

Chapter 2

Metric Spaces

2.1 Definition and Some Examples

Definition 2.1 Let X be a set and $d: X \times X \rightarrow \mathbb{R}^+$ a function satisfying the following properties. For all $x, y, z \in X$,

a) $d(x, y) = 0$ if and only if $x = y$.

b) $d(x, y) = d(y, x)$.

c) $d(x, z) \leq d(x, y) + d(y, z)$.

Then d is called a **metric** or **distance function** on X and $d(x, y)$ is called the **distance** from x to y . The set X with a metric d is called a **metric space** and is denoted by (X, d) .

Note that these properties are modeled on the distance functions that we have on $\mathbb{R}$ and $\mathbb{R}^2$. Doing so we usually call property (c) the *Triangle Inequality*.

Example 2.1.1 The real line, $\mathbb{R}$ is a metric space using the standard distance function, the absolute value: $d(a, b) = |a - b|$. The above properties are standard proofs about the absolute value function.

Example 2.1.2 The plane, $\mathbb{R}^2$, with the usual Euclidean distance formula is a metric space. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 2.1.3 These are special cases of the general Euclidean n -space,

$$\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{R}\}.$$

The distance formula here is the usual distance formula for Euclidean n -space:

$$d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}.$$

d is called the *usual metric* on $\mathbb{R}^n$.

To show that d is a metric, we need two standard results about vectors in $\mathbb{R}^n$. First, let $a \in \mathbb{R}^n$. The *norm* $\|a\|$ is the distance from a to the origin $O = (0, 0, \dots, 0)$:

$$\|a\| = d(a, O) = \left(\sum_{i=1}^n a_i^2 \right)^{1/2}.$$

Theorem 2.1 (Cauchy-Schwarz Inequality) For any points $a, b \in \mathbb{R}^n$

$$|a \cdot b| \leq \|a\| \|b\|.$$

Theorem 2.2 (The Minkowski Inequality) For any points $a, b \in \mathbb{R}^n$

$$\|a + b\| \leq \|a\| + \|b\|.$$

The distance between two points is given by $d(a, b) = \|a - b\|$.

The first two conditions making d a metric are easily seen to be satisfied. We only need check the Triangle Inequality. Let $x, y, z \in \mathbb{R}^n$

$$\begin{aligned} d(x, z) &= \|x - z\| = \|x - y + y - z\| \\ &\leq \|x - y\| + \|y - z\| \\ &= d(x, y) + d(y, z) \end{aligned}$$

Example 2.1.4 [The Taxicab Metric] Define a function $d': \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ as follows. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then

$$d'(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

This is called the taxicab metric because the distance is measured along line segments parallel to the coordinate axes.

Clearly, $d'(x, x) = 0$ and if $d'(x, y) = 0$, then $|x_1 - y_1| + |x_2 - y_2| = 0$ which means $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$. This implies that $x_1 = y_1$ and $x_2 = y_2$, and $x = y$. Because of the basic properties of the absolute value, it is obvious that $d'(x, y) = d'(y, x)$. The Triangle Inequality follows because of the validity of the Triangle Inequality with the absolute value on the real line.

What is the following set?

$$U = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid d'(x, O) = 1\}.$$

We can define an analogous metric, called the taxicab metric, on $\mathbb{R}^n$.

$$d'(x, y) = \sum_{i=1}^n |x_i - y_i|.$$

Example 2.1.5 [The Max Metric on $\mathbb{R}^n$] Another metric for $\mathbb{R}^n$ is given by taking the largest of the differences of the coordinates of x and y .

$$d''(x, y) = \max\{|x_i - y_i|\}_{i=1}^n.$$

Example 2.1.6 [The Discrete Metric] For any set X , define

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

This defines a metric on X , called the *discrete metric*. It is usually of little use, except for counterexamples. It does show, though, that every set can be assigned a metric.

Example 2.1.7 Let $\mathcal{C}[a, b]$ denote the set of all continuous real-valued functions defined on the interval $[a, b]$. For $f, g \in \mathcal{C}[a, b]$ define

$$\rho(f, g) = \int_a^b |f(x) - g(x)| dx.$$

The fact that ρ is a metric follows from the usual properties of the Riemann integral. This metric measures the distance between two functions to be the area between the two graphs from $x = a$ to $x = b$.

Example 2.1.8 For the set $\mathcal{C}[a, b]$ define ρ' by

$$\rho'(f, g) = \text{lub}\{|f(x) - g(x)| \mid x \in [a, b]\}.$$

The metric is called the *supremum metric* or the *uniform metric* for $\mathcal{C}[a, b]$. It measures the distance between f and g to be the supremum of the vertical distances from points $(x, f(x))$ to $(x, g(x))$ on the graphs of f and g on the closed interval $[a, b]$.

Definition 2.2 A number u is an **upper bound** for a set A of real numbers provided that $a \leq u$ for all $a \in A$. If there is a smallest upper bound u_0 for A , that is an upper bound that is less than or equal to all other upper bounds for A , then u_0 is called the **least upper bound or supremum** of A . The least upper bound for a set A is denoted by $\text{lub } A$ or $\sup A$.

Definition 2.3 A number ℓ is an **lower bound** for a set A of real numbers provided that $\ell \leq a$ for all $a \in A$. If there is a largest lower bound ℓ_0 for A , that is a lower bound that is less than or equal to all other lower bounds for A , then ℓ_0 is called the **greatest lower bound or infimum** of A . The greatest lower bound for a set A is denoted by $\text{glb } A$ or $\inf A$.

A very basic property of the real numbers is included in the following two statements:

The Least Upper Bound Property: Every non-empty set of real numbers which has an upper bound has a least upper bound.

The Greatest Lower Bound Property: Every non-empty set of real numbers which has a lower bound has a greatest lower bound.

We will accept the first property as an axiom of the real number system. The second property follows from the first.

Definition 2.4 Let (X, d) be a metric space and let A be a non-empty subset of X . If $\{d(x, y) \mid x, y \in A\}$ has an upper bound, then A is said to be **bounded**, and $\text{lub}\{d(x, y) \mid x, y \in A\}$ is called the **diameter** of A . For completeness, we define the diameter of the empty set to be 0. If the set X is bounded, then we call (X, d) a **bounded metric space**.

If $x \in X$, then the **distance** from x to A is defined by

$$d(x, A) = \text{glb}\{d(x, y) \mid y \in A\}.$$

Theorem 2.3 Let $\{(X_i, d_i)\}_{i=1}^n$ be a finite collection of metric spaces and let

$$X = \prod_{i=1}^n X_i.$$

For each pair of points $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$ in X , let $d: X \times X \rightarrow \mathbf{R}$ be defined by

$$d(x, y) = \left(\sum_{i=1}^n (d_i(x_i, y_i))^2 \right)^{1/2}.$$

Then (X, d) is a metric space. The metric d defined above is called the **product metric** on X .

2.2 Continuous Functions

In topology we are concerned with how spaces are changed when stretched, bent, twisted and modified — but not torn. We do so by studying the maps that do so. Our friend here is the continuous map. In your study of calculus, you saw that continuous functions did many things. At the time you were more interested in special continuous functions — the differential functions. We here are more interested in the more general function.

In calculus, we saw that a continuous function was one that did not do too much damage to the domain in the range. By this, we mean that if two points were close in the domain, then their images were not too far apart in the image. We saw this intuitively through looking at graphs and looking at limits. To insure specificity, we need the definition of continuity due to Cauchy and Weierstrauss. It is one with which you are familiar.

Definition 2.5 Let $f: (X, d) \rightarrow (Y, d')$ be a function between two metric spaces. Let $a \in X$. We say that f is continuous at a if given any $\epsilon > 0$ there is a $\delta > 0$ so that $d'(f(x), f(a)) < \epsilon$ whenever $d(x, a) < \delta$. We say that f is continuous if it is continuous at $a \in X$ for all $a \in X$.

This clearly depends on the metric in each of the two spaces. A change of metric *might* change the continuity of the given function. Will it? Is continuity that dependent on the metric in the domain or the range?

Let's check two well-known functions that we think should be continuous and make certain that they are continuous under this definition.

Example 2.2.1 Let $f: (X, d) \rightarrow (Y, d')$ be given by $f(x) = b$ for all $x \in X$ where $b \in Y$ is a constant. This is just the *constant function*.

To show that f is continuous, we need to show that if we are given any $\epsilon > 0$, then we can find a $\delta > 0$ so that whenever $d(x_1, x_2) < \delta$ then $d'(f(x_1), f(x_2)) < \epsilon$. In this case, this is easy. This is because $d'(f(x_1), f(x_2)) = d'(b, b) = 0 < \epsilon$ for any choice of $x_1, x_2 \in X$. Thus, it does not matter what we may choose for δ . You could take $\delta = \epsilon$ or $\delta = 1$. Regardless, whenever $d(x_1, x_2) < \delta$ then $d'(f(x_1), f(x_2)) = 0 < \epsilon$, and we are done.

Example 2.2.2 Let $1_X: (X, d) \rightarrow (X, d)$ denote the identity map from X to itself given by $1_X(x) = x$. We claim that this function is continuous.

Again, to show this we are given an $\epsilon > 0$. We then need to find a $\delta > 0$ so that whenever $d(x_1, x_2) < \delta$ then $d(1_X(x_1), 1_X(x_2)) < \epsilon$. However, since $1_X(x_1) = x_1$ and $1_X(x_2) = x_2$, it is easy to see that if we take $\delta \leq \epsilon$, then if $d(x_1, x_2) < \delta$ it follows that $d(1_X(x_1), 1_X(x_2)) = d(x_1, x_2) < \delta \leq \epsilon$. Thus, 1_X is a continuous function.

Example 2.2.3 This time we will be working with the same underlying set, but we will place a different metric on it. Will this make a difference?

Let $X = \mathbf{R}^n$ with the usual metric. Let $Y = \mathbf{R}^n$ with the maximum metric,

$$d''((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \max_{1 \leq i \leq n} \{|x_i - y_i|\}.$$

Define $h: (X, d) \rightarrow (Y, d'')$ by $h(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n)$. It is the identity map on the underlying set, but it does not carry the same metric information. Is h continuous? Is h^{-1} continuous?

It turns out that both are continuous! To prove this, let's first look at $h^{-1}: (Y, d'') \rightarrow (X, d)$. We are given an $\epsilon > 0$. We need to find a $\delta > 0$ so that if

$$d''((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) < \delta$$

then $d(h^{-1}(x_1, x_2, \dots, x_n), h^{-1}(y_1, y_2, \dots, y_n)) < \epsilon$.

To say that $d''((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) < \delta$ means that $|x_i - y_i| < \delta$ for all $i = 1, \dots, n$. Thus,

$$d(h^{-1}(x_1, x_2, \dots, x_n), h^{-1}(y_1, y_2, \dots, y_n)) = d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) \quad (2.1)$$

$$= \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2} \quad (2.2)$$

$$< \left(\sum_{i=1}^n \delta^2 \right)^{1/2} = \delta\sqrt{n} \quad (2.3)$$

Thus, we need $\delta\sqrt{n} < \epsilon$, or take $\delta < \frac{\epsilon}{\sqrt{n}}$.

Now, to show that h is continuous we are given an $\epsilon > 0$. We need to find δ so that whenever $d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) < \delta$, we have that

$$d''((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) < \epsilon.$$

To say that $d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) < \delta$ means that $(\sum_{i=1}^n |x_i - y_i|^2)^{1/2} < \delta$. Thus, each of the differences $|x_i - y_i|$ must be less than δ , and the largest of these differences is still less than δ . Thus, in order for $d''((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) < \epsilon$, we need only choose $\delta < \epsilon$.

2.3 Open Sets and Closed Sets

Definition 2.6 Let (X, d) be a metric space, $a \in X$, and $r > 0$ a positive real number. The **open ball** $B_d(a; r)$ with **center** a and **radius** r is the set

$$B_d(a; r) = \{x \in X \mid d(a, x) < r\}.$$

When there is only one metric under consideration, we will simplify the notation to $B(a; r)$.

Definition 2.7 Let (X, d) be a metric space and let $a \in X$. A subset $N \subset X$ is a **neighborhood of a** if there is a $\delta > 0$ so that $B(a; \delta) \subset N$. The collection $\mathfrak{N}_a$ of all neighborhoods of a point $a \in X$ is called a **complete system of neighborhoods of the point a** .

Definition 2.8 A subset U of a metric space (X, d) is an **open set** with respect to the metric d provided that U is a union of open balls. The family of all open sets defined in this way is called the **topology for X generated by d** . A subset $C \subset X$ is said to be **closed** (with respect to d) if its complement $X \setminus C$ is an open set (with respect to d).

Thus, a neighborhood of a and an open set containing a need not be the same thing. However, if U is an open set containing a , then U is a neighborhood of a .

Theorem 2.4 *The following statements are equivalent (TFAE) for a subset U of a metric space (X, d) .*

- a) U is an open set;
- b) for each $x \in U$ there is an $\epsilon_x > 0$ so that $B(x; \epsilon_x) \subset U$.
- c) for each $x \in U$, $d(x, X \setminus U) > 0$, if $U \neq X$.

PROOF: What this means is that Statements (a) and (b) are equivalent, (b) and (c) are equivalent, and (a) and (c) are equivalent. We can show this by proving that (a) is equivalent to (b) and then that (b) is equivalent to (c). In condition (c) we will assume that $U \neq X$ since the distance from the empty set is not defined.

Assume that U is an open set and let $x \in U$. Since U is the union of open balls, then $x \in B(a; r) \subset U$. Then $d(x, a) < r$. We want to center an open ball at x and have it contained in U . Choose $\epsilon_x \leq r - d(x, a)$. Then $B(x; \epsilon_x) \subset B(a; r)$ for the following reason: If $y \in B(x; \epsilon_x)$,

$$d(y, a) \leq d(y, x) + d(x, a) < \epsilon_x + d(x, a) \leq r - d(x, a) + d(x, a) = r.$$

Thus, $B(x; \epsilon_x)$ is an open ball of positive radius centered at x and contained in U . Thus (a) $\implies$ (b).

To show that (b) $\implies$ (a), since each $x \in U$ lies in an open ball contained in U , U is the union of these open balls.

To see that (b) $\implies$ (c), let $B(x; \epsilon_x) \subset U$. Then any point within distance ϵ_x of x is in U , so the distance from x to $X \setminus U$ must be at least ϵ_x . Thus, $d(x, X \setminus U) > 0$ for each $x \in U$.

Assuming that (c) holds, $d(x, X \setminus U) = \alpha_x > 0$ depending on x . This means that the distance from x to a point outside U must be at least α_x , so any point within distance α_x of x must be in U . This means $B(x; \alpha_x) \subset U$. ■

Note that we have just shown that for each $a \in X$ and for each $\delta > 0$, the open ball $B(a; \delta)$ is a neighborhood of each of its points.

2.3.1 Neighborhoods and Continuous Functions

How do we plan to use this information? While our definition of continuity is precise, it requires some specificity and does not look generalizable. What I mean by this is that the definition seems to rely specifically on the definition of the metric, and it will be hard to realign our definition when we have to move away from metric spaces.

Theorem 2.5 *Let $f: (X, d) \rightarrow (Y, d')$. f is continuous at $a \in X$ if and only if for each neighborhood M of $f(a)$ there is a corresponding neighborhood N of a , such that*

$$f(N) \subset M,$$

or equivalently

$$N \subset f^{-1}(M).$$

PROOF: First, let's suppose that f is continuous at $a \in X$ and let M be a neighborhood of $f(a)$. This means that for some $\epsilon > 0$ $B_{d'}(f(a); \epsilon) \subset M$. Since f is continuous at $a \in X$ we know that we can find $\delta > 0$ so that if $d(x, a) < \delta$ then $d'(f(a), f(x)) < \epsilon$. This says that $B_d(a; \delta) \subset f^{-1}(M)$ and we already have seen that $B_d(a; \delta)$ is a neighborhood of $a \in X$. Thus, if f is continuous, we have found a corresponding neighborhood to M .

Now, suppose that for any neighborhood, M , of $f(a)$ we can find a neighborhood N of a so that $f(N) \subset M$. Let $\epsilon > 0$ be given to you. You must find a $\delta > 0$ so that whenever $d(a, x) < \delta$ we have $d'(f(a), f(x)) < \epsilon$. Now, let $M = B_{d'}(f(a); \epsilon)$. M is a neighborhood of $f(a)$, so we know that there is a neighborhood $N \subset X$ of a so that $f(N) \subset M$. Since N is a neighborhood of a , it must contain a δ -ball centered at a , $B_d(a; \delta) \subset N$, by the definition of a neighborhood. Thus, if we have $x \in B_d(a; \delta)$, then $f(x) \in M = B_{d'}(f(a); \epsilon)$. The other way of writing this is: if $d(a, x) < \delta$ then $d'(f(a), f(x)) < \epsilon$. Therefore, f is continuous at $a \in X$. ■

2.4 Limits

Recall that a sequence is just a function $a: \mathbb{Z}^+ \rightarrow (X, d)$. We want to discuss what happens to the sequence as we let n go to infinity; in other words what happens to the sequence as we look further and further into the range of a . Let us first recall the definitions in the real numbers and then try to set them up so that we can easily generalize them to arbitrary metric spaces.

Let $\{a_i\}$ be a sequence of real numbers. A real number L is said to be the *limit of the sequence* $\{a_n\}$ if, given any $\epsilon > 0$, there is a positive integer N such that whenever $n > N$, $|a_n - L| < \epsilon$. In this case we say that the sequence *converges to* L and write

$$\lim_{n \rightarrow \infty} a_n = L.$$

How can we generalize this to an arbitrary metric space? It should not be hard, because all we used in the definition was the distance function in the real numbers. We will use the distance function in our metric space similarly.

Definition 2.9 Let $\{x_n\}$ be a sequence in the metric space (X, d) . We say that this sequence $\{x_n\}$ converges to $x \in X$ if given any $\epsilon > 0$ there is a positive $N \in \mathbb{Z}^+$ so that whenever $n > N$, $d(x, x_n) < \epsilon$. In this case we will write $\lim x_n = x$.

Lemma 2.1 Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X . Then $\lim x_n = x \in X$ if and only if for each neighborhood V of x there is an integer $N > 0$ so that $x_n \in V$ whenever $n > N$.

This is nothing but applying the definitions of convergence and neighborhood, and its proof will be omitted.

If S is a set of infinite points and **there is at most a finite number** of elements of S for which a certain statement is false, then the statement is said to be true for *almost all* of S . Thus, we may phrase the above lemma by saying that the sequence $\{x_n\}$ converges to x if each neighborhood of x contains almost all of the points of the sequence.

One reason for looking at sequences is the concept of continuity. In calculus we define a function to be continuous at $a \in \mathbb{R}$ if the following conditions were met:

1. $\lim_{x \rightarrow a} f(x)$ exists;
2. $f(a)$ exists;
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

It suffices to check this for all sequences approaching a (a fact to be proven later). Thus, we find that we can show that f is continuous at a if for each sequence $\{x_n\} \rightarrow a$, we have that $\{f(x_n)\} \rightarrow f(a)$.

We are able to extend this result to arbitrary metric spaces.

Theorem 2.6 *Let $f: (X, d) \rightarrow (Y, d')$. f is continuous at a point $a \in X$ if and only if whenever $\lim x_n = a$ we have $\lim f(x_n) = f(a)$.*

The proof is straightforward.

PROOF: Assume that f is continuous and let $\{x_n\} \rightarrow x$ in X . Let $\epsilon > 0$ and let $M = B_{d'}(f(x); \epsilon) \in Y$. There is a neighborhood U of x in X , such that $f(U) \subset M$. Since U is a neighborhood there is a $\delta > 0$ so that $B_d(x; \delta) \subset U$. Now, $\{x_n\} \rightarrow x$ so for this δ there is a positive integer N so that whenever $n > N$ we have $x_n \in B_d(x; \delta) \subset U$. Thus, $f(x_n) \in f(U) \subset M$. Therefore, for any neighborhood M of $f(x)$ there is a positive integer N so that whenever $n > N$ we have $d'(f(x), f(x_n)) < \epsilon$, which implies that the sequence $\{f(x_n)\}$ converges to $f(x)$. ■

To prove the other direction,

2.5 Open Sets and Closed Sets Revisited

Remember that we defined an open set as a set that is the union of open balls. A set is closed if its complement is open.

Theorem 2.7 *The open subsets of a metric space (X, d) have the following properties:*

1. X and $\emptyset$ are open sets.
2. The union of any family of open sets is open.

3. *The intersection of a finite family of open sets is open.*

PROOF: These are straightforward.

1. The whole space X is open since it is the union of all open balls with all possible centers and radii. The empty set is open since it is the union of the empty collection of open balls.
2. If $\{U_\alpha \mid \alpha \in A\}$ is a family of open sets in X , then each U_α is a union of open balls. Then $\bigcup_{\alpha \in A} U_\alpha$ is the union of all of the open balls that comprise each U_α and is hence open.
3. Let $\{U_i \mid i = 1, \dots, n\}$ be a finite collection of open sets and let $x \in \bigcap_{i=1}^n U_i$. Then, by our previous theorem there exist $\epsilon_i, i = 1, \dots, n$ so that $B_d(x; \epsilon_i) \subset U_i$ and

$$\bigcap_{i=1}^n B_d(x; \epsilon_i) \subset \bigcap_{i=1}^n U_i.$$

Let $\epsilon = \min\{\epsilon_i \mid i = 1, \dots, n\}$. Then, $\bigcap_{i=1}^n B_d(x; \epsilon_i) = B_d(x; \epsilon)$. Thus, $B_d(x; \epsilon)$ is an open ball centered at x and contained in $\bigcap_{i=1}^n U_i$. Thus, $\bigcap_{i=1}^n U_i$ is open. ■

Theorem 2.8 *The closed subsets of a metric space (X, d) have the following properties:*

1. X and $\emptyset$ are closed sets.
2. The intersection of any family of closed sets is closed.
3. The union of a finite family of closed sets is closed.

This follows from our previous theorem and complements.

Definition 2.10 *Let (X, d) be a metric space and A a subset of X . A point $x \in X$ is a **limit point** or **accumulation point** of A provided that every open set containing x contains a point of A distinct from x . The set of limit points of A is called its **derived set**, denoted by A' .*

Lemma 2.2 *Let (X, d) be a metric space and A a subset of X . A point $x \in X$ is a limit point of A if and only if $d(x, A \setminus \{x\}) = 0$.*

Lemma 2.3 *A subset A of a metric space (X, d) is closed if and only if A contains all its limit points.*

PROOF: Let A be closed and let x be a limit point of A . If $x \notin A$ then $X \setminus A$ is an open set containing x but containing no other point of A . Thus, x could not be a limit point of A . This means that if x is a limit point of A , then it must be a member of A .

Now suppose that A contains all of its limit points. To show that A is closed, we must show that $X \setminus A$ is open. If $x \in X \setminus A$, then x is not a limit point of A . Thus, there is some open set U_x containing x but no other point of A . Then $X \setminus A$ is the union of all of these sets. Hence, $X \setminus A$ is open and A is closed. ■

What is the connection between *limit points* and the *limit* of a sequence?

Theorem 2.9 *Let (X, d) be a metric space and A a subset of X .*

1. *A point $x \in X$ is a limit point of A if and only if there is a sequence of distinct points of A which converges to x .*
2. *The set A is closed if and only if each convergent sequence of points of A converges to a point of A .*

Corollary 1 *Let x be a limit point of a subset A of a metric space X . Then every open set containing x contains infinitely many members of A .*

2.6 Interior, Closure, and Boundary

Definition 2.11 *Let A be a subset of a metric space (X, d) . A point $x \in A$ is an **interior point** of A if there is an open set U which contains x and is contained in A ; $x \in U \subset A$. The **interior** of A , denoted $\text{int}A$, is the set of all interior points of A .*

Note that for the open set U in the definition, every point of U is an interior point of A . Thus, the interior of A contains every open set contained in A and is the union of this family of open sets. This means two things:

1. the interior of a set A is an open set, and
2. the interior of a set A is the largest open set contained in A .

Item (2) above means that if U is open and $U \subset A$, then $U \subset \text{int}A$.

Example 2.6.1

Let $X = \mathbb{R}$ with the usual metric.

1. For $a, b \in \mathbb{R}$ with $a < b$

$$\text{int}(a, b) = \text{int}[a, b) = \text{int}(a, b] = \text{int}[a, b] = (a, b).$$

2. The interior of a finite set is empty, since such a set cannot contain any open interval.
3. The interior of the set of irrational numbers is empty, since each open interval must contain some rational number. Likewise, the interior of the set of rationals is empty. If the rationals contained an open interval, then the set of rationals would have to be uncountable, since an open interval is uncountable.
4. $\text{int}\emptyset = \emptyset$; $\text{int}\mathbb{R} = \mathbb{R}$.

Definition 2.12 The **closure** $\overline{A}$ of a subset of a metric space (X, d) is the union of the set A and the set of its limit points:

$$\overline{A} = A \cup A'$$

where A' is the derived set of A .

Example 2.6.2

Let $X = \mathbb{R}$ with the usual metric.

1. For $a, b \in \mathbb{R}$ with $a < b$

$$\overline{(a, b)} = \overline{[a, b]} = \overline{(a, b]} = \overline{[a, b)} = [a, b].$$

2. The closure of a finite set is itself, since the set of limit points of a finite set is empty.
3. The closure of the set of rational numbers is $\mathbb{R}$. Likewise, the closure of the set of irrationals is $\mathbb{R}$. Since every open interval contains both rational and irrational numbers.
4. $\overline{\emptyset} = \emptyset$; $\overline{\mathbb{R}} = \mathbb{R}$.

While the interior of a set is the largest open set contained in the set, the closure has a similar property described in the next theorem.

Theorem 2.10 If $A \subset X$, then $\overline{A}$ is a closed set and is a subset of every closed set containing A .

This says that the closure of a set is the smallest closed set containing the set.

PROOF: To show that $\overline{A}$ is closed, we need to show that it contains all of its limit points. Suppose that $x \notin \overline{A}$. Then there is an open set U containing x so that $U \cap A = \emptyset$. Now, this means that U cannot contain a limit point of A either, since if an open set contains a limit point of A it must contain some other point of A also. Thus, U contains no point of $\overline{A}$, so x is not a limit point of $\overline{A}$. This means that all of the limit points of $\overline{A}$ must be contained in $\overline{A}$. Thus, $\overline{A}$ is closed.

Suppose now that F is a closed subset of X and $A \subset F$. Then we can show that $\overline{A} \subset \overline{F}$ and, since F contains all of its limit points, then $\overline{F} = F \cup F' = F$. Thus, $\overline{A} \subset F$ for every closed set F containing A . ■

Since this shows that $\overline{A}$ is the smallest closed set containing A , we can easily show that $\overline{A}$ is the intersection of all closed sets containing A .

Theorem 2.11 *Let A be a subset of the metric space (X, d) .*

1. *A is open if and only if $A = \text{int } A$.*
2. *A is closed if and only if $A = \overline{A}$.*

Definition 2.13 *Let A be a subset of the metric space (X, d) . A point $x \in X$ is a **boundary point** of A provided that $x \in \overline{A} \cap \overline{X \setminus A}$. The set of boundary points of A is called the **boundary** of A and is denoted by ∂A .*

The industrious reader will readily work to show that the following statements are equivalent for a subset A of X and a points x in the metric space (X, d) .

1. $x \in \partial A$,
2. $x \in (\overline{A} \setminus \text{int } A)$,
3. Every open set containing x contains a point of A and a point of $X \setminus A$.
4. Every neighborhood of x contains a point of A and a point of $X \setminus A$.
5. $d(x, A) = d(x, X \setminus A) = 0$.
6. $x \in \overline{A} \cap \overline{X \setminus A}$.

Example 2.6.3 1. Let $X = \mathbb{R}$ with the usual metric. For $a, b \in \mathbb{R}$ with $a < b$

$$\partial(a, b) = \partial[a, b) = \partial(a, b] = \partial[a, b] = \{a, b\}.$$

2. In $\mathbb{R}^n$

$$\partial B(a; \epsilon) = \{x \in \mathbb{R}^n \mid d(a, x) = \epsilon\}.$$

3. The boundary of the set of all points in $\mathbb{R}^n$ having only rational coordinates is $\mathbb{R}^n$.
4. For any metric space (X, d) ,

$$\partial \emptyset = \partial X = \emptyset.$$