Compactness, topology, convergence, and famous theorems PhL 1.11.08

Comments: Every time I review Stakgold, I get to page 114 and the word "compact" appears, and I go off and write a separate document about that concept because I remember always being unclear about it. After writing the stuff below, I found older notes from 2005 on this subject. I had downloaded a set of notes by David Royster on the general subject of topology, and then I made some doc comments on his lectures. At that time, I was separating PDF files from my own doc files for purposes of backup, and I now see that is not a good idea. I had today to go hunt up those Royster notes in the CD backup area.

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I am not very familiar with mathematics from a high level, so maybe some comments here:

Topology. In Stakgold, we sort of identified the notion of a topological space with that of a metric space, something that has a notion of distance between two points in the space. Consider, however, these more general definitions:

1. If you have some "set X of objects x" you can talk about the subsets of set X. There is some complete set of all the subsets of X. You might select from that complete set some smaller set of subsets. If you find such a smaller set of subsets T that have the simple properties listed below, then that set T of subsets is called a **topology** on X, and the combination T with X is called a **topological space**. The properties are simply these: T must contain the full set X and the null set φ, and the union or intersection of elements of T must lie in T.

The sets of such a T can be simply defined to be "**open sets**" without regard to our metric notion of what an open set is (you can put a ball around any point). Then the complements of the open sets are the closed sets. A **neighborhood** of a point x is any set that contains an open set containing x.

Every metric space can be given a **metric topology**, in which the basic open sets are open balls defined by the metric. This is **the standard topology** on any normed vector space. On a finite-dimensional vector space this topology is the same for all norms.

A **function between topological spaces** is said to be **continuous if** the inverse image of every open set is open. This is an attempt to capture the intuition that there are no "breaks" or "separations" in the function.

The wiki page http://en.wikipedia.org/wiki/Topology gives a general feel for the subject of topology. You want to learn about both global and local properties of objects (like solid objects in 3D space, for

example) which are independent of the detailed shape of the objects. You can imagine smoothly deforming a sphere to a cube, or a donut to a coffee cup. **Connectedness** is a subject of interest (singly or doubly connected in our examples). I think two objects have the same topology if you can map between them with a function which is continuous in both directions.

" If a continuous function is one-to-one and onto and if the inverse of the function is also continuous, then the function is called a **homeomorphism** and the domain of the function is said to be **homeomorphic to** the range. If two spaces are homeomorphic, they have identical topological properties, and are considered to be topologically the same. The cube and the sphere are homeomorphic, as are the coffee cup and the doughnut. But the circle is not homeomorphic to the doughnut."

The Heine-Borel Theorem discussed below equates two definitions of compactness. One involves a set being bounded and closed, perhaps the older definition of compactness for R_n. The other definition involves "open covers" which means covers of a set by a group of *open* sets. Thus, you see how the world of open sets connects to the concept of compactness, so we are making a connection between topological spaces and compactness.

I think now we are prepared to appreciate the opening paragraph of wiki on "topology":

Topology (Greek Topologia, Τοπολογία, from topos, τόπος, "place," and logos, λόγος, "study") is the branch of mathematics that studies the **properties of a space that are preserved by a continuous function that has a continuous inverse**. Topology grew out of geometry, but unlike geometry, topology is not concerned with metric properties such as distances between points. Instead, topology involves the study of properties that describe both the fine structure and global structure of space. Key concepts in topology include compactness, connectedness, and orientability.

The word topology is used both for the area of study, and for a family of sets with certain properties that are used to define a topological space, the most basic object studied in topology. Of particular importance in the study of topology are the functions called **homeomorphisms**, are continuous with a continuous inverse. Another class of functions that is important in topology are the **homotopy equivalences**, which stretch space without tearing it apart or sticking distinct parts together.

Historically, the topological space of interest was the 3D space in which we live, R_3. We can of course think of the open sets as going with the metric open ball concept we are used to, and this is the "standard topology" as noted above.

More generally, **modern geometry** concerns itself with the study of **manifolds** which are spaces that are locally Euclidean, but might be globally very strange. If your manifolds have enough structure to support doing calculus, you have a **differentiable manifold** and these are pretty big in physics. This general area is study is called **differential geometry**.

Again: a **topological manifold** is a topological space locally homeomorphic to a Euclidean space.

Comment: So lots of things are interrelated here. Topological spaces, open sets, manifolds, compactness, continuity, all part of set theory really.

Real analysis is a branch of mathematical analysis dealing with the set of real numbers. In particular, it deals with the analytic properties of **real functions** and **sequences**, including **convergence** and **limits** of sequences of real numbers, the **calculus** of the real numbers, and **continuity**, smoothness and related properties of real-valued functions.

Definitions of Compactness

Definition: compact for R^n ; In mathematics, a subset of Euclidean space R^n is called **compact** if it is **closed and bounded**. For example, in R, the closed unit interval [0, 1] is compact; the set of integers Z is not (it is not bounded) and neither is the half-open interval [0, 1) (it is not closed). So a compact set in R^n is just a volume that includes it's boundary

Note that a **set or subset** can be finite or infinite, regardless of the finiteness or infiniteness of the metric space the set is taken from.

Definition: closed and closure. The idea of "including the boundary" is generalized by saying that a set includes all its sequence limit points. For a metric space, this makes the space be **complete**, as we saw in Stakgold. For a set of points in a metric space, including all the limit points makes that set be **closed**. One usual says that you start with S and add the limit points and the **closure** is called $\overline{S}$.

Obviously the notion of being closed implies the limit points of convergent sequences and this implies we are talking about metric or topological spaces. So compactness is a property of a set in this world.

Modern definition of compact: Instead of the definition of compactness as being closed and bounded, the *modern* approach is to call a topological space **compact** if each of its open covers has a finite subcover.

Definition of sequential compactness**:** In mathematics, a topological space is **sequentially compact** if every sequence in the space has a convergent subsequence. Some people (Stakgold) just refer to this being the definition of "compact".

Fact: Any finite set is compact in the above sense since any sequence must repeat things.

A **limit point** is a point in a metric space which has an infinite number of neighboring points within distance ε, no matter how small you make ε. Also called an accumulation point. Sometimes a cluster point. Would be a point where some infinite sequence might converge.

An **open set** is one such that every point has a **neighborhood** that lies within the set. The complement of an open set is a **closed set**. It is possible for a set to be **neither** open nor closed, e.g., the half-closed interval (0,1].

Convergence is the idea that, given any ε, there is an N such that $| x_n - n | \equiv d(x_n,x) < \varepsilon$ for n>N.

In the notion of uniform convergence, the above sequence x is regarded as a function of some parameter t that has some range (a,b), or more generally, some t that is in some set S. Then :

Uniform convergence : for any ε, there is an N such that $d(x_n(t),x(t)) < \varepsilon$ for n>N *for all* $t \in S$. This sequence then "converges uniformly" over the set S.

The Weierstrass M-test: suppose $|x_n(t)| < M_n$ for all $t \in S$.
Then if M_n converges, $|x_n(t)|$ uniformly on S.

The Heine–Borel Theorem

In the topology of metric spaces the **Heine–Borel theorem** (1895 final form) named after Eduard Heine and Émile Borel, states:

For a subset S of Euclidean space R_n, the following two statements are equivalent:

* S is closed and bounded
* every open cover of S has a finite subcover, that is, S is compact.

In the context of real analysis, the former property is sometimes used as the defining property of compactness. However, the two definitions cease to be equivalent when we consider subsets of more general metric spaces and in this generality only the latter property is used to define compactness. In fact, the Heine–Borel theorem for arbitrary metric spaces reads:

A subset of a metric space is compact if and only if it is complete and totally bounded.

Definition: A metric space (or topological vector space) is said to have the **Heine-Borel property** if every closed and bounded subset is compact.

The Bolzano–Weierstrass Theorem

In real analysis, the Bolzano–Weierstrass theorem is a fundamental result about convergence in a *finite*-dimensional Euclidean space R^n. The theorem states that (two equivalent formulations)

- each bounded sequence in R^n has a convergent subsequence.
- a *subset* of R_n is sequentially compact if and only if it is closed and bounded.

My take on this? Imagine a sequence that is "bounded" in some region. If the sequence converges in the normal manner, then any tail of this sequence would be a convergent subsequence. If the sequence goes around through some finite number of points again and again, then the hits on any of those points form a convergent subsequence. This theorem is saying that if the sequence never repeats the points in this way, then somehow it must converge somewhere because it cannot keep going finding "new ground" because it is bounded.

The Bolzano–Weierstrass theorem is named after mathematicians Bernard Bolzano and Karl Weierstrass. It was actually first proved by Bolzano (1817??) , but this proof was lost. It was re-proven by Weierstrass (1860??) and became an important centerpiece of analysis. Later, it was discovered that Bolzano had in fact proved the theorem long before Weierstrass, hence the current name.

Comments on the BWT. In the first bullet, we are *not* talking about some arbitrary subset S in R_n, we are talking about *all* of R_n . We are saying that if you select a bounded sequence in R_n , then it will have a convergent subsequence. But of course R_n is closed, so we are just here saying that closed + bounded implies compact.

Good web site for information about mathematicians:

> The MacTutor History of Mathematics archive
> http://www-groups.dcs.st-and.ac.uk/~history/
> School of Mathematic and Statistics
> University of St Andrews Scotland (this is the st-and part
>
> http://www-groups.dcs.st-and.ac.uk/~history/BiogIndex.html // for individual math people