Royster Notes on Metric Spaces, Topo Spaces, and Compactness PhL 1.30.04

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Chapter 2: Metric Spaces

In Stakgold Chapter 2 we had a very minimal introduction to "metric spaces", and here Royster expands on the subject to about as far as I want to go right now. His chapter is 12 excellent pages.

2.1. Definitions and Examples.

We start out defining a **metric**, then we have lots of solid examples including new ones like the **Taxi Cab** metric and the **Max metric** and the **Discrete metric** (see later on this) and the **sup metric** on a function space and so on. Along the way he informally throws in the definition of a **norm** $\| x \|$ as d(x,0). He then makes very clear distinction between what he calls the **Cauchy-Schwarz Inequality**[1] **[~1830]** (which requires the notion of an inner product which he has not really introduced yet, but which we all know about at least in R^n), and the **Minkowski Inequality [1896]** which is the triangle rule on norms. These simple things are what you need to show that things are metrics.

He does me the favor of carefully defining **lub** = **sup** and **glb** = **inf**.

In general, most theorems and comments are about a **subspace A** of a metric space X, and not about X itself. This I think is a key thing to keep in mind. He defines what it means for a subspace A to be **bounded** (the metric d(x,y) has an upper bound), and he defines the **diameter** of A (max d(x,y). He talks about the **distance between a point x and a subspace A** as being the distance of smallest approach, and you write this as d(x,A), which is the glb. The **product metric** is mentioned in passing.

He makes very clear distinction between what he calls the Cauchy-Schwarz Inequality (which requires the notion of an inner product which he has not really introduced yet, but which we all know about at least in R^n), and the Minkowski Inequality which is the triangle rule on norms.

2.2. Continuous functions

We start off with the usual ε, δ definition of continuity of f(x). I never thought about this before, but suppose you have f: (X,d) → (Y,d') so X and Y have different metrics d and d'. Then what you are saying is that **continuity** at some point x_1 is true if,

" given some ε, you can find δ such that [$d(x,x_1) < \delta \Rightarrow d'(f(x),f(x_1)) < \varepsilon$]

and he calls this **"Cauchy-Weierstrass continuity"**. Notice that it depends on both metrics d and d'. He then does a few examples. In one example, he has d = "the usual metric" and d" = "the max metric".

2.3 Open Sets and Closed Sets

[1] Named for Augustin Cauchy, 1789-1857; and Hermann Amandus Schwarz, 1843-1921. Some call it the Schwarz Inequality, while others, including Russians, call it the Cauchy-Buniakowski [1859] Inequality.

We first define the notion of the **open ball $B(x_1; r)$** around some point x_1 [$d(x,x_1) < r$] and how dimly I remember this idea from those Sever Hall lectures. The *open* ball does not contain its boundary. Then any set **N is a neighborhood** of x_1 if we can wedge an open ball around x_1 with the entire ball inside N, no matter how small we have to make the ball. If there is some δ (no matter how small) such that $B(x; \delta)$ fits in N, then N is a neighborhood. Notice that the ball and the neighborhood are dependent on your metric d.

Then we can define an **open set** is any union of open balls.

Question: is a finite set an open set? Consider the set of the first 10 integers with the metric of the reals. You can certainly get an open ball around each set element. But that ball does not fit inside the set of the 10 integers! No ball can do this. Thus, the singleton elements are not open sets. So any set of integers is not an open set. The complement would be open, and so a finite set of integers is a closed set. All the points are boundary points I think. Suppose X is the integers and A is the positive integers with the usual metric. Note that $d(x,y) = 0$ for $x=y$ but $d(x,y) \geq 1$ otherwise. So again, you cannot wedge a ball around an integer of radius 1/2 and have it be in the set. So this set to is not an open set. Only infinite sets can have open subsets.

And then the **family of all open sets** is the **topology** for X generated by d.

Theorem (page 16). If U is an open set in X, then for x in U we have that $d(x,X\backslash U) > 0$. This says that the entire complement set is separated from U by a δ-thin boundary layer, hence the > 0.

Suppose $A \subset X$. Then $A^c = X \setminus A$ = the **complement** of set A. This notation $X \setminus A$ reminds you what is "outside of A", whereas A^c does not remind you of this.

A **closed set** is one whose complement is an open set. Only one of these can contain the boundary layer, and since an open set does not, the complement does. Notation: [0,1] is a closed set, (0,1) is open, and the set (0,1] is neither! It contains only part of its boundary!

2.3.1 Neighborhoods and Continuous Functions

We now have an alternate **"neighborhood definition of continuity"** which we show is equivalent to our earlier definition. A function f(x) is continuous at point x_1 if:

> If M is *any* neighborhood of $f(x_1)$, we can find a neighborhood N of x_1 such that $f(N) \subset M$.
> (or the punch line could be stated as: $N \subset f^{-1}(M)$, but this does require that f^{-1} exist)

The real idea is that "we can find a neighborhood N of x_1 *that is small enough* so that $f(N) \subset M$"

2.4 Limits

Think of a sequence x_n as a function x(n) with $x: Z^+ \rightarrow (X,d)$.

The notion of $a_n \rightarrow L$ is given relative to d in the usual ε notation: I can find n large enough so that $d(x_n,L) < \varepsilon$ no matter how small an ε you give me. So this is the **limit of a sequence** and if there is such a limit, then the sequence **converges to the limit**. [He does not mention the Cauchy sequence idea.]

The notion of something **"true for almost all of S"** means true for all but a finite number of points in S. Thus, if a sequence converges $x^n \rightarrow x$,. and we pick any ball around x (no matter how small), then almost all of the sequence lies in that ball. Only a finite number of sequence elements don't lie in ball!

We now have an alternate **"sequence definition of continuity"** which we show is equivalent to our earlier definitions. A function f(x) is continuous at point x_1 iff:

For *any* sequence $x_n \rightarrow x_1$ that you can create in X, $f(x_n) \rightarrow f(x_1)$.

where I imply the limit idea with my cheap notation $\rightarrow$.

2.5 Open Sets and Closed Sets Revisited

Properties of the open sets in some metric space X(d) : X and ϕ are open,
union of opens = open, intersection of finite number of opens = open

Properties of the closed sets in some metric space X(d) : X and ϕ are closed,
intersection of closed = closed, union of finite number of closed = closed

It is not clear to me how X and ϕ can be both open and closed. I think you always have to "extend" concepts to these two limiting cases of subsets of a set X, not really too meaningful. After all, X itself has no boundary that you can exclude or include. The idea of a boundary means something has to lie outside your set, your set has to be embedded in some larger set.

An **accumulation or limit point** x_1 in $A \subset X$ is one such that, if O is an open set containing x_1, then O contains at least one other element of A.

It seems to me that you can only have a limit point in a region of set A that is "continuous" in some sense (the sense of continuous spectrum). If X is a "continuous" space like R^n and you have a set A that contains some isolated points like { -1, -2 }, then no point in A can be a limit point because you can put a small ball around -1 and it will not contain any other elements of A. My notion of a continuous space here is that any ball contains an infinite number of set points. Not sure what the official name is.

So if A is a continuous subset of S, then if O is any open set well inside, of course any point in O will be a limit point. The big issues is what happens near a boundary of set A that lies in X. If you pick a point right on the boundary and put a ball around it, obviously that ball contains elements in A and in X\A. But since it contains points in X, it is a limit point.

It seems to me that the set of limit points of A (which set is called A') is really the closure of A. But that would not work for sets with isolated elements because, as noted, these won't be in A' Thus, you need to say this:

$$\bar{A} = A \cup A' \qquad = \text{the closure of A}$$

If a set A contains all its limit points, it is already closed.

Theorems about Limit Points.

1. [x = limit point of A] $\Leftrightarrow$ [$\exists\ x_n \to x$ within A]

2. [set A is closed] $\Leftrightarrow$ [all $x_n \to x$ within A]

2.6 Interior, Closure and Boundary

A point x_1 is **interior** to A if you can wedge a ball around it that fits in A, or if you can wedge any open set around x_1 which lies in A. (I think these are equivalent statements) , You call it **intA.**

The interior of A is open and is the largest open set in A. Example Int [a,b) = (a,b).

A finite set A has no interior and in fact contains no open sets at all, I think.

He then defines the closure of A as I have noted above. Note that $\bar{A}$ is the smallest closed set containing A.

Fact: [A is open] $\Leftrightarrow$ [A = intA]
Fact: [A is closed] $\Leftrightarrow$ [A = $\bar{A}$]

Finally, we can isolate our "boundary layer" as follows:

$$\text{boundary } \partial A = \text{closure of } A \cap \text{ closure of } X \setminus A$$

$$= \bar{A} / \text{intA}$$

Each of these is rather interesting, and there are other equivalent definitions of the boundary.

Finally, he notes that $\partial\phi = 0$ and $\partial X = 0$ -- neither of these limiting case sets has a boundary. That is why these sets can be both open and closed, but he fails to comment on that.

Comments: Let's review the topics mentioned in these few pages:

discrete metric, taxi metric, Minkowski ineq, C-S ineq, glb = inf, lub = sup
continuity three ways: ε,δ way, then neighborhood, then sequences
distance from a subset, diameter of a subset, product metric, notion of a bounded set (finite diameter)
the open ball , open and closed sets, a topology, complement of a set, family of sets
true for almost all of S, sequences and convergence
accumulation points = limit points, especially at a boundary
interior of A, closure of A, boundary of A

Everything here has to do with the notion of *distance* in the space X, as apropo for a chapter on Metric Spaces. The inner product is basically not mentioned at all. Nor is compactness or covers, that comes in a later section.

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Chapter 3: Topological Spaces

Unlike the previous section, I have not printed and studied this chapter, but I did browse it and will attempt here to summarize key points.

A goal for some reason is to be able to do things *without using a metric* d(x,y), I guess this is thought to be too limiting in some way. The plan is to replace d(x,y) and open balls with "open sets", but of course we will then need a new definition of an open set. This new world -- having no metric -- is no longer that of "metric spaces" ; it is that of "topological spaces", for want of a better term.

The word **family** now appears, as in "a family of subsets of space X". For a set of subsets of X to really be a "family" F, the subsets have to have these properties:

a) X and ϕ are in the family
b) union of any sets in the family is another set in the family
c) intersection of a (finite number) of sets in the family is another set in the family

If your set of subsets respects these rules, then that family F is, by definition, a **topology for X**, and the sets of the family are called **open sets**. The combination (X,F) is called a **topological space**.

So this was a lot of stuff in one shot. This is that "new definition" of an open set, for one thing. No mention of balls or metrics. Recall that (X,d) was a metric space, well now we have a brand new kind of space! The three rules shown above of course mimic the three rules we had for our earlier definition of an open set. The **metric space appeared** late, in **1906** (Maurice Frechet). In **1914** if was Hausdorff who coined the "topo space" phrase, so this is a slightly more recent subject that metric spaces.

Side note: nothing is said about the complement of a family set being required to be in the family. I think the word collection is also used in place of family.

Claim: for R^n, the usual metric, max metric *and* the taxicab metrics all result in the same topo space, so this will be the "usual topology" for R^n.

The **discrete metric** was mentioned earlier as being d(x,y) = 1 if $x \neq y$ else 0. In terms of balls, if you put a ball around any member x of a set X with ball $\varepsilon < 1/2$, say, then that element x is the only thing in the ball, and is therefore an open set. That is because open sets are any unions of balls. Thus, in the discrete metric, each x in X is an open set, and of course any union of elements is also an open set. For a set of sets, this means that your "family" contains all the **singleton sets** (and all their unions of course). But we know that this particular set is the **power set, usually written 2^X** which means for each singleton, you can either take it or not as you construct a set. Thus, if X had N singleton subsets, you would have 2^N sets in your power set. The family of sets you have in this case does in fact respect our three rules and is therefore a topo space, and it has a special name: the **discrete topology**. This is just another name then for the power set.

The **trivial topology** (aka the indiscrete topology) has only ϕ and X.

If you take as your family any A whose complement X\A is a finite set, you get the **finite complement topology**, a very good name. Also called the **cofinite topology** (a shortened form of the longer name)

As in metric world, a set A is **closed** if its complement X\A is open. This results in the same three rules for a closed set that we had in metric world!

Basically, we are going to "redo" everything now in "topo world" that we did earlier in "metric world", and I expect to see perhaps some small differences.

The **limit point** is defined exactly as before, and **A'** is the **derived set** of all limit points, as before.

A **subset A is closed** if it contains all its limit points, same as before.

A **sequence x_n converges** to a **limit** x in X if, for any open set we produce which includes x, the tail of the sequence above n=some N will be in that open set.

HOWEVER: sequences are not something you work with in topo world, they are just not useful. Author gives an example of what that is the case: with the cofinite topology defined above, author shows that every sequence x_n converges to every real number at once, and all sequences converge. So hard to imagine a sequence in this topo space as doing anything useful.

Interior point x in A means you can wedge an open set around it which entirely fits in A. If you can do so, then A is a neighborhood.

Closure and **boundary** are same as before.

Now comes something new: if A is in X and A-closure = X, then A is **dense** in X. For example, the rationals are dense in the reals, and so are the irrationals. Polys are dense in set of continuous functions.

If X contains a **countable dense set**, then X is **separable**. Since reals contain rationals, and since rationals are countable, the R is a separable space! And E^n is separable because the rational grid points can be counted. Things are making more sense now!

A set B is **nowhere dense** if interior(B-closure) = ϕ. Since a finite set has no interior, it would be nowhere dense.

In section 3.3 something completely new is introduced, the notion of a **basis of a topology**. The idea seems very simple. A **basis** for the family $\mathtt{F}$ is any set of steps from which you can generate all sets of the family by doing unions. Such a basis $\mathtt{B}$ consists then of the **basic open sets**. (definition). If a topo space has at least one basis which is countable, the space is called **second countable**.

Examples: open intervals in R form a basis. From these you could create any perforated open set. The set of open balls in a metric space is a basis there. The singletons always form a basis in the discrete topology. This particular basis does not seem countable, by the way, since could end on any real.

The notion of a **local basis** is tougher to see. A local basis $\mathtt{B}_a$ is associated with a particular element a of X. Of course $\mathtt{B}_a \subset \mathtt{F}$, the family which is the topology. The rules are this: (1) a is an element of all the

sets that make up the collection B_a ; (2) if an open set U contains a, it must contain at least one member of B_a . I think local means local to the point a in X.

My own example. Let a = 5.2, and consider to be all open intervals which include a. That makes rule (1) happy. And rule 2 seems OK too. For example, some disjoint open set containing a contains an interval which contains a, and that interval would be in B_a .

Suppose your topo space (X,F) can have a B_a for any a in X. Then it is **first countable**. If my example above is OK, then the reals with the usual topology is first countable.

Claim: if space is second countable (countable basis), then it is also first countable (any a has a local basis)

Claim**:** If space is **second countable** (countable basis), then it is **separable**. The proof here is that you construct an explicit subset A of X which is dense in X, and that makes X separable!

Claim: Every **metric space is first countable** (every a has a local basis).

Claim: Every **separable metric space is second countable**.

Theorem 3.8: B is a basis if, for any B_1 and B_2 in B, their intersection $B_1 \cap B_2$ contains some B_3 in B. I think the intersection may have to be non-null. Also, the union of all the B_i must of course give X.

Now we continue re-doing our metric space stuff in topo world. I copy from above one of our metric space definitions of continuity:

> "If M is *any* neighborhood of $f(x_1)$, we can find a neighborhood N of x_1 such that $f(N) \subset M$.
> (or the punch line could be stated as: $N \subset f^{-1}(M)$, but this does require that f^{-1} exist)"

Just replace the word neighborhood with open set and you have the new version on page 29. The neighborhood version above I think is OK in topo world just as it is (also), since we do have a topo world definition of neighborhood.

Now back to new things. A mapping is a **homeomorphism** if it is onto and 1-1. You can in fact have entire topo spaces which are homeomorphic.

A **topological property** of a space is one that will be true for any other space which is homeomorphic to your space. Examples are separability, first and second countability

A topo space is **metrizable** iff you can generate it from some metric, and this is a topological property.

Boundedness is not a topo property, because R and (0,1) are homeomorphic!

Now, can you sort of induce a topology on X down onto some subset A of X? You start with (X, F) and you want to end up with some topo space (A, F' ') . It would seem reasonable to take all the sets in F and just intersect them with A to get the open sets for F' ' . Remember that all sets in F are open sets, by our

definition of family F. If you so this, then (A, F' ') is called the **relative topology** (aka **subspace topology**). Members of F' ' are called **relatively open sets**, and (A, F' ') is a **subspace** of (X, F) .

A **hereditary property** of a topo space is one that passes down to all subspaces. Examples are first and second countability, but not separability.

Recall how a Hilbert Space is a vector space with a scalar product added (more or less).

A **Hausdorff Space** is a topo space with an added requirement: For any two points in the space, you have to be able to wedge an open set around each point such that the sets don't intersect. All metric spaces are Hausdorff because you can wedge two balls in a metric space of radius d(x,y)/2.

Hausdorffness is a topological property and a hereditary property!

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Chapter 4: Compactness

Again, no printed pages, just reading the PDF and commenting here.

A set being **compact** is a *generalization* of a set of the reals being both **closed and bounded**, like [0,1].

We first define an **open cover** of A to be a collection of open sets whose union contains all elements of A. A **subcover** would be any subset of the cover that has this same covering of A property.

Now the definition here of compact has always seemed distant to me:

"A set A in X is **compact** if every open cover of A contains at least one finite subcover."

Among other things, this seems to require that the set A be coverable with a finite number of open sets, but I guess that is not very restrictive. Just take one set being all of A and add some around the boundary if A is not open, then that one set would cover A, so that is not really the point. Exactly what the point is I hope to learn!

Cantor's Nesting theorem. This tells you to consider an every decreasing nested sequence of *closed* intervals which we write as $[a_n, b_n]$. If the diameter of the interval (I guess than means we need a metric, so we will use the usual metric on R) $\rightarrow 0$, then you must end up with a single point p. Otherwise you end up with some larger interval as your limit. The key fact here is that these nested intervals must be closed, not half-closed or open.

Claim: The interval (0,1) is not compact. [Note that it is bounded, but not closed.]

Proof : Consider the sequence of open intervals (1/n,1) for $n = 2 \rightarrow \infty$. That left endpoint starts at 1/2 and moves toward 0 and gets as close as you want. So this is an open cover. No finite subcover exists, so our interval (0,1) is therefore not compact.

Claim: The interval [0,1] *is* compact. [Note that it is bounded, and also closed.]

Proof: Suppose it were not so. Then you construct a nested set of intervals that tries to drill down onto the location in the interval where we fail to have a finite subcover for some starting infinite cover. By construction, the diameter $\rightarrow 0$ for our $[a_n, b_n]$ as we progress, so we will end up by Cantor with some point p in the interval. But it easy to cover such a point p with a single open set, just some open interval around p. So we must not have drilled down to where we have a failure. So our assumption of non-compact must have been wrong.

This proof seems a bit wobbly to me. Maybe at each choice you "fail" on both sides, so you then end up with an infinite number of points p. Then you need an infinite number of open sets to cover all these p points, and then proof fails. Also, what if you have p = an endpoint like a?

Let's go find another proof and see if we can find the missing logic. Well, I now learn that this Claim is really the famous **Heine-Borel Theorem**, usually it shows that [a,b] is compact. I have looked at various proofs, and have not found one yet that I can follow. These proofs require previously defined concepts in their respective worlds that I do not understand. So I will let this ride for now.

Claim: If a space is compact, so is every closed subset.

Claim: Every compact subset of a Hausdorf space is closed.

Claim: If X is compact and you find at least one f: $X \rightarrow Y$ that is continuous, then Y is also compact.

Claim: Compactness is a topological property.

A space is **locally compact at a** if you find an open set U containing such that U-closure is compact. If all points in your space have this property, then the space itself is **locally compact**.

Every compact space is locally compact. The real line is not compact, but it is locally compact.

STOP! I am not really learning anything useful here. I think for my purposes, I will just think of compact as being a shorthand for "closed and bounded". Royster is focusing on using a pure topological approach to compactness, maybe that is not good for me.

Metric Space version of Compact.

A set is **compact** if every sequence has a convergent subsequence which converges to something in the set.

Example 1: A finite set like { 1,2,3 } is compact. A convergence subsequence can only be a constant like 2,2,2,2,2... and obviously that is in the set. All finite sets are compact by this argument.

Example 2: The positive integers are not compact. Consider some sequence that runs to infinity. All subsequences of this sequence do also. But infinity is not in the set, so the set is not compact. The failure here is the lack of being closed!

Bolzano-Weierstrass Theorem: Consider an infinite sequence in a set A in R that is bounded. The theorem says that this sequence must contain a subsequence which converges. It does *not* say that the limit point is in the set A.

Proof: Let the bound be M, so sequence is in [-M,M]. This sequence we know has an infinite number of elements. If we divide this interval in half, could both halves have an infinite number? Perhaps, but we know for sure that at least one half must have an infinite number of elements of the sequence. So pick that half. Then keep doing this in a binary way. You eventually get down to an arbitrarily small interval which contains "an infinite number of elements". Thus, you have found a convergence point. however, it is possible that this point could be on the boundary of an open set, and so not be in the set. The sequence itself may not be converging here, but some "subsequence" of that sequence must be converging here! Perhaps the sequence has two subsequences that converge to different points, that is just fine. All we claim is that there is at least one convergent subsequence.

Note that we do not talk about the "tail" of the sequence being in the left half or the right half. This is because the sequence could jump around in some strange way. For example, if could be converging to a point, but every 10th element is some other point far away from that convergence point. We eliminate that jump point when we talk about our convergent subsequence.

Corollary of B-W: If a set in the reals is bounded and closed, then it must be compact. We know from B-W that there since bounded, there is a convergent subsequence for any sequence, and since our set is closed, that subsequence must converge to something in the set, hence compact.

Question: The B-W applies in R. What other spaces does it apply in?

Heine-Borel Theorem: Every subspace of R which is closed and bounded is compact.

This seems to be the same as our little corollary above, so I would say that

Bolzano-Weierestrass theorem $\Rightarrow$ Heine-Borel Theorem

This certainly agrees with the claim of math world.

I don't really see the connection between the two definitions of compact: the subsequence one, and the finite subcover one.

Def: A space is **sequentially compact** if every sequence has a convergent subsequence. I guess if the limit point is in the space, it is then regular compact.

Fact: A compact set A in any metric space X must be bounded. To prove this, you consider a set of "shells" which form an infinite cover of the set. If the set has no bound, these shells go out forever, and there will be no finite subcover. Well, why not just take one big sphere which goes out to infinity and call that a finite subcover. Fine, but that is not part of our assumed set of shells. Remember that EVERY cover must have a finite subcover for compactness. So we select as our cover a set of ever-larger shells.

Fact: A compact set A in any Hausdorff space X must be closed.

Comments: I think both definitions of "compact" have the "closed and bounded" flavor, so I guess it is at least not grossly surprising that you can prove one from the other.

For example, the convergent subsequence business seems to imply bounded and closed both. On the other hand, the finite subcover for any cover suggests bounded due to the shells argument above. And closed fits in as well, as the analysis of (0,1) being non-compact shows. Either being unbounded or being unclosed provide an outlet for something infinite to happen without a finite resolution, so to speak.

For example, either unbounded or unclosed gives an outlet for a sequence limit to go which is not in your subset. Similarly, either unbounded or unclosed allows for an infinite cover which has no finite subcover.

People talk about a function f: $X \rightarrow Y$ having **support**. We can think of the nullspace or kernel idea where $f(x) = 0$ for x in N(X). Then for x in $N(X)^{\perp}$ we know that $f(x) \neq 0$. This is the support. So I would say that support of a function is the region of the domain that we usually just call $N(X)^{\perp}$.

A function f(x) has **compact support** if the support in X is a compact set.