This is the Title PhL 1.11.15

This has been installed in Buck Chapter 4 notes at the start of Section 4.4.

4.4 Improper Integrals with a Parameter **[204]**

(a) A Review of Continuity and Convergence

In Section 2.3 Bucks discuss continuity and uniform continuity for f: $E^n \to E^m$. The two notions are very closely related and we say

For any $\varepsilon > 0$ and for a specific p_0 in D which lies in E^n,
we can find $\delta(p_0) > 0$ such that $|f(p) - f(p_0)| < \varepsilon$ when $|p-p_0| < \delta$
=> f(p) is continuous at the point p_0

For any $\varepsilon > 0$ and *all* p_0 in D which lies in E^n,
we can find $\delta > 0$ such that $|f(p) - f(p_0)| < \varepsilon$ when $|p-p_0| < \delta$
=> f(p) is uniformly continuous on E

In both the above we have the idea that $\lim_{p \to p_0} f(p) = f(p_0)$. In Theorem 1 [57] Bucks showed that if we have this kind of continuity at some point p_0, then for any point sequence $p_n \to p_0$ we have $f(p_n) \to f(p_0)$. When the word "sequence" appears in the discussion, so does the word "convergence". In the above, we are saying that if p_n converges to p_0 in the domain, then $f(p_n)$ converges to $f(p_0)$ in the range. The phrase is that "continuity preserves convergence". This convergence is $f(p_n) \to f(p_0)$ in both cases stated above, the difference is in the nature of the convergence with respect to whether δ depends on p_0 or not. It is true that this type of convergence does involve *functions* (like f).

Notation anomaly: The symbol p_0 is here overloaded since it represents our point of interest, but it also represents the first point in the sequence $\{p_n\}$. These are completely different meanings, so one must just keep this in mind.

Now, in the above we have $p_n \to p_0$ being a "sequence of points in E^n"
f(p) is a "function" such that f: $E^n \to E^m$.

E^n is of course a particular Hilbert Space in the language of Stakgold vol I p 110.

Stakgold discusses another Hilbert Space he calls $\mathcal{L}_2^{(r)}(a,b)$ and $\mathcal{L}_2^{(c)}(a,b)$ where are the spaces of real or complex valued functions on the interval (a,b) which are L_2 integrable over that interval. We could generalize this to talk about $\mathcal{L}_2^{(m)}(D)$ which would be functions taking values in E^m which are defined on a domain D in E^m, which again are square integrable over D. Another choice is $\mathcal{L}_1^{(m)}(D)$ which is functions for which $\int_D d^m x \,\|f(x)\|$ is integrable and then $\|f\|_1 = \int_D d^m x \,\|f(x)\|$, whereas in the previous

case we had $\| f \|_2 = (\int_D d^m x \| f(x) \|^2)^{1/2}$ and in general $\| f \|_p = (\int_D d^m x \| f(x) \|^p)^{1/p}$. Note that E^2 and complex are not quite the same, we let that pass.

Another norm of interest is this one: $\| f \|_\infty =$ max(||f(x)||) on D. I might call this "the max norm" but wiki seems to like the "infinity norm" because it is like the 1 and 2 norms except $p \to \infty$ (though I have not proven that fact). Officially it is the "supremum" norm, but I will call it the max norm.

This max norm is the one the Bucks use for functions, and they write it as $\| f \|_E$ over set E.

Back in Section 4.2 we have the following new concepts. The players are now functions $f_n(p)$ which form a sequence of functions $\{f_n(p)\}$ which perhaps converge to some function f(p)

For any $\varepsilon > 0$ and for a specific p in D which lies in E^n,

we can find $\delta(p) > 0$ such that $| f_n(p) - f(p)|_{En} < \varepsilon$ when $n > N$. // n is overloaded!

=> $f_n(p)$ converges pointwise to f(p) on D

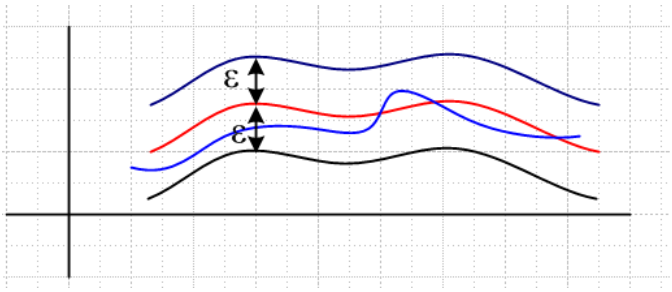
Notice that the norm used here is the geometric norm of E^n , it is not any of the fancy function space norms listed above. As we have learned, this pointwise convergence is not very "interesting" because it doesn't lead to nice facts like order interchange of things. What is really much more interesting is the following notion:

For any $\varepsilon > 0$

we can find $\delta > 0$ such that $\| f_n - f \|_\infty < \varepsilon$ when $n > N$.

=> sequence f_n converges uniformly to f over D

Now we are for the first time using this max norm, and that norm knows about the nature of $f_n(p) - f(p)$ for all values of p in D, because it needs to know that in order to compute the "max". So it seems best not to display the function argument p, since the norm here is for the function itself! The main idea here is that if you have $\| f_1 - f_2 \|_\infty$ very small, then the two functions are close together for ALL values of their arguments. If $\| f_1 - f_2 \|_\infty < \varepsilon$ then each function lies within the 2ε band of the other function:



Recap. In the above, we have discussed two totally different notions of convergence: We have first a convergence of points p_n or $f(p_n)$ which involves the single function f. And we have second a convergence of functions f_n to some function f. In more detail:

1. In the discussion of continuity and uniform continuity, we had $\lim_{p \to p0} f(p) = f(p_0)$ using the E^m norm saying that that $| f(p) - f(p_0)|_{Em} < \varepsilon$. We could then bring into the discussion a sequence of points p_n and

we could then say $\lim_{p_n \to p_0} f(p_n) = f(p_0)$ and we then have convergence of the sequence $\{f(p_n)\}$ which is a sequence of points in E^m. This convergence is associated with the continuity of the function f(p). In the case that the same δ works for all p_0, we had uniform continuity over E, and you *could* say that we have uniform convergence of $f(p_n) \to f(p_0)$ where $p_n \to p_0$ where p_0 is any point in E. In other words, we are saying that $f(p_n) \to f(p_0)$ converges to p_0 sort of "uniformly" for any p_0 in E. This use of the phrase "uniform convergence" is not used by the Bucks. You might call it (uniform convergence)^p since it involves a sequence of points. Note that, although f(p) is a function, here we are really talking about sequences of points, $\{ p_n \}$ and $\{ f(p_n) \}$. We are *not* talking sequences of function, only one function f is involved in the discussion.

2. In the above subsequent discussion of sequences of functions $f_n(p)$, we talked above about $f_n(p)$ converging pointwise to f(p) on D, which involved the E^n norm, and we talked about f_n converging uniformly to f using the max norm over D. We say that f_n converges uniformly to f over D, and this is the official definition of the phrase "uniform convergence". Maybe call this (uniform convergence)^f since it involves a sequence of functions.

(b) Application to a bystander parameter.

Imagine that p lies in some set E, and that t lies in some set T.
Consider a function of both variables f(p,t).
Then consider this ε δ statement:

1. For any $\varepsilon > 0$ and for a specific p in E and for a specific t_0 in T,
we can find $\delta(t_0,p) > 0$ such that $| f(t,p) - f(t_0,p)| < \varepsilon$ when $|t-t_0| < \delta$.

I would describe this as "continuity at a point t_0 in T with a bystander parameter p in E".
I would also say that f(t,p) "converges" to $f(t_0,p)$ as $t \to t_0$.
This means there are sequences t_n so that $f(t_n,p)$ "converges" to $f(t_0,p)$ as $t_n \to t_0$.

Our first "upgrade" of this concept would be:

2. For any $\varepsilon > 0$ and for *all* p in E and for a specific t_0 in T,
we can find $\delta(t_0) > 0$ such that $| f(t,p) - f(t_0,p)| < \varepsilon$ when $|t-t_0| < \delta$.

I would describe this as "continuity at a point t_0 in T which is uniform over E".
I would also say that $f(t_n,p)$ (converges uniformly)^p over E" to $f(t_0,p)$ as $t_n \to t_0$.
Notice that there is only one function here, it is called f.

A different upgrade of item 1 might be this:

3. For any $\varepsilon > 0$ and for a specific p in E and for *all* t_0 in T,
we can find $\delta(p) > 0$ such that $| f(t,p) - f(t_0,p)| < \varepsilon$ when $|t-t_0| < \delta$.

I would describe this as "uniform continuity" over T with a bystander parameter p".
One could also say we have (uniform convergence)^{p} of $f(t_n,p) \to f(t_0,p)$ as $t_n \to t_0$.
The sequence involved is a sequence of *points* $\{t_n\}$ or $\{ f(t_n,p) \}$; only function is f.

Now in the next item we combine both the above upgrades:

4. For any $\varepsilon > 0$ and for *all* p in E and for *all* t_0 in T,
we can find $\delta > 0$ such thatc when $|t-t_0| < \delta$.

I would describe this as "uniform continuity over T which is also uniform over P".
This latter is a strange new use of the word "uniform". The statement involves being uniform in two different spaces at the same time, E and T. The "continuity" aspect only involves space T.
One could also say we have (uniform convergence)^{p} of $f(t_n,p) \to f(t_0,p)$ as $t_n \to t_0$.
The sequence involved is a sequence of *points* $\{t_n\}$ or $\{ f(t_n,p) \}$; only function is f.
Notice that we still have the E^m norm, not the max norm. We still have only one function.
So even this double upgrade thing has nothing to do with (uniform convergence)^{f} of a set of functions $f_n \to f$ in the Hilbert Space of the max norm.

On page 204 the Bucks use the above double-upgrade concept, which involves to different underlying sets E and T. They want to say that the result is uniform convergence of $f(t_n,p) \to f(t_0,p)$ since it is uniform in both the T and E sense. I will call this (uniform convergence)^{p2} to indicate that we are still using the point norm for $f(t_n,p) \to f(t_0,p)$, but that we are uniform in 2 spaces, T and E.

So this really is a brand new definition of the term "uniform convergence".

Then f(t,p) is continuous at $t = t_0$ and p = p. We say there is convergence, $\lim_{t \to t0} f(t,p) = f(t_0,p)$.