

Problem Set 2

1. (Exercise 1, page 34) Let (X, d) be a metric space and let $k > 0$ be any positive real number. Let $d_k(x, y) = k \cdot d(x, y)$ for any two points $x, y \in X$. Show that (X, d_k) is a metric space.
2. (Exercise 8, page 35) Let $\mathbb{Z}$ be the set of integers. Let p be a positive prime integer. Given distinct integers $m, n \in \mathbb{Z}$ there is a unique integer $t = t(m, n)$ such that $m - n = p^t \cdot k$, where k is an integer not divisible by p . Define a function $d: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$ by

$$d(m, n) = \begin{cases} 0 & \text{if } m = n \\ \frac{1}{p^t} & \text{if } m \neq n \end{cases}$$

Prove that $(\mathbb{Z}, d)$ is a metric space. Let $p = 3$. What is the set of elements $n \in \mathbb{Z}$ such that $d(0, n) < 1$? What is the set of elements $m \in \mathbb{Z}$ such that $d(0, m) < \frac{1}{3}$?

3. (Exercise 1, page 39) Let X be the set of continuous functions $f: [a, b] \rightarrow \mathbb{R}$. Let d^* be the metric on X given by

$$d^*(f, g) = \int_a^b |f(t) - g(t)| dt,$$

for $f, g \in X$. For each element $f \in X$ set

$$I(f) = \int_a^b f(t) dt.$$

Prove that this function $I: (X, d^*) \rightarrow (\mathbb{R}, d)$ is continuous.

4. For $P = (-2, 1)$ and $Q = (3, 4)$ in $\mathbb{R}^2$, compute the distance from P to Q in each of the following metrics:
 - (a) usual;
 - (b) taxicab;
 - (c) maximum;
 - (d) discrete.
5. Let $B = \{P = (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$. Compute the diameter of B in each of the following metrics:
 - (a) usual;
 - (b) taxicab;
 - (c) maximum;
 - (d) discrete.