Confusion about the meaning of ω* PhL 6.29.15

I was trying to understand things in terms of tensor doc, and this led to confusions. My first two conjectures as to the meaning of ω* both failed to give correct results on BOTH pages 409 and 411, but the third conjecture seems to have worked in both these cases.

Here is the original text taken from Buck Ch 7 raw notes, just to archive this confusion:

Digression on page 409. What happens to differential forms under a transformation?

$\omega \equiv A_1 dx_1 + A_2 dx_2 = \mathbf{A(x)} \bullet d\mathbf{x}$ $\mathbf{x} = (x,y)$ $\mathbf{x}' = (u,v)$

$\omega' \equiv A'_1 dx'_1 + A'_2 dx'_2 = \mathbf{A'(x')} \bullet d\mathbf{x}'$ // using tensor doc notation

But this is NOT the kind of transformation they are talking about. They instead want you to think of

$\omega^* \equiv A'_1 dx_1 + A'_2 dx_2 = \mathbf{A'(x')} \bullet d\mathbf{x}$ **(*)** // my conjecture of ω* meaning

[Note 1: whereas on page 401 we had a 1-form ω going to a 2-form ω*, here we have a 1-form ω going to a 1-form ω* (clearly stated as being a 1-form). I conclude that this is a new and different meaning for the symbol ω* . But I have to decode how here they get from ω to ω*, it is not clear.]

Then we can use

$d\mathbf{x}' = R d\mathbf{x}$ $d\mathbf{x} = S d\mathbf{x}'$

to get

$\omega^* \equiv \mathbf{A'(x')} \bullet [S d\mathbf{x}'] = A'_i [S d\mathbf{x}']_i = A'_i S_{ij} d\mathbf{x}'_j$

$= [A'_1 S_{11} + A'_2 S_{21}] dx'_1 + [A'_1 S_{12} + A'_2 S_{22}] dx'_2$

$= [A' S_{11} + B' S_{21}] du + [A' S_{12} + B' S_{22}] dv$

Now from tensor doc,

$d\mathbf{x}' = R(\mathbf{x})\, d\mathbf{x}$ $R_{ik}(\mathbf{x}) \equiv (\partial x'_i / \partial x_k)$ $R = S^{-1}$ // $dx'_i = R_{ij}\, dx_j$

$d\mathbf{x} = S(\mathbf{x}')\, d\mathbf{x}'$ $S_{ik}(\mathbf{x}') \equiv (\partial x_i / \partial x'_k)$ $S = R^{-1}$ // $dx_i = S_{ij}\, dx'_j$. (2.1.6)

So then [see p 409 A]

$S_{11} = \partial_u x = \varphi_1$ $S_{21} = \partial_u y = \psi_1$ $S_{12} = \partial_v x = \varphi_2$ $S_{22} = \partial_v y = \psi_2$

Then you get

$\omega^* = [A' \ \varphi_1 + B' \psi_1]\, du + [A' \varphi_2 + B' \psi_2]\, dv$ = page 409 E

Since the above expression agrees with p 409 E (assuming A* means A' etc), I think my conjecture (*) above is correct.

OK, so we are on the same page (I think). Go back then to

$\omega \equiv \mathbf{A(x)} \bullet d\mathbf{x}$ $\mathbf{x} = (x,y)$ $\mathbf{x}' = (u,v)$

$\omega^* = \mathbf{A'(x')} \bullet d\mathbf{x}$ $\omega^* = A'(x')\, dx$ as simple example,

So they are defining ω* by transforming only the k-form coefficients, not the differentials. This is like the active rotation of a vector which acquires new components in the same coordinate system. They are allowed to define a "transformation of a k-form" any way they want, and so ω* is what they want to do.

Now one benefit of this definition of ω* is the claim

$d(\omega^*) = (d\omega)^*$

They show this with some examples, but maybe I can prove it more generally:

$\omega \equiv \mathbf{A(x)} \bullet d\mathbf{x}$ $d\omega = d[\mathbf{A(x)}] \bullet d\mathbf{x}$
$\omega^* = \mathbf{A'(x')} \bullet d\mathbf{x}$ $d(\omega^*) = d[\mathbf{A'(x')}] \bullet d\mathbf{x}$

Before doing more, let's work on these two results

$d\mathbf{A}(x_i) = \partial_j \mathbf{A}\, dx_j$

or

$dA_k(x_i) = \partial_j A_k\, dx_j$

Then since $d\omega = d[\mathbf{A(x)}] \bullet d\mathbf{x} = d[A_k(\mathbf{x})]dx_k = \partial_j A_k\, dx_j\, dx_k$

$d\omega = \partial_j A_k\, dx_j dx_k = [\partial_j A_k\, dx_j]\, dx_k = [(\partial_j \mathbf{A})\, dx_j] \bullet d\mathbf{x} = -[(\nabla A_k) dx_k] \bullet d\mathbf{x}$ // two ways
way #1 way #2

$(d\omega)^* = [(\partial'_j \mathbf{A'})\, dx'_j] \bullet d\mathbf{x} = -[(\nabla' A'_k) dx'_k] \bullet d\mathbf{x}$ // two ways
way #1 way #2

Meanwhile,

$d[\mathbf{A'(x')}] = \partial'_j \mathbf{A'}\, dx'_j$

or

$dA'_k(x'_i) = \partial'_j A'_k\, dx'_j$

Then

way #1

$d(\omega^*) = \partial'_j A'_k \, dx'_j \, dx_k = [\, (\partial'_j \mathbf{A'}) \, dx'_j] \bullet d\mathbf{x}$ = $(d\omega)^*$ as shown above !!!

So I have then proven the little theorem:

Theorem PL1: if $\omega \equiv \mathbf{A(x)} \bullet d\mathbf{x}$ and $\omega^* = \mathbf{A'(x')} \bullet d\mathbf{x}$, then $d(\omega^*) = (d\omega)^*$.

Comment: That fact that I get the result $d(\omega^*) = (d\omega)^*$ lends further credence to my conjecture (*) above about the "meaning of symbol ω^* ".

Theorem PL2: Now consider:

$\int_\gamma \omega^* = \int_\gamma [\mathbf{A'(x'(x))} \bullet d\mathbf{x}]$ // my interpretation of ω^* as above

$= \int_{\gamma'} [\mathbf{A'(x')} \bullet d\mathbf{x'}]$ // run $\mathbf{x}'$ along $\boldsymbol{\gamma}'$ rather than $\mathbf{x}$ along $\boldsymbol{\gamma}$

$= \int_{\gamma'} [\mathbf{A(x)} \bullet d\mathbf{x}]$ // dot product is scalar in E^n if A is a vector

$= \int_{\gamma'} \omega$ // my interpretation of 1-form ω

$= \int_{\gamma^*} \omega$ // their name for γ' curve.

The above is then my proof of (7-31) A. [but who says A is a vector ?]

Theorem PL3: Next, consider

$\int\int_D (d\omega)^* = \int\int_D [\, (\partial'_j \mathbf{A'(x'(x))} \, dx'_j(\mathbf{x})] \bullet d\mathbf{x}$ // one of my "two interp ways" above

$= \int\int_{D'} [\, (\partial'_j \mathbf{A'(x')}) \, dx'_j] \bullet d\mathbf{x'}$ // run $dx'_j dx'_k$ over D' instead of D

$= \int\int_{D'} [\, (\partial_j \mathbf{A(x)}) \, dx_j] \bullet d\mathbf{x}$ // dot product is scalar

$= \int\int_{D'} (d\omega)$ // my interp above

$= \int\int_{D^*} d\omega$ // their name for D'

The above is then my proof of (7-31) B. But there are some subtleties here! Consider

$\mathbf{V}'(\mathbf{x}') = (\partial'_j \mathbf{A}'(\mathbf{x}'))\, dx'_j$

$V'_k(x') = (\partial'_j A'_k(\mathbf{x}'))\, dx'_j = (\partial'_j A'_k) dx'^j$ = component k of a vector in x'-space

So our integrand is

$[\,(\partial'_j \mathbf{A}'(\mathbf{x}'))\, dx'_j] \bullet d\mathbf{x}' = V'_k(x')\, dx'^k$ = a true scalar for any transformation!

$= \mathbf{V}'(\mathbf{x}') \bullet d\mathbf{x}'$

$= \mathbf{V}(\mathbf{x}) \bullet d\mathbf{x}$ // since a true scalar

This occurs in the first proof as well where we have

$[\mathbf{A}'(\mathbf{x}') \bullet d\mathbf{x}'\,] = A'_k\, dx'^k$

but in order to really see this, you have to use covariant notation with up and down indices. I can see why Buck's don't want to get into that right here. Does Arapura address this stuff? The answer is NO. I would have to find some other source. I just found a pretty good source that talks about more advanced things like pullbacks and I think that does relate to the above, but let's not wander off right now!!

Status: I will go with my little derivations of page 410 A and B (7-31). These were hugely non-obvious and Bucks are a little dishonest in their presentation I would say. But let's now move on. The proof of Theorem 7 is now clear:

Theorem 7: If Green's Theorem holds in x-space, it also holds in x'-space.

Note that this is for an arbitrary transformation they call T, and I call **F**. Perhaps this would have some implication for curvilinear coordinates. So far we don't have much motivation for why Theorem 7 is important.

Theorem 8: Proof that $(d\omega)^* = d(\omega^*)$

This is a 3-page proof. But I think I did my own proof above as Theorem PL1. I wonder if there is any connection between our proofs? Mine is probably no good. My proof was only for 1-forms!

0-forms.

$\omega = f(\mathbf{x})$ $d\mathbf{x} = S d\mathbf{x}'$ contravariant

$\omega^* = f'(\mathbf{x}') = f(\mathbf{x}(\mathbf{x}'))$ // seems only possibility // same as page 411 A

$d[f(\mathbf{x}(\mathbf{x}'))] = \partial_i f\, dx_i = \partial'_i f'\, dx'_i$ // your choice, both ways are right

$d(\omega^*) = d[f(\mathbf{x}(\mathbf{x}'))] = \partial_i f \, dx_i = \nabla f \bullet d\mathbf{x} = \nabla f \bullet [Sd\mathbf{x}']$ // probably what B says

Let's try to explicitly confirm that the above agrees with p 411 B: the following based on p 411 F:

$S_{11} = \partial_u x = \varphi_1 \qquad S_{21} = \partial_u y = \psi_1 \qquad S_{31} = \partial_u z = \theta_1$
$S_{12} = \partial_v x = \varphi_2 \qquad S_{22} = \partial_v y = \psi_2 \qquad S_{32} = \partial_v z = \theta_2$

Then

$d(\omega^*) = \nabla f \bullet [Sd\mathbf{x}'] = \Sigma_{i=1}^{3} \partial_i f \, \Sigma_{j=1}^{2} S_{ij} \, dx'_j = \Sigma_{i=1}^{3} f_i \, \Sigma_{j=1}^{2} S_{ij} \, dx'_j$

$= f_1 [S_{11}dx'_1 + S_{12}dx'_2] + f_2 [S_{21}dx'_1 + S_{22}dx'_2] + f_3 [S_{31}dx'_1 + S_{32}dx'_2]$

$= [f_1 S_{11} + f_2 S_{21} + f_3 S_{31}] \, dx'_1 + [f_1 S_{12} + f_2 S_{22} + f_3 S_{32}] \, dx'_2$

$= [f_1 \varphi_1 + f_2 \psi_1 + f_3 \theta_1] \, du + [f_1 \varphi_2 + f_2 \psi_2 + f_3 \theta_2] \, dv$

and this DOES agree with p 411 B. We now continue:

$d\omega = df = \nabla f \bullet d\mathbf{x}$ = this is a 1-form // this is what C says

$(d\omega)^* = \nabla' f'(\mathbf{x}') \bullet d\mathbf{x}$ // according to my 1-form treatment above

I know that

$f'(\mathbf{x}') = f(\mathbf{x}(\mathbf{x}'))$

so then

$\nabla' f'(\mathbf{x}') = \nabla' f(\mathbf{x}(\mathbf{x}'))$

and then

$[\nabla' f'(\mathbf{x}')]_i = \partial'_i f(\mathbf{x}(\mathbf{x}')) = f_j \, \partial x_j / \partial x'_i = f_j \, S_{ji}$

and finally,

$(d\omega)^* = \nabla' f'(\mathbf{x}') \bullet d\mathbf{x} = [\nabla' f'(\mathbf{x}')]_i \, dx_i = f_j \, S_{ji} \, dx_i$

$= f_1(S_{11}dx_1 + S_{12}dx_2) + f_2(S_{21}dx_1 + S_{22}dx_2) + f_3(S_{31}dx_1 + S_{32}dx_2)$

$= f_1(\varphi_1 dx_1 + \varphi_2 dx_2) + f_2(\psi_1 dx_1 + \psi_2 dx_2) + f_3(\theta_1 dx_1 + \theta_2 dx_2)$

This looks like the first line of D, but they have dx_1' and dx_2' whereas I have dx_1 and dx_2. Something is wrong. We would agree if I used $(d\omega)^* = \nabla' f(\mathbf{x}') \bullet d\mathbf{x}'$ instead of $(d\omega)^* = \nabla' f(\mathbf{x}') \bullet d\mathbf{x}$, but then this conflicts with my initial conjecture (*) above!

Let's back up and try again from p 411 C:

$$d\omega = df = \nabla f \bullet d\mathbf{x} = \partial_i f \; dx_i = f_i \; dx_i$$

$$(d\omega)^* = \partial'_i f' \; dx_i = \partial f' / \partial x'_i \; dx_i = df/\partial x_j * \partial x_j/\partial x'_i \; dx_i = f_j \; \partial x_j/\partial x'_i \; dx_i = f_j \; S_{ji} \; dx_i$$

$$= f_j \; [Sd\mathbf{x}]_j = f_j \; [SSd\mathbf{x}']_j = f_j \; [S^2 \; d\mathbf{x}']_j$$

$$= f_1 \; [\; (S^2)_{11} \; dx'_1 + (S^2)_{12} \; dx'_2] \; + \; \text{etc} \quad = \text{a big mess which is just plain wrong.}$$

So I am unable to get from C to D on page 411. Note that

$$dx = \varphi_1 du + \varphi_2 dv$$

So in line D, Bucks are simply rewriting things with x,y,z just "replaced" with their x(u,v) and so-on functions. That is not my ω* conjecture!

Let's now back up to that conjecture and reexamine it. Here is the conjecture (*) which I made:

$$\omega \equiv A_1 dx_1 + A_2 dx_2 = \mathbf{A}(\mathbf{x}) \bullet d\mathbf{x} \qquad \mathbf{x} = (x,y) \qquad \mathbf{x}' = (u,v)$$

$$\omega^* \equiv A'_1 dx_1 + A'_2 dx_2 = \mathbf{A}'(\mathbf{x}') \bullet d\mathbf{x} \qquad (*) \qquad \text{// my conjecture of } \omega^* \text{ meaning}$$

But let's now consider a different conjecture for the meaning of ω*.

$$\omega \equiv A_1(\mathbf{x}) dx_1 + A_2(\mathbf{x}) dx_2 = \mathbf{A}(\mathbf{x}) \bullet d\mathbf{x} \qquad \mathbf{x} = (x,y) \qquad \mathbf{x}' = (u,v)$$

$$\omega^* \equiv A_1(\mathbf{x}(\mathbf{x}')) dx_1' + A_2(\mathbf{x}(\mathbf{x}')) dx_2' = \mathbf{A}(\mathbf{x}(\mathbf{x}')) \bullet d\mathbf{x}'$$

$$\equiv A_1^*(\mathbf{x}') dx_1' + A_2^*(\mathbf{x}') dx_2' = \mathbf{A}^*(\mathbf{x}') \bullet d\mathbf{x}' \qquad (**)$$

This then is a different meaning for A_1^*. We have

$$A_1^*(\mathbf{x}') \equiv A_1(\mathbf{x}(\mathbf{x}'))$$

This is not the same as tensor doc's thing

$$A_1'(\mathbf{x}') \equiv R_{1j} \; A_j(\mathbf{x}) \qquad \text{for a vector field}$$

Now, let's try out conjecture (**) on page 409:

$\omega \equiv A_1 dx_1 + A_2 dx_2 = \mathbf{A(x)} \bullet d\mathbf{x}$

$\omega^* \equiv A_1^*(\mathbf{x}')\, dx'_1 + A_2^*(\mathbf{x}') dx'_2 = \mathbf{A^*(x')} \bullet d\mathbf{x}'$

$= A_1^*(u,v)\, du + A_2^*(u,v)\, dv$

But this disagrees with p 409 C.

OK, let's now try a third conjecture:

$\omega \equiv A_1(\mathbf{x}) dx_1 + A_2(\mathbf{x}) dx_2 = \mathbf{A(x)} \bullet d\mathbf{x}$ $\quad \mathbf{x} = (x,y) \quad \mathbf{x}' = (u,v)$

$\omega^* \equiv A_1(\mathbf{x(x')}) dx_1 + A_2(\mathbf{x(x')}) dx_2 = \mathbf{A}(\mathbf{x}(\mathbf{x}')) \bullet d\mathbf{x}$

$\equiv A_1^*(\mathbf{x'}) dx_1 + A_2^*(\mathbf{x'}) dx_2 = \mathbf{A^*}(\mathbf{x}') \bullet d\mathbf{x}$ $\quad (***)$

This agrees with p 409 C. Now jump ahead to page 411 and see what conjecture (***) says there:

$d\omega = df = \nabla f \bullet d\mathbf{x}$ = this is a 1-form // this is what C says

Now let's define

$\mathbf{g}(\mathbf{x}) \equiv \nabla f(\mathbf{x})$ $\quad g_i(\mathbf{x}) = \partial_i f(\mathbf{x})$

$\Rightarrow \mathbf{g}^*(\mathbf{x}') \equiv \mathbf{g}(\mathbf{x}(\mathbf{x}'))$ $\quad g^*_i(\mathbf{x'}) = g_i(\mathbf{x}(\mathbf{x}'))$

This is a little tricky for stupid me, so try not to screw it up please:

$\mathbf{g}(\mathbf{x}) \equiv [\nabla f](\mathbf{x})$

$\mathbf{g}^*(\mathbf{x}') = [\nabla f](\mathbf{x}(\mathbf{x}')) = [\nabla f]^*(\mathbf{x}')$

It is fighting me hard. How do you "write this out"? Strong mental block in progress.

$[\nabla f](\mathbf{x}(\mathbf{x}'))$ = just express $[\nabla f](\mathbf{x})$ in terms of $\mathbf{x}'$ coordinates!

$[\partial_i f](\mathbf{x}(\mathbf{x}'))$ = just express $[\partial_i f](\mathbf{x})$ in terms of $\mathbf{x}'$ coordinates!

$\partial_i f = \partial f / \partial x_i = f_i(\mathbf{x}(\mathbf{x}'))$

$d\omega = df = \nabla f \bullet d\mathbf{x} = \partial_i f\ dx_i = f_i(\mathbf{x})\, dx_i$

$(d\omega)^* = f_i(\mathbf{x}(\mathbf{x}'))\, dx_i = f_i\, dx_i = f_i\ [S d\mathbf{x}']_i = f_i S_{ij} dx'_j$

$= f_1[S_{11}dx'_1 + S_{12}dx'_2]$ + 2 more terms

$= f_1[\varphi_1 dx'_1 + \varphi_2 dx'_2]$ + 2 more terms

$= f_1[\varphi_1 du + \varphi_2 dv]$ + 2 more terms = agrees with D (finally).

So now I need to push this confusion off to a separate doc and just do the correct version in these raw notes.

Recall from tensor doc

$\bar{\nabla}' = S^T \bar{\nabla}$. and $d\mathbf{x} = Sd\mathbf{x}'$

Therefore

$(d\omega)^* = [S^T \nabla f(x)] \bullet [Sd\mathbf{x}']$

$= [S^T \nabla f(x)]_i [Sd\mathbf{x}']_i$

$= [S^T_{ik} \partial_k f(x)][S_{ij} dx'_j]$

$= [S_{ki} \partial_k f(x)][S_{ij} dx'_j]$

$= [S_{ki} S_{ij}] \partial_k f(x)][dx'_j]$ = wrong

=

$= \nabla' f(\mathbf{x}') \bullet [Sd\mathbf{x}']$ // does NOT agree with D

I need to review this * operator going back to page 408 where it is first used.