Buck Chapter 6 Meta Notes PhL 5.16.15

Chapter 6: Applications to Geometry and Analysis [pp 296-365].. Error! Bookmark not defined.
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6.1 Transformations of Multiple Integrals **[296]**

This section is the Buck's version of deriving the Jacobian volume element for integration rule. Their approach is very different from my approach in tensor doc.

What they do in this section:

- **Claim** (no proof) the J = detS finite volume ratio rule for *linear* transformation L, **Theorem 1**.
- Example: verify this rule for a finite **tetrahedron** with one corner at the origin.
- Show a **complicated *area* transformation rule** for linear transformation L: $E^2 \rightarrow E^n$.
- Claim the **J volume rule for a *general* transformation** and prove this in **Theorems 2,3,4.**
- **Compare** the J rule with change of variables in 1D
- Restate their pet **box notation** for integration volume on page 306
- **Examples** 1,2,3 showing how to compute certain **2D integrals** two ways. One way is the brute force way doing integral as stated. Second way is to use some kind of simplifying transformation and then do the integral in the new variables with J added.
- Example 4 shows how you might deal with an integration whose area **includes** a place where J = 0.

In their proof of Theorem 3, they get to make use of their fancy set-derivative idea developed in an earlier chapter. They are adamant in their approach, just as I am with my N-piped approach.

The Bucks are much more interested than I am in fine details and rigor in this business. They even claim the subject was one of current research in 1956 when the first Buck was written. Phrase algebraic topology is mentioned, as is "measure theory". I think the Bucks know all about measure theory and the fancy methods, just as they know about complex variables, but they are trying to explain things in this chapter in their dumbed-down world for the student reader. They wanted to say at least something about the Jacobian Integration Rule and their section title is "transformations of multiple integrals".

I recall when I first encountered the J integration volume rule. I was very uncomfortable about it, teachers just used it without explaining why it worked in general. Students understood the idea in polars from simple geometry, but student had no notion of what I have presented in tensor doc on this subject.

The basic idea is very simple: (1) you could first show that *finite* volumes transform with J under a linear transformation, like their tetrahedron example. (2) If $J \neq 0$, then this would be true for a *differential* volume since the transformation is linear locally; (3) then you just make J(x) be a function of x for the general case. Now they never prove item (1) and neither have I, but I think I could. My method fills a finite region with abutting differential N-pipeds (think polars) and for each of these we have a little $J(\mathbf{x})$ rule.

6.2 Curves and arc length **[313]**

You should think of a curve as a *mapping* like $\gamma : E^1 \rightarrow E^n$. Different mappings can give the same trace, so do not think of the trace itself as the curve. The curve is $\boldsymbol{\gamma}(t)$ [bolded since vector in E^n].

The curve can cross itself with different tangents $\boldsymbol{\gamma}'(t)$ at the same point in E^n, for example, a simple crossing is a double point. Curve can be closed on itself so start and end are at same point. A curve with no self-crossings is a **simple curve** = a **Jordan curve**.

Think of a curve as **r**(t) maybe in better notation. The tangent at domain value t is d**r**/dt = **v**(t). Sometimes if the curve has a sharp corner, you find **v** = 0 at the corner, that is a **zero-speed point**. I made an example with a corner which is not a zero speed point, but they don't mention that possibility.

They have some nice examples.

Example 1 = page 315 trace V like	has a zero speed point
Example 2 = a straight line	
Example 3 = page 316 trace in 2D	has a double point, curve open
Example 4 = page 317 trace in 3D	has a double point, curve closed

Curvature. I rolled my own on this topic. For any smooth non-planar curve, I show that local to a point on the curve the curve is planar and can be fitted to a circle and thus has a curvature radius R, but the curvature itself is called k = 1/R. In (6.17) Bucks produce an expression for k in terms of the velocity and acceleration of the curve, that is, in terms of $\boldsymbol{\gamma}'(t)$ and $\boldsymbol{\gamma}''(t)$ at point t. They expend 3 pages proving this, but I did my own derivation with some Maple help and my result duplicates theirs.

Arc Length. Some curves have infinite arc length and are thus not **rectifiable**. The formula they give is fine by me:

$$L(\mathbf{r}(t)) = \int_a^b |\, d\mathbf{r}/dt|\, dt = \int_a^b v(t)dt = \int_a^b ds = \int_a^b |d\mathbf{r}| \qquad (6\text{-}20)$$

They give an elaborate derivation of this formula for arc length as Theorem 5. [321]

Example 4: [322] just shows the computation of a simple arc length integral, a form of helix in 3D.

Equivalent Curves. Have the same trace, one is just a respeeding of the other. One curve will have unit speed at every point.

Theorem 6; [323] Two equivalent curves have the same direction at the same point **p** in space.
Theorem 7; [323] Smoothly equivalent curves have the same arc length.

The comment that there is ongoing theory work on classes of curves.

Curve mapped through a transformation. Closed goes to closed, but map pick up extra loops. Curves placed on surfaces to make things clearer. Examples given.

Conformal. If the R matrix has the form $R = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ then the relation between d**x**' and dx says

$dx' = a\,dx + b\,dy \quad \Rightarrow \quad \partial x'/\partial x = a \quad \partial x'/\partial y = b$
$dy' = -b\,dx + a\,dy \quad \Rightarrow \quad \partial y'/\partial x = -b \quad \partial y'/\partial y = a$

One normally takes x' = u and y' = v so

$\partial u/\partial x = a \quad \partial u/\partial y = b \quad \Rightarrow \quad \partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$
$\partial v/\partial x = -b \quad \partial v/\partial y = a$ Cauchy Riemann equations

When z = x+iy and w = u+iv, these are the Cauchy Riemann equations and w(z) is analytic. In this case, the mapping above preserves angles as long as $J \neq 0$. Conformal means preserving angles. Concept cannot be applied to $E^3 \rightarrow E^3$ it turns out.

Theorem 8. [326] If a mapping $E^2 \rightarrow E^2$ is conformal, the linear matrix they call dT must have the form shown top page 327, which is $R = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. They want $J > 0$ as a condition. the elements of the R matrix can of course vary from point to point as can J.

6.3 Surfaces and surface area **[330]**

Whereas a curve is $\gamma : E^1 \rightarrow E^n$, a surface is a mapping $\Sigma : E^2 \rightarrow E^n$. As with curves, the surface is the mapping, not the trace. Different surfaces (mappings) can have the same trace (semantic problems). New feature is $\partial\Sigma$ which is the boundary of a surface. As with all mappings, there are "problem points" and for the surface map, those are where rank of R falls below 2.

A simple mapping Σ: $E^2(u,v) \rightarrow E^3(x,y,z)$ is $\mathbf{r} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$ which creates a plane in E^3 which has normal $\mathbf{n} = \mathbf{a}\times\mathbf{b}$ and passes through point $\mathbf{r}_0$. Can then write this plane as $\mathbf{n}\bullet\mathbf{r} = d$ where $d = \mathbf{n}\bullet\mathbf{r}_0$, so this is my familiar planar form. In this case, the normal components can be expressed in terms of little 2x2 Jacobians. For sample

$$n_1 = a_2b_3 - a_3b_2 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \begin{vmatrix} \partial y/\partial u & \partial y/\partial v \\ \partial z/\partial u & \partial z/\partial v \end{vmatrix} = \partial(y,z)/\partial(u,v)$$

But this is a 2x2 subdeterminant of the R matrix. If rank(R) = 2, then at least one of these Jacobians must not vanish, and so **n** does not vanish. If rank = 1, then you get **n** = 0 which does not sound healthy.

Theorem 9 [336] : For any curve **r(t)** passing through a point **r** on a smooth surface, one must have $\mathbf{r}'(t)\bullet\mathbf{n} = 0$, which is to say, $\mathbf{v} \bullet \mathbf{n} = 0$. The proof of this seemingly obvious fact is a bit messy as shown on page 336. I write up this proof in the raw notes. They have a fancy chain rule proof.

Page 337 B shows the equation of the **tangent plane** to a surface at some point on the surface. And then C shows you higher order correction terms as you might try to approximate the surface in a Taylor expansion manner.

Theorem 10: [338] If surface is defined by g(x,y,z) = 0, then ∇g is the normal. My ancient proof of this fact works just fine.

Area Formula. [338] Area = $\int$ dudv | **n**(u,v)| where **n** is the non-unit vector normal given in (6-29).

This is really a theorem that Buck's don't prove, but if I identify Bucks' **n** = **a** x **b** with **n** = $\mathbf{e}_1$ x $\mathbf{e}_2$ I verify the formula using tensor doc stuff.

$dA^3 = J\ d\mathcal{A}'^3\ \mathbf{e}^3 = J\ d\mathcal{A}'^3\ \mathbf{E}_3 = \det(S)\ d\mathcal{A}'^3\ \det(R)\ \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_1 \times \mathbf{e}_2\ d\mathcal{A}'^3 = \mathbf{n}\ d\mathcal{A}'^3$
$= \mathbf{n}(u,v)\ dudv$

$dA^3 = |\mathbf{n}(u,v)|\ dudv$ But $|\mathbf{n}|^2$ = sum of three squared 2x2 matrices as shown (6-30).

I show in the raw notes how this matrix sum form of tensor doc agrees with that Bucks give.

The new feature is this: In my work, dA^3 is the area of a face of a differential 3-piped, and you have to imagine a finite volume and its surface made up of lots of tiny such pipeds (think spherical surface), so this integral then gives you the area of a finite arbitrary surface.

Bucks then do some Example 1 and Example 2 area calculations of real world areas, p 341.

Note 1: Bucks comment that "area theory" is still in the process of development, their books 1956 and 1965 editions. Hard to believe such a claim but must be true. Again, the world of a mathematician.

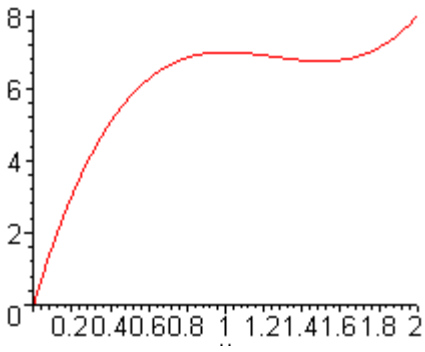
Bucks close this section with a rather obscure discussion of **manifolds**. They want you to think of a 2-manifold as a certain set or class of mappings (not just as a surface), but I don't grok that idea much. A certain fancier statement is that if you consider a certain combination of transformations from domain to range, if you find that a certain Jacobian never vanishes, then the manifold is **differentiable**. And if the Jacobian always has the same sign, then the manifold is **orientable**. The latter means that you cannot smoothly get from one side of the trace surface to the other. They show that a Mobius strip is not orientable because those Jacobians are not always the same sign, and of course you CAN smoothly get from one side to the other. A spherical surface is a **closed** manifold (no boundary!), but one with a hole in it is **open**. I guess a Mobius strip is then open.

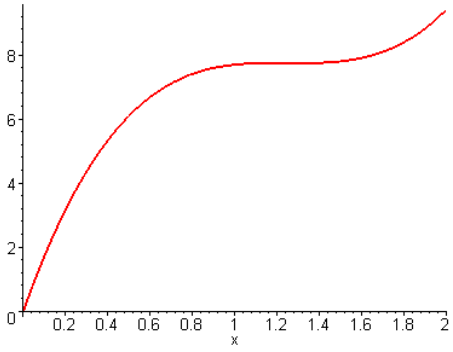
This is all a very cursory discussion and it is algebraic topology. They are just giving the student reader a little kick in the pants (5 pages), a little view of what is beyond this book.

6.4 Extremal Properties of functions of several variables **[349]**

A **critical point** of the function f(x,y,,,) is a place where $f_1 = f_2 \ldots = f_n = 0$.

Example 1: [350] Take a simple **1D** function $f(x) = 4x^3-15x^2+18x$, graph on the left:





(1) There are visibly two critical points (zero slope places) in the interval, **but** the true max and min on the interval are at the endpoints! If you could not "see" this plot, you would check each of the critical points' f values and the two boundary point values before you concluded where the maximum of f was on this interval.

(2) There are two interior places where f'(x) = 0, one is a local max, the other is a local min.

Example 1A: $f(x) \approx 4x^3-15x^2+18.7x$ shown on the right above. This has a single place where f'(x) = 0, but that place is neither a local min nor a local max! It is just a pause. It is also an inflection point where the curvature changes. So this shows that

f'(x) = 0 at interior point $\not\Rightarrow$ that point is a local min or max

I don't think you can call this a 1D saddle point.

Example 1A demonstrates:

Theorem 12: [352] (**1D Local Min or Max Theorem**). Consider f which is C" on [a,b] and f'(c) = 0 for some interior c, so c is then a critical point. Then:

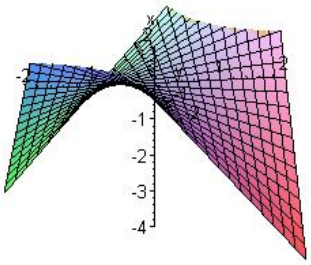
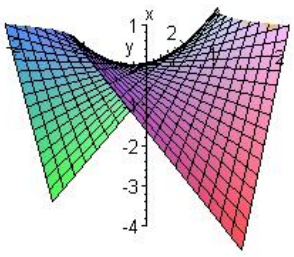
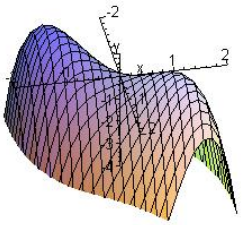
f"(c) < 0	=>	c is a *local* maximum point
f"(c) = 0	=>	c is not a local minimum

Example 2: [351] **2D** $f(x,y) = 4xy - 2x^2 - y^4$. What are the critical points? Graph on the right above:

$f_1 = 4y-4x$	So solve these equations:	y-x = 0	y=x	
$f_2 = 4x - 4y^3$	**plot** on right below	$x - y^3 = 0$	$x(1-x^2) = 0$	x = 0,1,-1

so there are three critical points which are (0,0), (1,1),(-1,-1) which lie in our ±2 domain square. Two of these critical points are local and global max's, the one at (0,0) is a saddle where curves up in one direction and down in the cross direction. The true min is somewhere on the boundary.

Example 3: [352] f = xy Here is a plot of xy where (0,0) is a critical point

At the origin you have $f_1 = f_2 = f_{11} = f_{22} = 0$. The origin definitely is a saddle point, but the up and down are at 45 degrees. This example demonstrates:

saddle $\not\Rightarrow$ curvatures must be + on one axis and - on the other axis

Theorem 13: [353] (**2D Local Min or Max Theorem**) Let $\Delta = f_{12}^2 - f_{11}f_{22}$ at some critical point. That is, we also know that $f_1 = 0$ and $f_2 = 0$. Then

1	$\Delta > 0$	=>	saddle point	
2	$\Delta < 0$	=>	local min or max	(max if $f_{11} < 0$, min if $f_{11} > 0$)
			// $f_{11} = 0$ or $f_{22} = 0 \Rightarrow \Delta \geq 0 \Rightarrow$ not Case 2	
3	$\Delta = 0$	=>	no conclusion	

Apply this theorem to Example 2 above where we already know the three critical points:

$f_1 = 4y-4x$ $\quad f_{11} = -4$ $\quad f_{12} = -4$ $\quad \Delta = 16 - (-4)(-12y^2) = 16 - 48y^2$

$f_2 = 4x - 4y^3$ $\quad f_{22} = -12y^2$

(0,0):	$\Delta = 16$, so saddle point
(1,1):	$\Delta = -32$ so extremum, and $f_{11} < 0$ so its a max
(-1-1):	same as above case

Pages 355 and 356 show **topo maps** of a peak, of a saddle point, and of a monkey saddle, and finally a saddle next to a peak which is Example 3 $f(x,y) = x^2 + y^3 - 3xy$.

On page 356 Bucks consider f(a,b) as the cost function for a **linear fit** F(x) = ax + b to some data. What straight line fit causes f(a,b) to be a global minimum? They find the single critical point, and at that point find $\Delta < 0$ and $f_{11} > 0$ so it really is a local min. It is also a global min and it solves this problem. They mention **gradient ascent** and **gradient descent** as iterative methods of finding such min and max.

Theorem 14 [358] If $\nabla^2_{2D} f(x,y) = 0$ on bounded and open domain D and if f is C", then the min and max of f must occur on the boundary ∂D. That is to say, a harmonic function has its max and min on the boundary ∂D.

Constraint Problem Example 1. You have u = f(x,y,z) and v = g(x,y,z). Constraint is g(x,y,z) = 0 and you want to find the max of f(x,y,z). What do you do?

First, a thought experiment. *Suppose* you could solve g(x,y,z) = 0 in the form z = φ(x,y). Then your problem is to find a max of the function F(x,y) ≡ f(x,y,φ(x,y)) which we know how to do! The critical points are obtained by solving

$F_1 = f_1 + f_3\varphi_1 = 0$
$F_2 = f_2 + f_3\varphi_2 = 0$

and then you could compute F_{11} F_{22} and F_{12} to get Δ and you have it. Example 1 on page 359 is handled in exactly this manner.

First aside: Think of these equations u = f(x,y,z) and v = g(x,y,z) as a transformation where perhaps we could add arbitrary w = 0 to get a tensor doc transformation. Then x' = (u,v,w). What does the R matrix look like?

$R_{ik}(\mathbf{x}) \equiv (\partial x'_i/\partial x_k)$
$S_{ik}(\mathbf{x}') \equiv (\partial x_i/\partial x'_k)$

Then $R_{11} = \partial u/\partial x = \partial_1 u$ so

$$R = \begin{matrix} \partial_1 u & \partial_2 u & \partial_3 u \\ \partial_1 v & \partial_2 v & \partial_3 v \\ 0 & 0 & 0 \end{matrix} = \begin{matrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ 0 & 0 & 0 \end{matrix}$$

S**econd aside**: On the constraint surface we know is we move d**r** along the surface, then

$dg = g_1dx + g_2dy + g_3dz = \nabla g \bullet d\mathbf{r} = 0$

Now if we really can find z = φ(x,y) as above, then $dz = \varphi_1 dx + \varphi_2 dy$ as rule for staying on the surface. So install this dz to get

$dg = g_1dx + g_2dy + g_3[\varphi_1 dx + \varphi_2 dy] = 0$

or

$dg = [g_1 + g_3\varphi_1]\, dx + [g_2 + g_3\varphi_2]\, dy = 0$

I am not sure where this leads, so just stop here.

Third aside: We can also define G(x,y) = g(x,y,φ(x,y)) = 0. This is valid for all x and y in the domain. Then

$G_1 = g_1 + g_3\varphi_1 = 0$
$G_2 = g_2 + g_3\varphi_2 = 0$

and these equations are parallel to those for F shown above!

Right now I am not sure any of these "asides" is relevant.

Theorem 15: [360] . Imagine that there are some point **r** such that g(x,y,z) = 0 [**r** on surface S] *and* f(x,y,z) is a local max or min at this same **r** which is not on the boundary of S. The theorem claims that **r** must be a "critical point" of the combined transformation

$$u = f(x,y,z) \qquad E^3 \rightarrow E^2$$
$$v = g(x,y,z)$$

"Critical point" here means a place where the R matrix rank drops down from maximum. As I just showed above, the non-critical rank of the R matrix is 2, so the theorem claims that if you find a point **r** which solves your problem of max f with constraint g = 0, then at that point the rank drops below 2.

Proof #1. We have already shown above that (the G stuff from aside #3 and critical points from first step)

$$F_1 = f_1 + f_3\varphi_1 = 0 \qquad G_1 = g_1 + g_3\varphi_1 = 0$$
$$F_2 = f_2 + f_3\varphi_2 = 0 \qquad G_2 = g_2 + g_3\varphi_2 = 0$$

Solve the top left for $\varphi_1 = -f_1/f_3$ and then top right says $g_1 + g_3[-f_1/f_3] = 0$ or $f_3g_1 - f_1g_3 = 0$.
Solve the bot left for $\varphi_2 = -f_2/f_3$ and then bot right says $g_2 + g_3[-f_2/f_3] = 0$ or $f_3g_2 - f_2g_3 = 0$.

Rewrite these as: $f_1g_3 - f_3g_1 = 0 \Rightarrow g_1 = g_3(f_1/f_3)$
$f_2g_3 - f_3g_2 = 0 \Rightarrow g_2 = g_3(f_2/f_3)$

Now consider $f_1g_2 - f_2g_1 = f_1\,[g_3(f_2/f_3)] - f_2\,[g_3(f_1/f_3)] = 0$

Thus we have shown that

$$\begin{vmatrix} f_1 & f_3 \\ g_1 & g_3 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} f_2 & f_3 \\ g_2 & g_3 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix} = 0$$

Recall that

$$R = \begin{matrix} \partial_1 u & \partial_2 u & \partial_3 u \\ \partial_1 v & \partial_2 v & \partial_3 v \end{matrix} = \begin{matrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{matrix}$$

so we have just shown all three three 2x2 determinants vanish, and therefore rank(R) = 1 or 0.

We have thus shown that if you find a solution to your problem, then that solution is a point where rank drops off! We have not proven the converse however. But if you are hunting for a solution, you need only consider points where rank drops off!! The reduces the scaled of the search dramatically!

Proof #2 of Theorem 15 is quite interesting, I summarize it in the raw notes and won't repeat that here. It is a proof based on contradiction.

The Lagrange Multiplier Solution to this same class of problem.

Define $H(x,y,z,\lambda) = f(x,y,z) + \lambda\, g(x,y,z)$. Then consider these four equations

$$H_1 = f_1 + \lambda g_1 = 0$$
$$H_2 = f_2 + \lambda g_2 = 0$$
$$H_3 = f_3 + \lambda g_3 = 0$$
$$H_4 = g = 0$$

These four equations define a critical point for the new function H of its four variables. Recall again,

$$R = \begin{matrix} \partial_1 u & \partial_2 u & \partial_3 u \\ \partial_1 v & \partial_2 v & \partial_3 v \end{matrix} = \begin{matrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{matrix}$$

The first three H equations say that

$$(f_1,f_2,f_3) + \lambda(g_1,g_2,g_3) = 0 \qquad \text{or} \qquad \mathbf{f} + \lambda\mathbf{g} = 0$$

This says that the two rows of R are linearly dependent, and that means rank drops off from 2, and that in turn means that the **r** that makes these H equations be true is a candidate **r** for a solution point. Note that the last equation enforces g = 0, the constraint.

Since we know f and g, we know all the derivatives so we can of course write out the four H equations shown above. We must then find a value of x,y,z,λ that solves these equations! That is, **r**,λ. Although the equations are linear in λ, they in general are non-linear on x,y,z so **not a slam dunk** to solve the equations.

Example: Suppose $f(x,y,z) = 4x^2 + y^2 + z^2$
$g(x,y,z) = 2x+3y+z-12$ constraint Ex 1 p 359.

We first compute: (notice how simple this example is, in general that won't be the case)

$$f_1 = 8x \qquad g_1 = 2$$
$$f_2 = 2y \qquad g_2 = 3$$
$$f_3 = 2z \qquad g_3 = 1$$

The four H equations are then:

$$H_1 = f_1 + \lambda g_1 = 0 \;=\; 8x \;+ 2\lambda$$
$$H_2 = f_2 + \lambda g_2 = 0 \;=\; 2y + 3\lambda$$
$$H_3 = f_3 + \lambda g_3 = 0 \;= 2z + \lambda$$
$$H_4 = g = 0 \;=\; 2x+3y+z-12$$

What next? We have 4 equations in 4 unknowns. Replace $\lambda = -2z$ from the H_3 and then we have

$$H_1 = f_1 + \lambda g_1 = 0 \;=\; 8x \;- 4z \qquad \Rightarrow \qquad z = 2x$$

$H_2 = f_2 + \lambda g_2 = 0 = 2y - 6z \quad \Rightarrow \quad y = 3z = 6x$

$H_4 = g = 0 = 2x + 3y + z - 12 \quad \Rightarrow \quad 2x + 18x + 2x - 12 = 0$

So we then have to solve,

$22x = 12$ or $11x = 6$ $x = 6/11$

Then $z = 2x = 12/11$ $y = 6x = 36/11$ $\lambda = -2z = -2 (12/11) = -24/11$

My solution is then

$x = 6/11$
$y = 36/11$
$z = 12/11$ // x,y,z agree with p 360 top
$\lambda = -24/11$

I will write up the whole Lagrange Multiplier Theory in a separate document, as if it were to go online.

So Chapter 6 meta notes are concluded!