

Problem Set 5

1. Let $f: X \rightarrow Y$ be a function. Then f is an **open mapping** if for each open set $O \subset X$, $f(O)$ is open in Y .
 - (a) Give an example of a mapping that is continuous, but not open.
 - (b) Give an example of a mapping that is open, but not continuous.
 - (c) Prove that a one-to-one, onto mapping $f: X \rightarrow Y$ is a homeomorphism if and only if f and f^{-1} are open mappings.
2. Prove that a finite subset A of a Hausdorff space X has no limit points. Conclude that A must be closed.
3. If X is a space which is homeomorphic to a subspace A of a space Y , then X is said to be **embedded** in Y . Give an example of spaces A and B for which A can be embedded in B and B can be embedded in A , but A and B are not homeomorphic. (Simple examples can be found in $\mathbb{R}$.)
4. Prove that every countable subset of $\mathbb{R}$ is totally disconnected.
5. Give examples of subsets A and B in $\mathbb{R}^2$ to illustrate each of the following. A drawing is sufficient.
 - (a) A and B are connected, but $A \cap B$ is disconnected.
 - (b) A and B are connected, but $A \setminus B$ is disconnected.
 - (c) A and B are disconnected, but $A \cup B$ is connected.
 - (d) A and B are connected and $\overline{A} \cap \overline{B} \neq \emptyset$, but $A \cup B$ is disconnected.
6. **Definition:** A Hausdorff space X is **0-dimensional** if X has a basis $\mathcal{B}$ of sets which are simultaneously open and closed.

Prove that every 0-dimensional space is totally disconnected.
7. Prove:
 - (a) The property of being totally disconnected is a topological invariant but not a continuous invariant.
 - (b) The property of being totally disconnected is hereditary.