Curve Integration PhL 2.12.16

What I mean by the title is this: "integrating something along a curve".

The Buck discussion of "curves" is spread out in multiple chapters. Here are notes from three places.

Section 6.2. Curves and arc length

I have general Ch 6 notes on this section. Their first task is to define what they mean by a curve γ in E^n. It is a mapping, though we often confuse a curve with the trace of that mapping. They then define a "simple" curve (does not self-intersect) and then a "smooth" curve (no points of zero velocity). We learn that if $\boldsymbol{\gamma}(t)$ is a curve [$\mathbf{r}(t)$], then $\boldsymbol{\gamma}'(t) = \mathbf{v}(t)$ is the velocity vector which is a vector tangent to the curve at any point. It is unfortunate that my gamma character does not "bold" very well. Bucks give several nice examples of curves to illustrate various situations, and then they note that at any point a curve has a curvature. On page 320 they first talk about arc length of a curve. They then define the notion of curves which are parametrically equivalent which they call γ and γ^* on page 322 bottom. It is more than just saying two curves have the same trace. One curve is basically what I call a "respeeding" of the other, but it is not quite so simple and I added more notes today in Ch 6 notes to show what it means for two curves to be equivalent in this sense. Bucks then show page 323 bottom that two equivalent curves had the same arc length. You could gather all equivalent curves with a given trace into a "class" if you wanted. Bucks then finish up with comments on preservation of angles with a given set of conditions and this we know is related to analytic functions.

Section 7.1 Integrals of functions over curves.

They address the basic question: what does one MEAN by integrating a function "along a curve". We write this as $\int_{\boldsymbol{\gamma}} f$, where $\boldsymbol{\gamma}$ is the curve and f is the scalar function to be integrated along γ. This means that f is a scalar function of a vector argument so $f: E^n \rightarrow R$ and we talk about $f(\mathbf{r})$. We are instructed to break the curve $\boldsymbol{\gamma}$ into tiny short nearly-straight curves γ_i and compute $\Sigma_i\, f(\mathbf{p}_i)\, \Delta L(\gamma_i)$ where $\mathbf{p}_i$ is a point on the curve $\boldsymbol{\gamma}_i$ and $\Delta L(\gamma_i)$ is the length of that tiny curvelet. In the limit that we take more and more elements in the set of $\boldsymbol{\gamma}_i$ we get a limit and we can say

$$\int_{\boldsymbol{\gamma}} f \;= \text{limit}_{i\rightarrow\infty}\; \Sigma_i\, f(\mathbf{p}_i)\, \Delta L(\gamma_i)$$

So this is the MEANING Bucks give to the object $\int_{\boldsymbol{\gamma}} f$. This sounds good to me. Everything is scalar. Notice that the integral only involves the function f evaluated ON the curve, no other values are used. Now we already know that

$$L_i(\gamma) = \int_{t_1}^{t_2} |\boldsymbol{\gamma}'(t)|\, dt \qquad \text{so} \qquad \Delta L(\gamma_i) = |\boldsymbol{\gamma}'(t)|\, dt$$

This last item is the arc length of a tiny curvelet γ_i going from t_i to t_i+dt in parameter. Then you get in the limit that

$$\int_\gamma f = \int_{t_1}^{t_2} f(\boldsymbol{\gamma}(t)) \, |\, \boldsymbol{\gamma}'(t) \,| \, dt$$ and this is Buck page 367 (7-1)

Bucks relegate to Exercise 5 (p375) a proof of the fact that this integral object gives the same result for all curves in the same equivalence class with the curve (mapping) $\gamma(t)$. I added my proof of this fact into the Chapter 6 notes just now. Trivially true.

Now suppose we choose our equivalent curve γ^* to be one for which $|\, \boldsymbol{\gamma}^{*\prime}(t) \,| = 1$ and also one for which we select $(\alpha,\beta) = (0,L)$ where L is the arc length of the curve. This means that we are selecting the parameter t to be the arc length parameter. Then we can write

$$\int_\gamma f = \int_0^L f(\gamma^*(t)) \, dt = \int_0^L f(\gamma^*(s)) \, ds$$ and this is Buck (7-2)

But now rename this curve just to be $\gamma(s)$ where s is the arc length parameter, then you have

$$\int_\gamma f = \int_0^L f(\boldsymbol{\gamma}(s)) \, ds = \int_0^L f(\mathbf{r}(s)) \, ds = \int_0^L f(\mathbf{r}) \, ds$$

The section then turns to surfaces which I ignore for now.

Section 7.2 Integrals of 1-forms over curves

Here we write a general 1-form in E^2

$$\omega = A(x,y)dx + B(x,y)dy$$

and then we assume that we can express x and y in terms of a common parameter t according to some

$$x = \varphi(t) \qquad y = \psi(t)$$

which I like to call "a transformation from t-space to E^2 ". Naively one can write

$$dx = \varphi'(t) \, dt \qquad dy = \psi'(t)dt$$

so that

$$\omega = A(\varphi(t),\psi(t)) \, \varphi'(t)| \, dt + B(\varphi(t),\psi(t)) \, \psi'(t)dt$$

and then

$$\int_\gamma \omega = \int_{t_1}^{t_2} [\; A(\varphi(t),\psi(t)) \, \varphi'(t) \, dt + B(\varphi(t),\psi(t)) \, \psi'(t)dt \,]$$

and this right side is regarded then as the DEFINITION of the integral of a 1-form along curve γ. It is something we know how to compute, and Bucks to several nice examples with various A and B functions, and with various φ and ψ functions. It seems to me that if you take the special case A = 0 or B = 0, then this integral is the same as what was considered in the previous section AND you could select an equivalent curve so that the integral can be written with φ'(t) = 1 (in the case B = 0). I don't think however that for the general case where A and B are both non-zero that you can have dt be the arc length for both parts of the integral at the same time, though Bucks do not comment on this. Bucks go on do write a general 1-form in E^3 and everything is similar and in fact an example is given.

Now that we are talking about 1-forms, we can make contact with Sjamaar Chapter 4. Suppose we first rewrite the Buck result above in this way,

$\omega = A(x,y)dx + B(x,y)dy \quad \rightarrow \quad \alpha = \Sigma_i \, f_i(\mathbf{x})dx^i$ // the 1-form

$x = \varphi(t),\ y = \psi(t)$ OR $\boldsymbol{\gamma} = \mathbf{r} = (\varphi(t),\psi(t)) \ \rightarrow \quad \mathbf{c} = (c_1(t),c_2(t))$ // the curve

$$\int_\gamma \omega = \int_{t_1}^{t_2} [\ A(\varphi(t),\psi(t))\ \varphi'(t)\ dt + B(\varphi(t),\psi(t))\ \psi'(t)dt\ ]$$

$$\rightarrow \int_c \alpha = \int_{t_1}^{t_2} [\ \Sigma_i\ f_i(\mathbf{c}(t))\ (dc_i(t)/dt)\ dt\ ]$$

In his earlier and more general notation Sjamaar would write

$\mathbf{c} = (c_1(t),c_2(t)) \quad \rightarrow \quad \mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ $\mathbf{y}$ = on manifold (the curve in E^n)

Sjamaar's general pullback idea for our 1-form is then this

$$\int_c \alpha = \int_{t_1}^{t_2} [\ \Sigma_i\ f_i(\mathbf{c}(t))\ (dc_i(t)/dt)\ dt\ ] \quad // = \int_{t_1}^{t_2} c^*\alpha$$

where the right side is the integral *pulled back* from E^n space (y-space) to t-space (x-space). The pull back form serves really as the definition of the integral of a 1-form $\int_c \alpha$. The abstract 1-form integral here is some kind of "complicated" integral over a piece of a Manifold, whereas the pull back form is an operational form where you integrate over a "flat patch" (real interval) and you can actually compute a result.

So the point is the Bucks are really showing pull backs but they never use that phrase. Bucks are talking about the integral of a 1-form over a 1-dimensional manifold and they write it as $\int_\gamma \omega$.

Later in section 7.3 Bucks will show that the differential df of a 0-form is a 1-form, and this is related to the gradient operator in E^n. I don't think they do much else with 1-forms.

Comparison of 1-form integration to earlier Buck curve integrations

Here is the 1-form integration for γ in E^3 written in adjusted Buck notation,

$$\int_\gamma \omega = \int_{t_1}^{t_2} [\, \Sigma_{i=1}^{3} f_i(\gamma(t))\, (d\gamma_i(t)/dt)\, dt\,] = \int_{t_1}^{t_2} [\, \mathbf{f}(\gamma(t)) \bullet \gamma'(t)]\, dt$$

$$= \int_{p}^{q} \mathbf{f}(\gamma) \bullet d\gamma = \int_{p}^{q} \mathbf{f} \bullet d\boldsymbol{\ell}$$

Written in the last form, you see this type of integral is the classic "**line integral**" between two points **p** and **q** in E^n . Buck on page 376 confirms the phrase "line integral". The only "integral theorem" I have regarding such integrals is this one:

$$\int_{p}^{q} \nabla\varphi \bullet d\boldsymbol{\ell} = \varphi(q) - \varphi(p) \qquad \text{where } \mathbf{f} = \nabla\varphi\ .$$

If all components of f are zero except the first, we then have

$$\int_\gamma \omega = \int_{t_1}^{t_2} f_1(\gamma(t))\, (d\gamma_1(t)/dt)\, dt\ .$$

Even this integral is *completely different* from the earlier Buck integral,

$$\int_\gamma f = \int_{t_1}^{t_2} f(\gamma(t))\, |\, \gamma'(t)\, |\, dt\ . \qquad \text{and this is Buck page 367 (7-1)}$$

One very obvious difference is that the second has $|\, \gamma'(t)\, | = \sqrt{(\gamma_1')^2 + (\gamma_2')^2 + (\gamma_3')^2}$ whereas the first has instead the factor (γ_1') . So perhaps following Buck's hint on page we would call this last type of integration an "integral of a scalar function over a curve with respect to arc length" whereas we call the 1-form integral a "line integral of a vector function".

Now here is a special case of the 1-form line integral where I select $\mathbf{f}(\gamma) = \gamma$. Then we get

$$\int_\gamma \omega = \int_{t_1}^{t_2} [\, \gamma(t) \bullet \gamma'(t)]\, dt = (1/2) \int_{t_1}^{t_2} d_t\, [\, \gamma(t)\,]^2\, dt = (1/2) \int_{t_1}^{t_2} d_t\, |\, \gamma(t)\, |^2 dt$$

$$= (1/2) \int_{t_1}^{t_2} 2\, |\, \gamma(t)\, |\, |\, \gamma'(t)\, |\, dt = \int_{t_1}^{t_2} |\, \gamma(t)\, |\ |\, \gamma'(t)\, |\, dt$$

and then this 1-form line integral reduces to something that is an arc-length integral. From the first line I can compute this integral as

$$\int_\gamma \omega = (1/2) \int_{t_1}^{t_2} d_t \, |\gamma(t)|^2 dt = |\gamma(t)|^2 \Big|_{t_1}^{t_2} = |\gamma(t_2)|^2 - |\gamma(t_1)|^2 .$$

A second special case would be to take $f(\gamma) = \gamma'$. Then we would get

$$\int_\gamma \omega = \int_{t_1}^{t_2} [\gamma'(t) \bullet \gamma'(t)] \, dt = \int_{t_1}^{t_2} [\gamma'(t)]^2 \, dt = \int_{t_1}^{t_2} |\gamma'(t)|^2 \, dt$$

$$= \int_{t_1}^{t_2} |\gamma'(t)| \; |\gamma'(t)| \, dt$$

This also reduces to an arc length type integral. But we know that such an integral is unchanged if we replace $|\gamma'(t)| \rightarrow 1$ and then dt is the arc length variable. But this then sets the entire integrand to 1, and we find that arc length $L = t_2 - t_1$. In the other form we would have said

$$\int_\gamma \omega = \int_p^q \mathbf{v} \bullet d\boldsymbol{\ell} = L = t_2 - t_1$$

though I must admit this form does not make the result very obvious.

I guess my main point here is to distinguish the notion of integrating a 1-form over a curve from the notion of integrating a scalar function over a curve with respect to arc length. Although both these notions involve "integration over a curve", they are very different notions.