

Topology
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Chapter 1

Introduction to Topology

1.1 History¹

Topology is thought of as a discipline that has emerged in the twentieth century. There are precursors of topology dating back into the 1600's. Gottfried Wilhelm Leibniz (1646–1716) was the first to foresee a geometry in which position, rather than magnitude was the most important factor. In 1676 Leibniz use the term *geometria situs* (geometry of position) in predicting the development of a type of vector calculus somewhat similar to topology as we see it today.

The first practical application of topology was made in the year 1736 by the Swiss mathematician Leonhard Euler (1707–1783) in the Königsberg Bridge Problem.

Carl F. Gauss (1777–1855) predicted in 1833 that *geometry of location* would become a mathematical discipline of great importance. His **study** of closed surfaces such as the sphere and the torus and surfaces much like those encountered in multi-dimensional calculus may be considered as a harbinger of general topology. Gauss was also interested in knots, which are of current interest today in topology.

The word **topology** was first used by the German mathematics Joseph B. Listing (1808–1882) in the title of his book *Vorstudien zur Topologie (Introductory Studies in Topology)*, a textbook published in 1847. Listing book dealt with knots and surfaces but failed to generate much interest in either the name or the subject matter. Throughout much of the nineteenth and early twentieth centuries, much of what now falls under the auspices of topology was studied under the name of *analysis situs* (analysis of position).

Bernard Riemann (1826–1866) was the first mathematician to foresee topology in the generality it has achieved today. He initiated the study of connectivity of a surface, or the arrangement of holes in a surface. He used concepts in which the number of dimensions exceeded three, which at that time was generally conceded to be the maximum number of dimensions involved with any geometric object.

Present-day topology can be traced to two primary sources: the development of non-euclidean geometry and the process of putting calculus on a firm mathematical foundation.

¹The information here is taken from *Principles of Topology* by Fred H. Croom.

1.2 Sets and Set Operations

We need some basic information about sets in order to study the logic and the axiomatic method. This is not a formal study of sets, but consists only of basic definitions and notation.

Braces $\{$ and $\}$ are used to name or enumerate sets. The *roster* method for naming sets is simply to list all of the elements of a set between a pair of braces. For example the set of integers 1, 2, 3, and 4 could be named

$$\{1, 2, 3, 4\}.$$

This does not work well for sets containing a large number of elements, though it can be used. The more common method for this is known as the *set builder notation*. A property is specified which is held by all objects in a set. $P(x)$, read P of x , will denote a sentence referring to the variable x . For example,

$$x = 23$$

x is an odd integer.

$$1 \leq x \leq 4.$$

The set of all objects x such that x satisfies $P(x)$ is denoted by

$$\{x \mid P(x)\}.$$

The set $\{1, 2, 3, 4\}$ can be named

$$\{x \mid 1 \leq x \leq 4, x \in \mathbb{Z}\} = \{x \in \mathbb{Z} \mid 1 \leq x \leq 4\}.$$

From hence forth, the words *object*, *element*, and *member* mean the same thing when referring to sets. Sets will be denoted mainly by capital Roman letters and elements of the sets by small letters. The following have the same meaning:

$$a \in A$$

a is in set A

a is a member of set A

a is an element of set A

Likewise, $a \notin A$ means that a is **not** an element of set A .

A is a *subset* of B if every element of A is also an element of B . The following have the same meaning:

$$A \subset B$$

Every element of A is an element of B

If $a \in A$, then $a \in B$

A is included in B

B contains A

A is a subset of B

Note that a set is always a subset of itself.

If A and B are sets, then we say that $A = B$ if A and B represent the same set:

$A = B$

A and B are the same set

A and B have the same members

$A \subset B$ and $B \subset A$

The set which contains no elements is known as the *empty set*, and is denoted by $\emptyset$. Note that for each set A , $\emptyset \subset A$.

The *intersection* of two sets A and B is the set of all elements common to both sets. The intersection is symbolized by $A \cap B$ or $\{x \mid x \in A \text{ and } x \in B\}$. The *union* of two sets A and B is the set of elements which are in A or B or both. The union is symbolized by $A \cup B$ or $\{x \mid x \in A \text{ or } x \in B\}$.

1.2.1 Universal Sets and Compliments

When we are working in an area or on a certain problem, we always have a frame of reference in which we are working called a *universal set*. In our geometry course, it will be the set of points that lie on a plane. In calculus we consider the set of real numbers, the set of real functions, the set of differentiable functions, and the set of continuous functions as universal sets.

The *complement* of a set A is defined to be the set of all elements of the universal set which are not in A , and is symbolized by $C A = A' = A^c$. Note that $A \cup A^c$ is always the universal set, while $A \cap A^c = \emptyset$.

The *set difference* of the sets A and B is defined to be all of those elements in A which are not in B . It is denoted by

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

Note that $A \setminus B$ and $B \setminus A$ will usually be different, and that even though $A \setminus B = \emptyset$ it need not follow that $A = B$.

1.3 Products Sets

Let X and Y be sets. The set of all ordered pairs $\{(x, y) \mid x \in X \text{ and } y \in Y\}$ is the *product set* $X \times Y$, or *Cartesian product* or *direct product* of X and Y . A *slice* of this product set is $\{x\} \times Y$ or $X \times \{y\}$ for a given $x \in X$ or $y \in Y$. Examples of common product spaces are the plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$, 3-space $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R}^2$, a right circular cylinder, $S^1 \times [0, 1]$, or the torus, $S^1 \times S^1$.

Theorem 1.1 *Let X and Y be sets and let $A, C \subset X$ and $B, D \subset Y$.*

- a) $A \times (B \cap D) = (A \cap B) \times (A \cap D)$.
- b) $A \times (B \cup D) = (A \cup B) \times (A \cup D)$.
- c) $A \times (Y \setminus D) = (A \times Y) \setminus (A \times D)$.
- d) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
- e) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.
- f) $(X \times Y) \setminus (A \times B) = (X \times (Y \setminus B)) \cup ((X \setminus A) \times Y)$

The concept of the product of two sets can be extended to more than two factors. If $\{X_i\}_{i=1}^n$ is a finite collection of sets, then their product is

$$X_1 \times X_2 \times \cdots \times X_n = \prod_{i=1}^n X_i = \{(x_1, x_2, \dots, x_n) \mid x_i \in X_i \text{ for each } i = 1, 2, \dots, n\}.$$

For an infinite collection of sets, the product is defined by

$$\prod_{i=1}^{\infty} X_i = \{(x_1, x_2, x_3, \dots) \mid x_i \in X_i \text{ for each } i = 1, 2, \dots\}.$$

1.4 Functions

A *function* $f: X \rightarrow Y$ is a rule which assigns to each $x \in X$ a unique $y \in Y$ and we say $y = f(x)$. If $y = f(x)$ then y is called the *image* of x and x is called the *preimage* of y . The set X is the *domain* of f and Y is the *range* or *codomain* of f .

Let $A \subset X$. The set $f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}$ is called the *image* of A . The set $f(X)$ is called the *image of f* . For $B \subset Y$, the set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

is the *inverse image* of B under f . The set of points

$$\Gamma = \{(x, f(x)) \in X \times Y \mid x \in X\}$$

is called the *graph* of the function f .

A function $f: X \rightarrow Y$ is *injective* if for distinct elements $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$ in Y . Another way to think of this is to say that f is injective if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

If $f(X) = Y$, the function f is said to be *surjective*.

A function that is surjective and injective is called a *bijection*. In this case we have that $f: X \rightarrow Y$ is a bijection provided that each member of Y is the image under f of exactly one member of X . In this case the *inverse function* $f^{-1}: Y \rightarrow X$ exists assigning to each element $y \in Y$ its unique preimage $x = f^{-1}(y)$ in X .

The *identity function* $i_X: X \rightarrow X$ from a set X to itself is the function defined by $i_X(x) = x$ for all $x \in X$. This function is often denoted by 1_X .

If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are functions, then the *composite function* $g \circ f: X \rightarrow Z$ is defined by $g \circ f(x) = g(f(x))$, for $x \in X$.

Definition 1.1 Let X be a set. A **sequence** in X is a function $f: \mathbb{Z}^+ \rightarrow X$ whose domain is the set of all positive integers, $\mathbb{Z}^+$ or the set of positive integers less than or equal to some given positive integer N . The sequence is called *finite* if its domain is $\{1, 2, \dots, N\}$ and *infinite* if its domain is all positive integers.

1.5 Equivalence Relations

Let X be a set. A *relation* R on X is a subset of $X \times X$. If $(x, y) \in R$ we will say that x is related to y by R and to write xRy .

A relation R on a set X is called *reflexive*, *symmetric*, or *transitive* if it satisfies the corresponding property below.

- (a) *The Reflexive Property:* xRx for all $x \in X$.
- (b) *The Symmetric Property:* If xRy , then yRx .
- (c) *The Transitive Property:* If xRy and yRz , then xRz .

Definition 1.2 An **equivalence relation** on a set X is a relation on X which is reflexive, symmetric, and transitive.

Definition 1.3 Let $\sim$ denote an equivalence relation on X . For $x \in X$ the set $[x] = \{y \in X \mid y \sim x\}$ is **called the equivalence class** of x .

Proposition 1.1 Let X be a set and let $\sim$ denote an equivalence relation on X .

- a) $x \in [x]$ for each $x \in X$.
- b) $x \sim y$ if and only if $[x] = [y]$.
- c) $x \not\sim y$ if and only if $[x] \cap [y] = \emptyset$.
- d) For $x, y \in X$, $[x]$ and $[y]$ are either identical or disjoint.

1.6 Cardinality

We are often interested in how big sets are in relation to one another. Clearly, we can tell the difference in sizes of two finite sets, but how do we differentiate between two infinite sets? Are there different sizes of infinite sets? How do we compare sets to tell if one has a greater number of members?

Definition 1.4 *a set A is **finite** if A is empty or if there is a bijection between A and the set of integers from 1 to N for some positive integer N . In the latter case, A is said to have N members. If a set is not finite, it is called **infinite**.*

Definition 1.5 *A set is **denumerable** or **countably infinite** if there is a bijection between the set and the positive integers. A set which is either finite or denumerable is called **countable**. A set which is not countable is called **uncountable**.*

Lemma 1.1 a) *Each subset of a finite set is finite.*

b) *Each subset of a countable set is countable.*

c) *Each set which contains an infinite set is infinite.*

d) *Each set which contains an uncountable set is uncountable.*

Example 1.6.1 1. The set $\mathbb{Z}^+ \cup \{0\}$ of all non-negative integers is countable. The bijection is $f: \mathbb{Z}^+ \cup \{0\} \rightarrow \mathbb{Z}^+$ given by

$$f(n) = n + 1, n \in \mathbb{Z}^+ \cup \{0\}.$$

2. The set of all integers, $\mathbb{Z}$ is countable. The bijection $g: \mathbb{Z} \rightarrow \mathbb{Z}^+$ is given by

$$g(n) = \begin{cases} 2n - 1 & \text{if } n \text{ is positive} \\ -2n & \text{if } n \text{ is negative} \end{cases}$$

3. The product set $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable. One method is to use the Cantor Diagonalization Method to count the ordered pairs (m, n) . A second method is to define the function $g: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ by

$$g(m, n) = 2^m 3^n, (m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+.$$

Now, g is not surjective, but the Fundamental Theorem of Arithmetic on the unique factorization into primes guarantees that the function g is injective. Thus, there is a bijection from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to a subset of $\mathbb{Z}^+$. Since every subset of a countable set is countable, we have that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Theorem 1.2 a) If $\{A_i\}_{i=1}^N$ is a finite collection of finite sets, then both $\bigcup_{i=1}^N A_i$ and $\prod_{i=1}^N A_i$ are finite.

(Finite unions and finite products of finite sets are finite.)

b) If $\{A_i\}_{i=1}^\infty$ is a countable collection of countable sets, then $\bigcup_{i=1}^\infty A_i$ is countable.

(Countable unions of countable sets are countable.)

c) If $\{A_i\}_{i=1}^N$ is a finite collection of countable sets, then both $\prod_{i=1}^N A_i$ is countable.

(Finite products of countable sets are countable.)

Why didn't we claim that a countable product of countable sets is countable? Mainly because it is not true, as is seen in the following example.

Example 1.6.2 Let $A_i = \{0, 1\}$ for $i = 1, 2, \dots$. Let

$$U = \prod_{i=1}^{\infty} A_i = \{(a_1, a_2, a_3, \dots) \mid a_i = 0 \text{ or } 1\}.$$

Assume that U is countable. Then there is a bijection $f: \mathbb{Z}^+ \rightarrow U$. For an element $a = (a_1, a_2, a_3, \dots) \in U$ we shall refer to a_1 as the first coordinate, a_2 as the second coordinate, and so forth.

We can list all of the elements in U using the bijection f . They are $\{f(1), f(2), f(3), \dots\}$. Consider the following element in U . Define $x = (x_1, x_2, x_3, \dots)$ as follows:

$$x_i = \begin{cases} 0 & \text{if the } i^{\text{th}} \text{ coordinate of } f(i) \text{ is } 1 \\ 1 & \text{if the } i^{\text{th}} \text{ coordinate of } f(i) \text{ is } 0 \end{cases}$$

Then, we have that for each positive integer i , $x \neq f(i)$ for they differ in the i^{th} coordinate. This means that x is not in the exhaustive list of elements we have in our bijection. That is this bijection is not surjective. This contradiction show us that U cannot be countable.

Theorem 1.3 The set of rational numbers is countable.

PROOF: There are several ways of proving this. One method is to use the Cantor Diagonalization Method to count the rationals.

(METHOD 1) List the positive rational numbers in rows where the first row consists of all positive rational numbers with 1 as a denominator, the second row lists all positive rational numbers with 2 as a denominator, and so on:

$$\begin{array}{ccccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots & & \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots & & \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots & & \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \end{array}$$

Clearly, we have each positive rational number in here numerous times, but the diagonalization method will still show that there are a countable number of elements in this array. The positive rationals form a subset of this array, thus there must be a countable number of positive rationals. This will yield that there are a countable number of rational numbers.

(METHOD 2) Every rational number can be expressed uniquely in lowest terms as m/n where m and n are integers with no common positive divisor other than 1, and n is positive. Consider the function $m/n \mapsto (m, n)$ from the set of rational numbers into $\mathbb{Z} \times \mathbb{Z}$. This function is injective since the ordered pair (m, n) determines only one rational number m/n . Thus, the set of rational numbers is equivalent to a subset of the countable set $\mathbb{Z} \times \mathbb{Z}$, and is hence countable. ■

Theorem 1.4 *The set of real numbers is uncountable.*

PROOF: We will make use of the example of the countable product above. Each element in $\prod_{i=1}^{\infty} 0, 1$ is a sequence consisting of 0's and 1's. Each of these sequences represents a unique real number between 0 and 1, by the correspondence

$$(a_1, a_2, a_3, \dots) \mapsto 0.a_1a_2a_3\dots$$

This is a one-to-one correspondence. Thus, the set of real numbers between 0 and 1 representable as a decimal using only 0's and 1's is an uncountable set. Thus, $\mathbb{R}$ contains an uncountable set and hence is uncountable. ■

Theorem 1.5 *The set of irrational numbers is uncountable.*

PROOF: Since the set of real numbers is the union of the set of rational numbers and the set of irrational numbers, if the set of irrationals were countable, then we would have that the real numbers are countable. That failing to be true, implies that the irrationals must be uncountable. ■

Chapter 2

Metric Spaces

2.1 Definition and Some Examples

Definition 2.1 Let X be a set and $d: X \times X \rightarrow \mathbb{R}^+$ a function satisfying the following properties. For all $x, y, z \in X$,

a) $d(x, y) = 0$ if and only if $x = y$.

b) $d(x, y) = d(y, x)$.

c) $d(x, z) \leq d(x, y) + d(y, z)$.

Then d is called a **metric** or **distance function** on X and $d(x, y)$ is called the **distance** from x to y . The set X with a metric d is called a **metric space** and is denoted by (X, d) .

Note that these properties are modeled on the distance functions that we have on $\mathbb{R}$ and $\mathbb{R}^2$. Doing so we usually call property (c) the *Triangle Inequality*.

Example 2.1.1 The real line, $\mathbb{R}$ is a metric space using the standard distance function, the absolute value: $d(a, b) = |a - b|$. The above properties are standard proofs about the absolute value function.

Example 2.1.2 The plane, $\mathbb{R}^2$, with the usual Euclidean distance formula is a metric space. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 2.1.3 These are special cases of the general Euclidean n -space,

$$\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{R}\}.$$

The distance formula here is the usual distance formula for Euclidean n -space:

$$d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}.$$

d is called the *usual metric* on $\mathbb{R}^n$.

To show that d is a metric, we need two standard results about vectors in $\mathbb{R}^n$. First, let $a \in \mathbb{R}^n$. The *norm* $\|a\|$ is the distance from a to the origin $O = (0, 0, \dots, 0)$:

$$\|a\| = d(a, O) = \left(\sum_{i=1}^n a_i^2 \right)^{1/2}.$$

Theorem 2.1 (Cauchy-Schwarz Inequality) For any points $a, b \in \mathbb{R}^n$

$$|a \cdot b| \leq \|a\| \|b\|.$$

Theorem 2.2 (The Minkowski Inequality) For any points $a, b \in \mathbb{R}^n$

$$\|a + b\| \leq \|a\| + \|b\|.$$

The distance between two points is given by $d(a, b) = \|a - b\|$.

The first two conditions making d a metric are easily seen to be satisfied. We only need check the Triangle Inequality. Let $x, y, z \in \mathbb{R}^n$

$$\begin{aligned} d(x, z) &= \|x - z\| = \|x - y + y - z\| \\ &\leq \|x - y\| + \|y - z\| \\ &= d(x, y) + d(y, z) \end{aligned}$$

Example 2.1.4 [The Taxicab Metric] Define a function $d': \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ as follows. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then

$$d'(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

This is called the taxicab metric because the distance is measured along line segments parallel to the coordinate axes.

Clearly, $d'(x, x) = 0$ and if $d'(x, y) = 0$, then $|x_1 - y_1| + |x_2 - y_2| = 0$ which means $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$. This implies that $x_1 = y_1$ and $x_2 = y_2$, and $x = y$. Because of the basic properties of the absolute value, it is obvious that $d'(x, y) = d'(y, x)$. The Triangle Inequality follows because of the validity of the Triangle Inequality with the absolute value on the real line.

What is the following set?

$$U = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid d'(x, O) = 1\}.$$

We can define an analogous metric, called the taxicab metric, on $\mathbb{R}^n$.

$$d'(x, y) = \sum_{i=1}^n |x_i - y_i|.$$

Example 2.1.5 [The Max Metric on $\mathbb{R}^n$] Another metric for $\mathbb{R}^n$ is given by taking the largest of the differences of the coordinates of x and y .

$$d''(x, y) = \max\{|x_i - y_i|\}_{i=1}^n.$$

Example 2.1.6 [The Discrete Metric] For any set X , define

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

This defines a metric on X , called the *discrete metric*. It is usually of little use, except for counterexamples. It does show, though, that every set can be assigned a metric.

Example 2.1.7 Let $\mathcal{C}[a, b]$ denote the set of all continuous real-valued functions defined on the interval $[a, b]$. For $f, g \in \mathcal{C}[a, b]$ define

$$\rho(f, g) = \int_a^b |f(x) - g(x)| dx.$$

The fact that ρ is a metric follows from the usual properties of the Riemann integral. This metric measures the distance between two functions to be the area between the two graphs from $x = a$ to $x = b$.

Example 2.1.8 For the set $\mathcal{C}[a, b]$ define ρ' by

$$\rho'(f, g) = \text{lub}\{|f(x) - g(x)| \mid x \in [a, b]\}.$$

The metric is called the *supremum metric* or the *uniform metric* for $\mathcal{C}[a, b]$. It measures the distance between f and g to be the supremum of the vertical distances from points $(x, f(x))$ to $(x, g(x))$ on the graphs of f and g on the closed interval $[a, b]$.

Definition 2.2 A number u is an **upper bound** for a set A of real numbers provided that $a \leq u$ for all $a \in A$. If there is a smallest upper bound u_0 for A , that is an upper bound that is less than or equal to all other upper bounds for A , then u_0 is called the **least upper bound or supremum** of A . The least upper bound for a set A is denoted by $\text{lub } A$ or $\sup A$.

Definition 2.3 A number ℓ is an **lower bound** for a set A of real numbers provided that $\ell \leq a$ for all $a \in A$. If there is a largest lower bound ℓ_0 for A , that is a lower bound that is less than or equal to all other lower bounds for A , then ℓ_0 is called the **greatest lower bound or infimum** of A . The greatest lower bound for a set A is denoted by $\text{glb } A$ or $\inf A$.

A very basic property of the real numbers is included in the following two statements:

The Least Upper Bound Property: Every non-empty set of real numbers which has an upper bound has a least upper bound.

The Greatest Lower Bound Property: Every non-empty set of real numbers which has a lower bound has a greatest lower bound.

We will accept the first property as an axiom of the real number system. The second property follows from the first.

Definition 2.4 *Let (X, d) be a metric space and let A be a non-empty subset of X . If $\{d(x, y) \mid x, y \in A\}$ has an upper bound, then A is said to be **bounded**, and $\text{lub}\{d(x, y) \mid x, y \in A\}$ is called the **diameter** of A . For completeness, we define the diameter of the empty set to be 0. If the set X is bounded, then we call (X, d) a bounded metric space.*

*If $x \in X$, then the **distance** from x to A is defined by*

$$d(x, A) = \text{glb}\{d(x, y) \mid y \in A\}.$$