|_E \to 0$ // Cauchy tail

Theorem 16 (**Weierstrass Comparison M-Test**). [184] Consider a series $\Sigma_k u_k(x)$. Suppose you know that $\| u_k(x) \|_E \le M_k$ for all k (perhaps in an infinite tail) and suppose you also know that the series $\Sigma_k M_k$ converges. Then $\Sigma_k u_k$ is *uniformly* convergent on E. Norm here is the max norm.

Example: Consider $f_k(x) = k^2 x\, e^{-kx}$ on [0,1] which we just saw has max = k/e and is not uniformly convergent on [0,1]. Then $f_k(x) \le k/e$ so you could try $M_k = k/e$, but alas, $\Sigma_k M_k$ does not converge so test is a no go.

Theorem 17: [184] Suppose $f_n(x) \to F(x)$ uniformly on E. If the $f_n(x)$ are *continuous*, so is F.

Suddenly the word "continuous" pops out of nowhere in this last. The premise here is that the sequence of functions shown converges uniformly to F(x). Restate this for a series:

Theorem 17A: Suppose $\Sigma_{k=1}^{n}\ u_k(x) \rightarrow F(x)$ uniformly on E. If the $u_k(x)$ are continuous, so is F(x).
[Corollary 1 top of page 185]

These both say that if a series or sequence converges uniformly to F(x) on [a,b], then F(x) must be continuous on [a,b] so it cannot be infinite at one of the endpoints (I think).

Theorem 18: [185] Let E be closure of S which is open, like (0,1). Suppose f_n converges uniformly on S, and the f_n are continuous on E. Then f_n converges uniformly on E.

This says if uniformly convergent on (a,b), then so also on [a,b].

The notion of uniform convergence is strongly related to the notion of **order interchange** in various contexts. Here is our first order interchange theorem:

Theorem 19. [186] (**order interchange for sequence and integral in E^2**). The domain D of the integration is closed and bounded (compact) in E^2. The sequence of functions $\{f_n(x)\}$ converges uniformly on D to some F(x), so $\lim_{n\rightarrow\infty} f_n(x) = F(x)$.

Then: $\lim_{n\rightarrow\infty} [\ \int_D f_n(x,y)dA] \ = \ \int_D [\ \lim_{n\rightarrow\infty} f_n(x,y)]\ dA$ which is $\int_D F(x)\ dA$

This says you can take the $n\rightarrow\infty$ right through the integral if premise conditions are met. In other words, you can interchange the order of "sequence limit" and "integration" which itself is a certain limit.

Corollary [186] : (**order interchange for series and integral in E^2**) If $u_k(x)$ are continuous on D and if $\Sigma u_k(x) \rightarrow F(x)$ converges uniformly on D, then if you need to integrate F(x) over D, you may do so "term by term".

This is just the above Theorem 19 restated as

$\lim_{n\rightarrow\infty} [\ \int_D (\Sigma_{k=1}^{n} u_k(x,y)\ )dA] \ = \ \int_D [\ \lim_{n\rightarrow\infty}\ (\Sigma_{k=1}^{n} u_k(x,y)\ )]\ dA$ which is $\int_D F(x)\ dA$

term by term integration

Theorem 20. [188] (**Arzela-Osgood-Lebesgue**) Assume $f_n(x) \rightarrow F(x)$ *pointwise* on bounded D and f_n and F are integrable on D. Suppose *also* that $\| f_n(x) \|_D \leq M$ (a constant) for all n. Then

$\lim_{n\rightarrow\infty} [\ \int_D f_n(x)\ ] \ = \int_D [\lim_{n\rightarrow\infty} f_n(x)]$ perhaps D is in E^n ?

and you can do order interchange between sequence limit and integration. So this is a refinement of Theorem 19 which does not require uniform convergence, but requires something else.

Comment: The above three items are treated where domain D lies in E^2. Bucks are strangely silent about whether the theorems are valid for E^1 or for general E^n. I presume all is OK for full E^n. The next theorem is stated for E^1.

Theorem 21 [188] (**Series/Derivative Order Interchange**)

Suppose $\Sigma u_n(x) \rightarrow F(x)$ pointwise on [a,b].

Suppose $u'_n(x)$ exists and is continuous on [a,b].

Suppose $\Sigma u_n'(x) \rightarrow G(x)$ uniformly on [a,b].

Then $G(x) = F'(x)$. That is to say, $\lim_{n\rightarrow\infty} [\Sigma_{k=1}^{n} \partial_x u_n(x)] = \partial_x[\lim_{n\rightarrow\infty} \Sigma_{k=1}^{n} u_n(x)]$

$G(x) = \partial_x [F(x)]$

term by term ∂_x

Notice that this theorem requires F(x) pointwise converge, but F'(x) uniform converge, and more.

Bucks consider on page 191 a certain Van Der Waerden function which is continuous but at the same time is **nowhere differentiable**. A grab bag item. Here is another:

Tietze's Extension Theorem [193]

You have E closed and bounded in E^n and f(p) [real] is continuous and bounded on E (so $|f(p)| \leq M$). Then there is a way to extend f(p) to all of E^n (call it F(p)) such that F(p) = f(p) on E *and* such that F(p) is continuous and bounded on all of E^n. The proof constructs a solution to the problem.

This is easy to see in one dimension, and plausible in 2D though not obvious there, not to mention higher dimensions. In 1D you just could just put linear tapering walls outside some [a,b] = E. This would not be differentiable at the walls, but it would be continuous and that is all that is required.

4.3 Power Series **[196]** // **theorems 22→ 25**

Theorem 22: [196] Power series $\Sigma a_n x^n$ has some R **radius of convergence** so $|x| < R$ gives convergence. You can find R from either the ratio test or root test, as shown top p 197. Could find R = 0 or $R = \infty$.

You could restate this replacing x by g(x) so $|g(x)| < R$ gives you some convergence domain in x.

Corollary [197] You can generalize trivially to $\Sigma a_n(x-c)^n$.

Theorem 23. [197] The power series $\Sigma a_n x^n$ with R converges *uniformly* on $|x| \leq b < R$.

Theorem 24. [198] For $|x| < R$, you can differentiate $f(x) = \Sigma a_n x^n$ term by term to get f'(x). The series f'(x) has the same R as f(x), and the same uniform convergence region.

Corollary 1 [198] For $f(x) = \Sigma a_n(x-c)^n$, you can identify $a_n = f^{(n)}(c)/n!$.

You can keep diffing using Thm 24 and then just set x = c in the result to prove this corollary.

Corollary 2 [198] $\Sigma a_n(x-c)^n = \Sigma b_n(x-c)^n$ in ball near x, then $a_n = b_n$.

This says that a function analytic near x = c can have only one power series there, and it is the Taylor series we studied earlier.

Notice how the Bucks have set us up for easy treatment of the above theorems by previously discussing the notions of uniform convergence and order interchange for differentiation. Power series is just an application of all that previous stuff to series with terms $f_n(x) = a_n x^n$.

Theorem 25 [199] (**Abel**) If $f(x) = \Sigma a_n x^n$ has radius of convergence R, and if f(x) converges at x = R, then f(x) is *uniformly* convergent on [0,R]. Same idea if you do R → -R.

Corollary [200]. If $f(x) = \Sigma a_n x^n$ has radius of convergence R, and if f(x) converges at x = R to some value S, then you must get this same value S if you take the $\lim_{x\to R}[\Sigma a_n x^n]$.

The section closes with various example, one of which is a series definition of e^x called E(x).

4.4 Improper Integrals with a Parameter **[204] // theorems 26→29**

My notes begin with a long review of notions of continuity and convergence, each of the non-uniform and uniform varieties. Here is then the raw notes summary regarding convergence:

1. In the discussion of continuity and uniform continuity, we had $\lim_{p\to p_0} f(p) = f(p_0)$ using the E^m norm saying that that $|f(p) - f(p_0)|_{E^m} < \varepsilon$. We could then bring into the discussion a sequence of points p_n and we could then say $\lim_{p_n\to p_0} f(p_n) = f(p_0)$ and we then have convergence of the sequence $\{f(p_n)\}$ which is a sequence of points in E^m. This convergence is associated with the continuity of the function f(p). In the case that the same δ works for all p_0, we had uniform continuity over E, and you *could* say that we have uniform convergence of $f(p_n) \to f(p_0)$ where $p_n \to p_0$ where p_0 is any point in E. In other words, we are saying that $f(p_n) \to f(p_0)$ converges to p_0 sort of "uniformly" for any p_0 in E. This use of the phrase "uniform convergence" is not used by the Bucks. You might call it (uniform convergence)^p since it involves a sequence of points. Note that, although f(p) is a function, here we are really talking about sequences of points, $\{ p_n \}$ and $\{ f(p_n) \}$. We are *not* talking sequences of function, only one function f is involved in the discussion.

2. In the above subsequent discussion of sequences of functions $f_n(p)$, we talked above about $f_n(p)$ converging pointwise to f(p) on D, which involved the E^n norm, and we talked about f_n converging uniformly to f using the max norm over D. We say that f_n converges uniformly to f over D, and this is the official definition of the phrase "uniform convergence". Maybe call this (uniform convergence)^f since it involves a sequence of functions.

The real distinction is this. In case 1 we are dealing with points in the space of points E^n with the E^n norm, whereas in case 2 we are dealing with functions in the space of functions called $\mathcal{L}_{max}(D)$.

The topic now changes toward consideration of functions of two variables in a certain sense. The functions are not like f(x,y) where (x,y) = **r** = a point in E^2. In that case, we usually treat the two variables x and y as the single variable **r**. Rather, we consider here f(p,t) where p is going to be what I call a bystander parameter and where p in E and t in T are "unrelated variables". I found it useful in the raw notes to ponder how one might discuss the notion of continuity for functions of two somewhat unrelated variables f(p,t). The Bucks seemed a little abrupt on this transition to treatment of bystander parameters and a new use of the word "uniform convergence". I do this as a set of "upgrades" and shall copy the raw notes stuff here:

Then consider this ε δ statement:

1. For any $\varepsilon > 0$ and for a specific p in E and for a specific t_0 in T,
   we can find $\delta(t_0,p) > 0$ such that $| f(t,p) - f(t_0,p)| < \varepsilon$ when $|t-t_0| < \delta$.

I would describe this as "continuity at a point t_0 in T with a bystander parameter p in E".
I would also say that f(t,p) "converges" to $f(t_0,p)$ as $t \to t_0$.
This means there are sequences t_n so that $f(t_n,p)$ "converges" to $f(t_0,p)$ as $t_n \to t_0$.

Our first "upgrade" of this concept would be:

2. For any $\varepsilon > 0$ and for *all* p in E and for a specific t_0 in T,
   we can find $\delta(t_0) > 0$ such that $| f(t,p) - f(t_0,p)| < \varepsilon$ when $|t-t_0| < \delta$.

I would describe this as "continuity at a point t_0 in T which is uniform over E".
I would also say that $f(t_n,p)$ (converges uniformly)^p over E to $f(t_0,p)$ as $t_n \to t_0$.
Notice that there is only one function here, it is called f.

A different upgrade of item 1 might be this:

3. For any $\varepsilon > 0$ and for a specific p in E and for *all* t_0 in T,
   we can find $\delta(p) > 0$ such that $| f(t,p) - f(t_0,p)| < \varepsilon$ when $|t-t_0| < \delta$.

I would describe this as "uniform continuity over T with a bystander parameter p".
One could also say we have (uniform convergence)^p of $f(t_n,p) \to f(t_0,p)$ as $t_n \to t_0$.
The sequence involved is a sequence of *points* $\{t_n\}$ or $\{ f(t_n,p) \}$; only function is f.

Now in the next item we combine both the above upgrades:

4. For any $\varepsilon > 0$ and for *all* p in E and for *all* t_0 in T,
   we can find $\delta > 0$ such that $| f(t,p) - f(t_0,p)|$ when $|t-t_0| < \delta$.

I would describe this as "uniform continuity over T which is also uniform over E".
This latter is a strange new use of the word "uniform". The statement involves being uniform in two different spaces at the same time, E and T. The "continuity" aspect only involves space T.
One could also say we have (uniform convergence)^p of $f(t_n,p) \to f(t_0,p)$ as $t_n \to t_0$.

The sequence involved is a sequence of *points* $\{t_n\}$ or $\{ f(t_n,p) \}$; only function is f.
Notice that we still have the E^m norm, not the max norm. We still have only one function.
So even this double upgrade thing has nothing to do with (uniform convergence)^f of a set of functions
$f_n \rightarrow f$ in the Hilbert Space of the max norm.

Now can I connect what Bucks say on page 204 with one of the "upgrades" shown above? I first thought they were doing the complicated upgrade #3, but now I think it is upgrade #2 where we have ordinary continuity at the point t_0 and where we are uniform in the bystander space p in E. From above:

> 2. For any $\varepsilon > 0$ and for *all* p in E and for a specific t_0 in T,
> we can find $\delta(t_0) > 0$ such that $| f(t,p) - f(t_0,p)| < \varepsilon$ when $|t-t_0| < \delta$.
>
> I would describe this as "continuity at a point t_0 in T which is uniform over E".
> I would also say that $f(t_n,p)$ (converges uniformly)^p over E to $f(t_0,p)$ as $t_n \rightarrow t_0$.
> Notice that there is only one function here, it is called f.

Restate this as

> For any $\varepsilon > 0$ and for a for specific t_0 (in some space T in E^n)
> one can find $\delta > 0$ such that
> $| f(t,p) - f(t_0,p) |_{Em} < \varepsilon$ when $|t-t_0|_{En} < \delta$, where δ works for all p in E.

In the Buck definition of "uniform convergence with a parameter", instead of saying one can find a δ where $|t-t_0| < \delta$ (a neighborhood ball), they say one can find a neighborhood N around t_0. For me, the distinction is very minor, though N is allowed to be more general in shape than a ball. I can restate the above ε,δ definition of "uniform convergence with a parameter" using their N :

> For any $\varepsilon > 0$ and for a for specific t_0 (in some space T in E^n)
> one can find a neighborhood N around t_0 such that
> $| f(t,p) - f(t_0,p) |_{Em} < \varepsilon$ when t lies in N, where N works for all p in E.

As noted above, it took me a long time to get happy with the above paragraph and to make the distinction between its implied (converges uniformly)^p over E which differs from (converges uniformly)^f which is a different kind of uniform convergence involving a function space rather than a point space.

The first example of this concept is this:

Example: Point p is now called x, it is the bystander variable and lies in some interval E. Consider

$f(t,x) = \sin(xt)/ [t(1+x^2)]$ and $t_0 = 0$ is the specific point of interest in T

$f(t_0,x) = f(0,x) = \lim_{t \rightarrow 0} \{\sin(xt)/ [t(1+x^2)]\} = x/(1+x^2) \equiv F(x)$

Our goal of the example is to prove that $\lim_{t\to 0} f(t,x) \to F(x)$ in fact (converges uniformly)^{p} for all real x. You always know it is not (converges uniformly)^{f} if only a single function is involved. I have lots of raw notes on this example.

But the main interest the Bucks have in this (converges uniformly)^{p} of f(t,p) as $t\to t_0$ is in regard to integration and differentiation. We do integration first, then differentiation after that.

In the first case, f(t,p) has the form $g(t,p) = \int_c^t du\, f(p,u)$ where t is taken to be $t_0 = \infty$. Here the bystander parameter p is just some variable of the integrand function that is not integrated over in the integral. The big question here is whether you can take a limit $\lim_{p\to p0}$ through the integral or not. It is then really a case of order interchange of two limits, one being $\lim_{p\to p0}$ and the other being $\lim_{t\to\infty}$ for the upper integration endpoint. A directly analogous situation would be when the function f(t,p) has the form $g(t,p) = \Sigma_{k=1}^{t} a_n(p)$ where t is taken to be $t_0 = \infty$ on the integer lattice In the first case t = r, whereas in the second case t = n. Several theorems are developed:

Theorem 26 [206] (**Comparison Test**).
If:

- f(p,u) is continuous in p over E, and in u on [c,∞)
- $|f(p,u)| \le$ some g(u) for all p in E and for all u in [c,∞)
- $\int_c^\infty du\, g(u)$ converges

Then:

$\lim_{r\to\infty} \int_c^r du\, f(p,u)$ (converges uniformly)^{p} for all p in E to some $F(p) \equiv \int_c^\infty du\, f(p,u)$

Theorem 27 [206] (**Follow On**). Assume the premises shown above so $F(p) \equiv \int_c^\infty du\, f(p,u)$ converges uniformly on E. Then F(p) is *continuous* at p.

Corollary 27 [206]. (**order interchange of limits**) Assume $F(p) \equiv \int_c^\infty du\, f(p,u)$ converges uniformly on E. Theorem 27 then shows that $\lim_{p\to p0} F(p) = F(p_0)$, which is the meaning of F(p) is continuous at p_0. This says that

$$\lim_{p\to p0}\left[\int_c^\infty du\, f(p,u)\right] = \int_c^\infty du\, f(p_0,u) = \int_c^\infty du\, \{\lim_{p\to p0}[f(p,u)]\}$$

$$F(p) \qquad\qquad F(p_0)$$

and that in turn says you are allowed to move the limit through the integration *in this case*! The two limits interchanged are $p\to p_0$ and upper endpoint $r \to \infty$. The text of the Corollary adds that p_0 must be a cluster point in E. I think this is added to make sure that p is a continuous variable in some sense, and not perhaps some isolated lattice point in E.

Example 1: Uses p = x in E^1 and $f(x,u) = x^2/[1 + x^2u^2]$ and c = 1. Using the comparison test above, it is easily shown that $\int_1^\infty du\ f(x,u)$ converges uniformly for all x, so $[-\infty,\infty] \equiv E$. Now take the point p_0 to be $x = \infty$ which lies in E. Then the Corollary above says

$$\lim_{x\to\infty} [\ \int_1^\infty du\ f(x,u)] \ = \ \int_1^\infty du\ f(\infty,u) \ = \ \int_1^\infty du/u^2 \ = 1.$$

This same result is shown to obtain by just "doing the integral" naively and using l'Hospital, but here we are carefully justifying the process.

Theorem 28 [207] (**order interchange of integrations**)
If :

- f is continuous in u for $[c,\infty)$ and in x for [a,b]
- $\int_c^\infty du\ f(x,u)$ converges uniformly for x in [a,b]

then $\int_c^\infty du\ [\ \int_a^b dx\ f(x,u)\ ] \ = \ \int_a^b dx\ [\ \int_c^\infty du\ f(x,u)\ ]$

This is a generalization of Theorem 8 of Section 3.2 to the fancier case that one upper endpoint is infinite, which involves a limit $\lim_{r\to\infty}$. Here is that Theorem 8 where the ∞ is the finite value d :

$$\int_R f \ = \ [\ \int_a^b \int_c^d dx\ dy\ ]\ f(x,y) \ = \ \int_a^b dx\ [\ \int_c^d dy\ f(x,y)\ ] \ = \text{could do in other order}$$

It is probably true that $\int_c^d dy\ f(x,y)$ converges uniformly on x in [a,b] if f is continuous, so the second bullet item is automatically met for finite endpoint d. In Chapter 3 it was shown that the rectangle integral $\int_R f(x,y)$ always exists and is well defined as long as f is continuous on the rectangle. This was Theorem 1 p 100, later generalized by Theorem 9 p 114 showing that you can do either single-variable integration first, and then do the other second. Since you can do the single integration first, it must converge for all values of the other variable, and that means uniform convergence on x in [a,b] which is the second bullet premise above.

In Theorem 28 we only have the $r\to\infty$ limit, apart from the implied Riemann integration limits, so this theorem is not an "order interchange of limits" the way Corollary 27 is above.

Bucks give a simple counterexample showing how lack of uniform convergence can result in a double integral giving different results when done in the two orders.

Well, let's try one more time. Imagine that integration on a,b is an integral operator I,

$$\int_a^b dx\ f(x,u) \ = \ [I\ f](u)$$

$$\int_a^b dx\, h(x) = I\, h$$

Then Thm 28 says (reversing the sides)

$$I\left[\int_c^\infty du\ f(x,u)\right] = \int_c^\infty du\ [I\, f](u)$$

and this says we can move the integral operator I through the du integration, so that we have order interchange between the integral operator I and the integration on du which involves the $r\to\infty$ limit.

The following theorem is the corresponding theorem where the integral operator I is replaced by the differential operator ∂_x :

Theorem 29 : [208]
If:

- f is continuous in u for $[c,\infty)$ and in x for [a,b]
- $\int_c^\infty du\ f(x,u)$ converges uniformly for x in [a,b]
- $\partial_x f$ is continuous in u for $[c,\infty)$ and in x for [a,b]
- $\int_c^\infty du\ \partial_x f(x,u)$ converges uniformly for x in [a,b]

Then:

$$\partial_x\left[\int_c^\infty du\ f(x,u)\right] = \int_c^\infty du\ [\partial_x f(x,u)] \qquad \text{for any x in [a,b]}$$

The first two premises are the same as for Theorem 28, but we have to add the two corresponding premises regarding $\partial_x f$. This theorem is stricter than Theorem 27 because differentiation is a less smooth operation that integration and requires these extra conditions.

Example: Start with $1/x = \int_0^\infty e^{-xu}\, du$ and apply ∂_x to both sides and apply Thm 29 to get the new equation $1/x^2 = \int_0^\infty u\, e^{-xu}\, du$. Can do this again and again. Lots of other examples treated in Buck and in my raw notes.

4.5 The Gamma Function **[213]** // **theorems 30→33**

An idea presented here is the definition of a function in terms of an integral of another function. The first example is log(x) as integral of 1/x. Second example is arctan(x) as an integral of $1/(1+x^2)$, and how given that integral, you can derive various properties of the arctan function. But the big example is the Gamma function integral shown in (4-20). Cleverly, Bucks already computed this integral for integer x and showed it as the n! result as top page 214. This example is going to continue now for 4 pages.

Theorem 30 [214] $\Gamma(x+1) = x\Gamma(x)$ for any $x > 0$.

This familiar fact is proved from the integral definition (4-20) using parts integration.

By changing integration variables in (4-20) you can write down lots of other forms of the integral definition of Γ(x) and four such alternatives are given on mid p 214.

Theorem 31 [214] **Half integer formulas**. We know Γ(1/2) just doing the integral, so Γ(n/2) can be done using Theorem 30's little formula. They then talk about "extending" the definition of Γ(x) to negative half integer values in the obvious manner. They don't mention the fact that really the parts integration provides an analytic continuation into a new convergence region, perhaps they have that in their appendix and I have seen it perhaps in Ahlfors or elsewhere. It then exposes the poles on the left. In this section the term pole does not appear since they have not talked at all about complex variables.

Theorem 32. [216] Derivation of **Stirling's** Formula for Γ(x).

The proof is a long one using a certain approximation method of breaking an integral into 3 terms and showing that only the middle one matters for large x. Two Lemmas are required along the way. I did some of it, but really it is not critical, nothing magic is done here.

Theorem 33. [218] **Beta** B(p,q) is defined as a certain integral is shown equal to Γ(p)Γ(q)/Γ(p+q).

The proof is pretty easy. It starts by writing Γ(p)Γ(q) as a double integral as shown in A. This is converted to polar coordinates, and further shuffling gives the desired result as shown p 219 B.

I am quite familiar with "special functions" being defined by their integral representations as well as by their series, and of using things like connection formulas between the integrals to extend the domain of the function. For example F(a,b,c;z) hypergeometric, an old friend (now out of practice).