The Respeeding Function for a Curve PhL 6.27.15

This seems a trivial concept, but I am having trouble writing it down in a credible manner.

We start with a vector-valued mapping of a scalar variable called "the parameter",

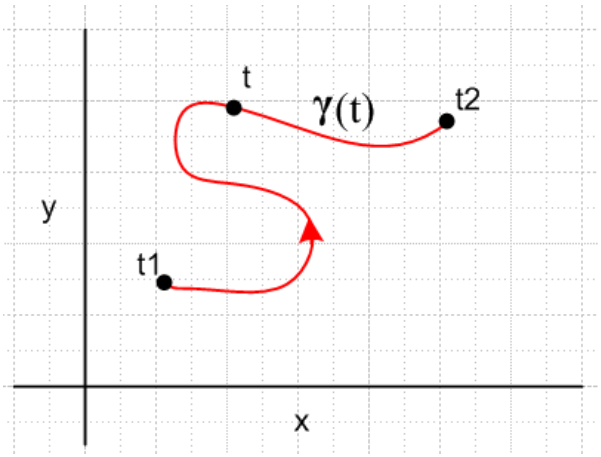
$\mathbf{r} = \gamma(t)$ $\gamma: E^1 \rightarrow E^n$ domain t in (t_1,t_2)

For such a mapping, we could obviously compute these quantities

$d\gamma/dt = \gamma'(t) = \mathbf{v}(t)$ and $|d\gamma/dt| = |\gamma'(t)| = |\mathbf{v}(t)| = v(t)$, the "speed" at t

and in general we will find that $|\gamma'(t)| \neq 1$, all fine.

Here is a picture where we assume $E^n = E^2$:



Now define a new mapping in this manner (new curve, but *same trace*)

$\mathbf{r} = \mathbf{\Gamma}(\tau)$ $\mathbf{\Gamma}: E^1 \rightarrow E^n$ domain τ in (τ_1,τ_2)

where

$\mathbf{\Gamma}(\tau) = \gamma(f(\tau))$ $t = f(\tau)$

where

$f(\tau)$ = monotonic increasing function of τ so that $f'(\tau) > 0$ at all points on the curve.

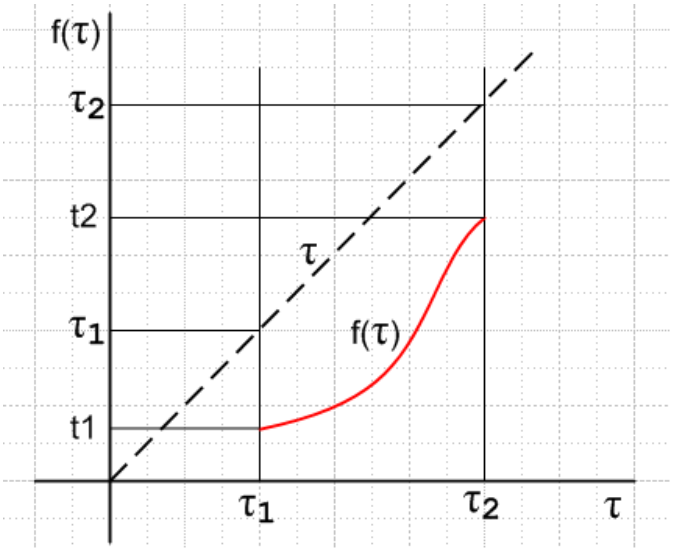
The fact that $t = f(\tau)$ tells us that

$t \equiv f(\tau)$ $t_1 = f(\tau_1)$ $t_2 = f(\tau_2)$

Then compute for the newly defined mapping,

$\mathbf{\Gamma'}(\tau) = \boldsymbol{\gamma'}(t(\tau))\ f'(\tau)$ // just the chain rule $d\gamma/dt * dt/d\tau$

Pause here just to draw a picture of a possible respeeding function $f(\tau)$,



Note that $f(\tau)$ is monotonic. The dashed line shows the function $f(\tau) = \tau$ which corresponds to not doing any respeeding since for this function $f'(\tau) = 1$ and $t = \tau$.

Now suppose we select a very special function $f(\tau)$, one which has this property:

$|\boldsymbol{\gamma'}(f(\tau))|\ f'(\tau) = 1$ for all τ on the curve

If such an $f(\tau)$ would be found, you would have

$|\mathbf{\Gamma'}(\tau)| = |\boldsymbol{\gamma'}(t(\tau))|\ f'(\tau) = 1 = |\mathbf{v}(\tau)|$ // we have "respeeded" so now "speed" = 1.

It is not obvious to me at this point exactly *how* you would find such a function $f(\tau)$! I think this is my point of confusion warranting this entire document digression. Can we express everything in terms of τ?

$t = f(\tau)$ $dt = f'(\tau)\ d\tau$ $f'(\tau) = dt/d\tau$

Then our equation is

$|\boldsymbol{\gamma'}(t)|\ dt/d\tau = 1$ or $d\tau = |\boldsymbol{\gamma'}(t)|\ dt$

Integrate both sides,

$\tau - \tau_1 = \int_{t_1}^{t} |\boldsymbol{\gamma'}(t')|\ dt'$

In theory, this integral can be computed and in that way we find

$$\tau(t) = \tau_1 + \int_{t_1}^{t} |\boldsymbol{\gamma}'(t')|\, dt' = \tau(t\,;\, t_1) = f^{-1}(t)$$

Notice that $\tau(t)$ is monotonic in τ because the integrand is positive! Since the inverse is found by reflecting in the 45 degree line (as Bucks showed earlier) $t = f(\tau)$ will also be monotonic in τ. You would have to draw a few pictures to nail this down, but I am OK with it for now. If $dy/dx > 0$ at all points on a curve, then $dx/dy > 0$ as well.

So we have now at least found the inverse function $\tau = f^{-1}(t)$. In theory, we can the invert this known function to find the function $t = f(\tau)$ and then our problem is solved.

Now that we have a way to solve this problem, let's back up to our previous statement:

Now suppose we select a very special function $f(\tau)$, one which has this property:

$$|\boldsymbol{\gamma}'(f(\tau))|\, f'(\tau) = 1 \quad \text{for all } \tau \text{ on the curve} \qquad t = f(\tau)$$

We know how to find such an $f(\tau)$, so assume that it has been found. We then have

$$|\boldsymbol{\Gamma}'(\tau)| = |\boldsymbol{\gamma}'(t(\tau))|\, f'(\tau) = 1 = |\mathbf{v}(\tau)| \qquad \text{for all points } \tau \text{ on the curve.}$$

So what, you ask? Well, we know that $\Gamma(\tau)$ and $\gamma(t)$ have the same "trace" as drawn above in red. We can use either of these "curves" to compute the arc length. But we have not yet even mentioned how one does that in general.

So assume again the curve $\mathbf{r}(t) = \boldsymbol{\gamma}(t)$. For some small dt we have

$$d\mathbf{r}(t) = \boldsymbol{\gamma}'(t)\, dt$$

This is a tiny vector tangent to the curve. The contribution of this little $d\mathbf{r}$ to the arc length is

$$ds = |d\mathbf{r}(t)| = |\boldsymbol{\gamma}'(t)|\, dt$$

Thus, the arc length is given by

$$L = \int_{\gamma} ds = \int_{t_1}^{t_2} |\boldsymbol{\gamma}'(t)|\, dt \ .$$

However, we can also compute the same arc length using the curve $\boldsymbol{\Gamma}(\tau)$ and we find

$$L = \int_{\Gamma} ds = \int_{\tau_1}^{\tau_2} |\boldsymbol{\Gamma}'(\tau)|\, d\tau = \int_{\tau_1}^{\tau_2} d\tau = (\tau_2 - \tau_1)$$

Since in this case the parameter τ IS the arc length at any point along the curve, the final result is very simple.

So what exactly is the point being made here?

1. For a given curve *trace*, it is always possible to find a *curve* (mapping) like $\mathbf{\Gamma}(\tau)$ such that the parameter τ is the same as the arc length.

2. In the above work, we showed how, if you start with some curve $\gamma(t)$ where t is NOT arc length, you can construct from this a curve $\mathbf{\Gamma}(\tau)$ whose parameter τ IS arc length. The key need is to identify this new parameter τ and in fact we showed that

$$\tau(t) = \tau_1 + \int_{t_1}^{t} |\boldsymbol{\gamma}'(t')|\, dt' = f^{-1}(t)$$

3. In the opening section of Chapter 7, Bucks talk about integrating a scalar function F along a curve.

$$I \equiv \int_{\gamma} ds\, F(\mathbf{r}) = \int_{\gamma} ds\, F(\boldsymbol{\gamma}) = \int_{t_1}^{t_2} dt\, |\boldsymbol{\gamma}'(t)|\, F(\boldsymbol{\gamma}(t))$$

We can evaluate this same integral using instead the curve Γ (same trace) to get

$$I \equiv \int_{\Gamma} ds\, F(\mathbf{r}) = \int_{\Gamma} ds\, F(\mathbf{\Gamma}) = \int_{\tau_1}^{\tau_2} d\tau\, F(\mathbf{\Gamma}(\tau))$$

Since this form is simpler, one might as well always use the curve Γ whose parameter is arc length rather than some other curve γ whose parameter is not arc length.

Extra Credit Question

Consider this 1-form situation

$$\int_{\gamma} \omega = \int_{t_1}^{t_2} [\, A(\boldsymbol{\gamma}(t))\gamma'_x(t) + B(\boldsymbol{\gamma}(t))\gamma'_y(t)]\, dt = \int_{\gamma} [\, A(\mathbf{r})dx + B(\mathbf{r})dy\,]$$

Notice that we don't have $|\boldsymbol{\gamma}'(t)|$ here, we have the components of $\boldsymbol{\gamma}$ appearing instead. Now suppose we carry out our process as described above and we define a new parameter τ and new curve Γ such that the above becomes

$$\int_{\Gamma} \omega = \int_{t_1}^{t_2} [\, A(\mathbf{\Gamma}(\tau))\Gamma'_x(\tau) + B(\mathbf{\Gamma}(\tau))\Gamma'_y(\tau)]\, d\tau = \int_{\gamma} [\, A(\mathbf{r})dx + B(\mathbf{r})dy\,]$$

Using the method above, it seems that we might carefully define $\tau(t)$ so that $\Gamma'_x(\tau) = 1$ for all τ. Then the above becomes

$$\int_{\Gamma} \omega = \int_{t_1}^{t_2} [\, A(\mathbf{\Gamma}(\tau)) + B(\mathbf{\Gamma}(\tau))\Gamma'_{\mathbf{y}}(\tau)] \; d\tau = \int_{\gamma} [\, A(\mathbf{r})dx + B(\mathbf{r})dy \,]$$

But now we cannot "clear out" the factor $\Gamma'_{\mathbf{y}}(\tau)$ without altering τ. I think the point is that you cannot clear out both speed factors for a general 1-form. For that reason, you cannot express this integral trivially as an integral over ds because $d\tau$ is only ds for the first term and not for the second term.

Conclusion added 4.14.16. Respeeding only works for the arc length (and surface area integrals). It does not work for integrating *functions* on curves and surfaces as one does with differential forms. Thus, it works for arc length, but as shown above it does NOT work for a general 1-form integration. Bucks say

Theorem 7; [323] Smoothly equivalent curves have the same arc length.

which means that respeeding the curve does not change arc length.

BUT!!!!! : All I have shown above is that respeeding with a 1-form cannot allow you to set both functions $\gamma'_{\mathbf{x}}(t)$ and $\gamma'_{\mathbf{y}}(t)$ to 1 and write a 1-form integral with $d\tau$ as the arc length parameter. This does not contradict the theorem Bucks later show on page 386 which is this (and which does not appear in my Ch 7 raw notes, and as yet I have no meta notes!!)

Theorem 2 page3 386: respeeding affects neither 1-form nor 2-form integrals!!!