

Chapter 3

Topological Spaces

3.1 Definition and Some Examples

We want to generalize the concepts that we developed in studying the metric spaces. We want to remove our reliance on a distance function. We were able to define most of what we wanted to do in metric spaces by defining our concepts in terms of the open sets. This was especially true of our study of continuous functions.

We will use the results that we proved about open sets as our basis for the generalization. We will define open sets as sets that satisfy certain conditions.

Definition 3.1 *Let X be a set and $\mathcal{T}$ a family of subsets of X satisfying the following properties.*

- a) The set X and $\emptyset$ belong to $\mathcal{T}$,*
- b) The union of any family of members of $\mathcal{T}$ is a member of $\mathcal{T}$.*
- c) The intersection of any finite family of members of $\mathcal{T}$ is a member of $\mathcal{T}$.*

*Then $\mathcal{T}$ is called a **topology** for X and the members of $\mathcal{T}$ are called **open sets**. The ordered pair $(X, \mathcal{T})$ is called a **topological space**, or simply a **space**.*

If we use the terminology *open sets* instead of *member of $\mathcal{T}$* , then the definition of a topological space may be restated as follows: *A family of subsets of X is a topology for X means that:*

- a) Both X and $\emptyset$ are open sets.*
- b) The union of any family of open sets is an open set.*
- c) The intersection of any finite family of open sets is open.*

Example 3.1.1 The *usual topology* for the real line $\mathbb{R}$ is the topology generated by its usual metric. We shall refer to the real line with the usual topology as simply the real line or $\mathbb{R}$.

Example 3.1.2 The *usual topology* on $\mathbb{R}^n$ is the topology generated by the usual metric on $\mathbb{R}^n$. It is also the topology generated by the taxicab metric and the max metric. Thus, the usual topology does not distinguish the metric determining it from the other two. We shall refer to $\mathbb{R}^n$ with the usual topology as *Euclidean n -space*, or simply $\mathbb{R}^n$.

Example 3.1.3 For any set X we take $\mathcal{T} = 2^X$ to be the set of all subsets of X . This clearly satisfies all of the properties of a topology, since we have included every possible subset in the topology. This is called the *discrete topology*. Note that it is the topology generated by the discrete metric. Also, note that this is the largest possible collection of open subsets of X .

Example 3.1.4 At the opposite extreme, we may take $\mathcal{T} = \{\emptyset, X\}$. This is called the *trivial topology*, or *indiscrete topology*, on X . This is the smallest collection of open sets on X .

Example 3.1.5 Let X be a set. We shall take $\mathcal{T}$ to consist of $\emptyset$, X , and all sets U so that $X \setminus U$ is a finite set. Then $\mathcal{T}$ is a topology on X called the *cofinite topology*, or *finite complement topology*. This is really of interest only when X is an infinite set. When X is a finite set, this is the same as the discrete topology.

Definition 3.2 A subset F of a topological space X is **closed** if $X \setminus F$ is an open set.

Theorem 3.1 The closed sets of a topological space X have the following properties:

- a) X and $\emptyset$ are closed.
- b) The intersection of any family of closed sets is closed.
- c) The union of any finite family of closed sets is closed.

Definition 3.3 Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. A point x in X is a **limit point** of A if every open set containing x contains a point of A distinct from x . The set of limit points of A is called the **derived set** of A , denoted A' .

Example 3.1.6 Let $X = \{a, b, c, d\}$. Let $\mathcal{T}_0$ be the indiscrete topology; $\mathcal{T}_1$, the discrete topology; $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$; and $\mathcal{T}_3 = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. The reader should verify that $\mathcal{T}_2$ and $\mathcal{T}_3$ are topologies on X . Let $A = \{a, b\}$, $B = \{c\}$, and $C = \{d\}$. We want to find the limit points of these sets in the different topologies.

	$\mathcal{T}_0$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$
A	X	$\emptyset$	$\{c, d\}$	$\{d\}$
B	$\{a, b, d\}$	$\emptyset$	$\{d\}$	$\{d\}$
C	$\{a, b, c\}$	$\emptyset$	$\{c\}$	$\emptyset$

Theorem 3.2 *A subset A of a topological space X is closed if and only if A contains all of its limit points.*

This is no surprise, and is proven exactly the way in which we proved it earlier in a metric space. We were careful there not to use the distance function, but to use the open sets.

Definition 3.4 *Let X be a topological space and let $\{x_n\}$ be a sequence of points in X . We say that $\{x_n\}$ **converges** to the point $x \in X$, or x is the **limit** of the sequence, if for each open set U containing x there is a positive integer N so that $x_n \in U$ for all $n \geq N$.*

Sequences are not as fundamental in general topological spaces as they are in metric spaces. The following example may show why.

Example 3.1.7 Consider $\mathbb{R}$ with the cofinite topology. Let $\{x_n\}$ be any sequence of real numbers. Let $a \in \mathbb{R}$ be **any** real number. Then $\{x_n\}$ converges to a , because if U is any open set containing a , then $\mathbb{R} \setminus U$ is a finite set. Since $\{x_n\}$ is an infinite set, we must have that infinitely many members of $\{x_n\}$ lie in U . Thus, there is a positive integer N such that if $n \geq N$ $x_n \in U$. Thus, $\{x_n\}$ converges to a . However, a was any arbitrary real number. This means that $\{x_n\}$ converges to **every real number**. What is more (and maybe worse) is that $\{x_n\}$ was an arbitrary sequence. This means that every sequence converges to every real number. There are no non-convergent sequences and sequences do not have unique limits.

3.2 Interior, Closure and Boundary

Just as before we will define the interior, closure, and the boundary.

Definition 3.5 *Let A be a subset of the topological space X . A point $x \in A$ is an **interior point** of A if there is an open set U so that $x \in U \subset A$. A is called a **neighborhood** of x . The **interior** of A , denoted by A° , is the set of all interior points of A .*

The **closure**, $\bar{A}$, of A is the union of A and its set of limit points:

$$\bar{A} = A \cup A'.$$

A point $x \in X$ is a **boundary point** of A if $x \in \bar{A} \cap \overline{X \setminus A}$. The set of boundary points of A is called the **boundary** of A and is denoted by ∂A .

Theorem 3.3 For any subsets A, B of a topological space X

- a) The interior of A is the union of all open sets contained in A and is the largest open set contained in A .
- b) A is open if and only if $A = A^\circ$.
- c) If $A \subset B$, then $A^\circ \subset B^\circ$.
- d) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

PROOF: We will offer only a proof for (d). The others follow closely from what we did in the case of a metric space.

Since $A \cap B$ is a subset of both A and B , then by (c) $(A \cap B)^\circ \subset A^\circ \cap B^\circ$. Now, $A^\circ \cap B^\circ$ is an open set and is a subset of $A \cap B$. Thus by (a), $A^\circ \cap B^\circ \subset (A \cap B)^\circ$. This completes the proof. ■

Theorem 3.4 For any subsets A, B of a topological space X

- a) The closure of A is the intersection of all closed sets containing A and is the smallest closed set containing A .
- b) A is closed if and only if $A = \bar{A}$.
- c) If $A \subset B$, then $\bar{A} \subset \bar{B}$.
- d) $\overline{(A \cup B)} = \bar{A} \cup \bar{B}$.

We leave this to the reader to prove.

Theorem 3.5 Let A be a subset of a topological space X .

- a) $\partial A = \bar{A} \cap \overline{X \setminus A} = \partial(X \setminus A)$.
- b) ∂A , A° , and $(X \setminus A)^\circ$ are pairwise disjoint sets whose union is X .
- c) ∂A is a closed set.
- d) $\bar{A} = A^\circ \cup \partial A$.
- e) A is open if and only if $\partial A \subset (X \setminus A)$.

f) A is closed if and only if $\partial A \subset A$.

g) A is open and closed if and only if $\partial A = \emptyset$.

PROOF: Parts (a)–(d) follow immediately from the definitions.

(e) If A is open then $A = A^\circ$. Now by (b) A° and ∂A are disjoint. Thus, A and ∂A are disjoint. This implies that $\partial A \subset X \setminus A$. Now, if $\partial A \subset X \setminus A$ then no point of A is a boundary point of A . Thus, every point of A is an interior point of A and $A = A^\circ$. Thus, A is open.

(f) This follows from our duality of open and closed sets.

(g) If A is both open and closed, then $\partial A \subset A \cap (X \setminus A) = \emptyset$. If $\partial A = \emptyset$ then clearly $\partial A \subset A$ — meaning A is closed — and $\partial A \subset (X \setminus A)$ — meaning A is open. ■

Definition 3.6 A set A in a topological space X is **dense** if $\overline{A} = X$. If X has a countable dense set, then X is a **separable** space.

Example 3.2.1

1. The reals with the usual topology is separable, since the rationals are dense.
2. Euclidean n -space is separable, since the set of points having only rational coordinates is dense and countable.
3. The reals with the cofinite topology is separable, since every countable infinite set is dense.

Definition 3.7 A subset B of a space X is **nowhere dense** if $(\overline{B})^\circ = \emptyset$.

Note that a finite subset of a metric space is nowhere dense. In other topological spaces, we will see more interesting examples.

3.3 Basis for a Topology

It appears that a topology can be relatively large. In fact, for an infinite set the discrete topology consists of all subsets of the space, so it would be prohibitive to have to check all subsets. We have seen though that we can get by with just checking some of the sets. For the discrete topology we have usually only checked the singleton sets. For a metric space we were able to do everything we wanted by working with the open balls. In fact we defined all open sets in terms of the open balls. Can we do this in general? Can we find a certain collection of subsets that will generate all of the elements of the topology, just like the open balls generate the metric topology? The answer is yes, because we can take $\mathcal{T}$ as this generating set. This begs the answer, because we are looking for a *smaller* collection than the whole topology.

Definition 3.8 Let $(X, \mathcal{T})$ be a topological space. A **basis** $\mathcal{B}$ for $\mathcal{T}$ is a subcollection of $\mathcal{T}$ with the property that each member of $\mathcal{T}$ is a union of members of $\mathcal{B}$. The members of $\mathcal{B}$ are called **basic open sets** and $\mathcal{T}$ is the topology **generated by** $\mathcal{B}$.

Example 3.3.1 Most of what we have seen is based on metric spaces.

1. The collection of all open intervals is a basis for the usual topology on the reals.
2. The collection of all open balls is a basis for the metric topology on the metric space (X, d) .
3. For any set X the collection of all singleton sets $\{x\}$ is a basis for the discrete topology.

Definition 3.9 Let $(X, \mathcal{T})$ be a topological space. A **local basis** at $a \in X$ is a subcollection $\mathcal{B}_a$ of $\mathcal{T}$ such that

- a) a belongs to each member of $\mathcal{B}_a$, and
- b) each open set containing a contains a member of $\mathcal{B}_a$.

Definition 3.10 A space X is **first countable** if there is a countable local basis at each point of X . The space X is **second countable** if the topology for X has a countable basis.

Note that every second countable space is first countable because if there is a countable basis $\mathcal{B}$, then the number of these sets containing any given point $a \in X$ is at most countable.

Theorem 3.6 Every second countable space is separable.

PROOF: Let X be a second countable space with a countable basis $\mathcal{B}$. Let A be a set formed by choosing one element from each non-empty element of $\mathcal{B}$. Each point of X is a limit point of some point in A by the definition of a basis. Thus, A is dense in X . ■

Theorem 3.7 a) Every metric space is first countable.

b) Every separable metric space is second countable.

The proof is left to the reader.

We have been starting with a topology and asking if there is a basis for it. We could be starting with a collection of open sets and asking if it forms a basis for a topology. Not every collection of open sets will work. When is a collection of open sets a basis for a topology on X ?

Theorem 3.8 *A family $\mathcal{B}$ of subsets of a set X is a basis for a topology on X if and only if both of the following hold:*

- a) *The union of members of $\mathcal{B}$ is X .*
- b) *For each $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is a member B_x of $\mathcal{B}$ such that $x \in B_x \subset B_1 \cap B_2$.*

Example 3.3.2 [Sorgenfrey Line] Let $\mathcal{B}$ be the collection of all half-open intervals of $\mathbb{R}$ of the form $[a, b)$, $a < b$. Clearly, the union of all of these intervals is $\mathbb{R}$. If we take two of these sets and intersect them, we can find another of these sets in the intersection. Thus, these sets form a basis for a topology on the real line, called the *half-open interval topology* $\mathcal{T}'$ for $\mathbb{R}$. $\mathbb{R}$ with this topology is called the *Sorgenfrey line*. The Sorgenfrey line has the property that it is first countable and separable, but not second countable.

3.4 Continuous Functions

We were able to move away from the epsilon-delta definition of continuity of a function in between two metric spaces by using open balls. We will use this as our starting point for general topological spaces.

Definition 3.11 *A function $f: (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$ is **continuous** at a point $a \in X$ if for each open set V in Y containing $f(a)$ there is an open set $U \subset X$ containing a so that $f(U) \subset V$, or equivalently $U \subset f^{-1}(V)$.*

Now, we are more interested in the situation where the function is continuous at every point of X . This means that for every open set V in Y and every point $a \in X$ with $f(a) \in V$, there is an open set $U_a \subset X$ with $a \in U_a$ and $f(U_a) \subset V$. Equivalently, we have $a \in U_a \subset f^{-1}(V)$. This means that for each point in $f^{-1}(V)$ we can find an open set containing that point and contained in $f^{-1}(V)$. Thus, $f^{-1}(V)$ must be open in X for each open set $V \subset Y$. This leads us to a more general definition.

Definition 3.12 *A function $f: (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$ is **continuous** if for each open set V in Y $f^{-1}(V)$ is an open set in X .*

Theorem 3.9 *Let $f: X \rightarrow Y$ be a function on the topological spaces X and Y and let $a \in X$. The following are equivalent.*

- a) *f is continuous at a .*
- b) *For each open set $V \in Y$ containing $f(a)$, there is an open set U in X such that $a \in U \subset f^{-1}(V)$.*
- c) *For each neighborhood V of $f(a)$, $f^{-1}(V)$ is a neighborhood of a .*

The proof is left to the reader.

Theorem 3.10 *Let $f: X \rightarrow Y$ be a function of topological spaces. The following are equivalent.*

- (i) f is continuous.
- (ii) For each closed subset $C \subset Y$, $f^{-1}(C)$ is closed in X .
- (iii) For each subset $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.
- (iv) There is a basis $\mathcal{B}$ for the topology of Y so that $f^{-1}(B)$ is open in X for each basic open set $B \in \mathcal{B}$.

PROOF: To show that (i) implies (ii) will require the duality between open and closed sets. If $C \subset Y$ is closed, the $Y \setminus C$ is open in Y . Since f is continuous, $f^{-1}(Y \setminus C)$ is open in X . Hence $X \setminus (f^{-1}(Y \setminus C))$ is closed. If $x \in X \setminus (f^{-1}(Y \setminus C))$ then $f(x) \notin Y \setminus C$, or $f(x) \in C$. Thus, $X \setminus (f^{-1}(Y \setminus C)) \subset f^{-1}(C)$. The opposite inclusion is clear. Thus, $f^{-1}(C) = X \setminus (f^{-1}(Y \setminus C))$ is closed. A similar analysis shows that (ii) implies (i).

To show that (ii) implies (iii), let $A \subset X$. Then $\overline{f(A)}$ is a closed subset of Y . Hence, $f^{-1}(\overline{f(A)})$ is a closed subset of X . Now, $A \subset f^{-1}(\overline{f(A)})$ so $\overline{A} \subset f^{-1}(\overline{f(A)})$. Thus, $f(\overline{A}) \subset \overline{f(A)}$.

To show that (iii) implies (ii), let C be a closed subset of Y . Then,

$$f(\overline{f^{-1}(C)}) \subset \overline{ff^{-1}(C)} \subset \overline{C} \subset C$$

so $\overline{f^{-1}(C)} \subset f^{-1}(C)$ making $f^{-1}(C)$ a closed set.

For the last equivalence, (i) clearly implies (iv). We need to prove the opposite implication. Let O be an open set in Y . Then by the definition of a basis, $O = \cup_{\alpha \in I} B_\alpha$ for some subcollection $\{B_\alpha\}_{\alpha \in I}$ of the basis $\mathcal{B}$. Then

$$f^{-1}(O) = f^{-1}\left(\bigcup_{\alpha \in I} B_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(B_\alpha).$$

Since each $f^{-1}(B_\alpha)$ is open in X and the union of any family of open sets is open, then $f^{-1}(O)$ is open in X and f is continuous. ■

Theorem 3.11 *If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are continuous functions, then $g \circ f: X \rightarrow Z$ is continuous.*

Definition 3.13 *A function $f: X \rightarrow Y$ is a **homeomorphism** if*

- a) f is one-to-one, (**injective**)
- b) f is onto, (**surjective**)

c) f is continuous,

d) f^{-1} is continuous.

Topological spaces are **topologically equivalent** or **homeomorphic** if there homeomorphism from f from X onto Y .

The continuity of the function **and** its inverse are extremely important! A property P of topological spaces is a **topological property** or **topological invariant** provided that if space X has property P , then so does every space which is homeomorphic to X .

Theorem 3.12 *Separability is a topological property.*

Theorem 3.13 *First countability and second countability are topological properties.*

Definition 3.14 *A topological space is **metrizable** provided that the topology on X is generated by a metric.*

Theorem 3.14 *Metrizability is a topological property.*

Since $\mathbb{R}$ and $(0, 1)$ are homeomorphic, the property of being a bounded metric space is not a topological property. Likewise, distance is not a topological invariant.

3.5 Subspaces

Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X . The **relative topology** or **subspace topology** $\mathcal{T}'$ on A determined by $\mathcal{T}$ consists of all sets of the form $O \cap A$ for which O is an open set of $\mathcal{T}$.

$$\mathcal{T}' = \{O \cap A \mid O \in \mathcal{T}\}.$$

The members of $\mathcal{T}'$ are called **relatively open sets** in A , and $(A, \mathcal{T}')$ is called a **subspace** of $(X, \mathcal{T})$.

Note that this is actually a topology for A .

$$\emptyset = \emptyset \cap A \quad A = X \cap A,$$

so both $\emptyset$ and A are open in A . If $\{U_\alpha\}$ are open in A , then $U_\alpha = O_\alpha \cap A$ and

$$\bigcup_{\alpha} U_{\alpha} = \bigcup_{\alpha} (O_{\alpha} \cap A) = \left(\bigcup_{\alpha} O_{\alpha} \right) \cap A$$

is relatively open since the union of any family of open sets is open in X . For any finite family of open sets $U_i = O_i \cap A$, we have

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (O_i \cap A) = \left(\bigcap_{i=1}^n O_i \right) \cap A$$

is relatively open since the intersection of any finite family of open sets is open in X .

Lemma 3.1 Let $(A, \mathcal{T}')$ be a subspace of the topological space $(X, \mathcal{T})$. A subset F of A is closed in the subspace topology on A if and only if $F = C \cap A$ for some closed subset C of X .

Example 3.5.1 1. The closed interval $[a, b]$ with $a < b$ is a subspace of $\mathbb{R}$ with the usual topology. The open sets containing a are sets of the form $[a, c)$ with $a < c < b$.

2. The subset of $\mathbb{R}^{n+1}$ consisting of all $(n+1)$ -tuples $(x_1, x_2, \dots, x_n, x_{n+1})$ with $x_{n+1} = 0$ is homeomorphic to $\mathbb{R}^n$.

3. Let $a < b < c < d$. Let $A = [a, b] \cup (c, d)$ be considered as a subspace of the real line. Then the subset $[a, b]$ of A is both relatively open and relatively closed. It is clearly closed because $[a, b] = [a, b] \cap A$ and $[a, b]$ is closed in the real line. It is open because for $0 < \epsilon < c - b$, $[a, b] = (a - \epsilon, b + \epsilon) \cap Y$. Thus, we see that since (c, d) is the complement of this set that is both relatively open and relatively closed, then we see that (c, d) is both relatively open and relatively closed.

Definition 3.15 A property P to topological spaces is **hereditary** provided that if X has property P , then every subspace of X has this property.

Example 3.5.2 1. First countability and second countability are hereditary properties. If X has a countable basis, then intersecting these basis elements with A will give a countable basis for the subspace topology. First countable is similar.

2. Separability is not hereditary. Let $A \subset \mathbb{R}^2$ consist of the x -axis and the point $a = (0, 1)$. Define a topology $\mathcal{T}$ on A by taking the empty set and all subsets of A that contain the singleton set $\{a\}$. Then $(X, \mathcal{T})$ is separable because the singleton set $\{a\}$ is dense. Every point except a is a limit point of $\{a\}$. However, the subspace topology on $\mathbb{R}$ (the x -axis) is the discrete topology, so $(\mathbb{R}, \mathcal{T}')$ is not separable.

3.6 Hausdorff Spaces

A topological space X is a **Hausdorff space** if for each pair of distinct points $a, b \in X$ there exist disjoint open sets U and V such that $a \in U$ and $b \in V$.

Example 3.6.1 1. Every metric space is Hausdorff. We proved this as a homework problem, but it is simple. Let $r = d(a, b)$ and then take two open balls of radius $r/2$ centered at a and b respectively.

2. The real line with the co-finite topology is not Hausdorff. Likewise, the real line with the co-countable topology is not Hausdorff.

3. Take any set with more than one point and give it the indiscrete (trivial) topology. It is not Hausdorff.
4. The space in Example 2 is not Hausdorff, because you can never separate a from any other point.
5. **[The Zariski Topology]** Let n be a positive integer and consider the family $\mathcal{P}$ of all polynomials in n real variables $x_1, x_2, \dots, x_n$. For $p \in \mathcal{P}$ let $Z(p)$ denote its solution set in $\mathbb{R}^n$:

$$Z(p) = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid p(x_1, x_2, \dots, x_n) = 0\}.$$

Let $\mathcal{B}$ be the collection of sets that are complements of some $Z(p)$ for some $p \in \mathcal{P}$. This forms a basis for a topology on $\mathbb{R}^n$ called the *Zariski topology*.

For the real line, $n = 1$, this is just the co-finite topology. This is because each finite set of real numbers is the solution set for some polynomial in one real variable. If $A = \{a_1, a_2, \dots, a_n\}$, then

$$p(x) = (x - a_1)(x - a_2) \dots (x - a_n)$$

is a polynomial with A as its solution set. Likewise, the set of solutions of a polynomial in one real variable of dimension n is at most n .

For $n > 1$ this is not the co-finite topology. For example, the line $y = a$ in $\mathbb{R}^2$ is the solution set to the polynomial in two variables

$$p(x, y) = y - a.$$

Note that this is not a finite set. However, each finite set can serve as the solution set of a polynomial.

Now, $\mathbb{R}^n$ with the Zariski topology is not Hausdorff. Assume that $P = (a_1, a_2, \dots, a_n)$ and $Q = (b_1, b_2, \dots, b_n)$ are two distinct points in $\mathbb{R}^n$.

Theorem 3.15 1. *The property of being a Hausdorff space is topological and hereditary.*

2. *In a Hausdorff space a sequence $\{x_n\}_{n=1}^{\infty}$ cannot converge to more than one point.*

PROOF: We will prove 1 only and leave the other to the reader.

Suppose that X is Hausdorff and $f: X \rightarrow Y$ is a homeomorphism. For $a \neq b \in Y$, we have that $f^{-1}(a)$ and $f^{-1}(b)$ are distinct points in X . Thus, there are disjoint open sets $U, V \subset X$ so that $f^{-1}(a) \in U$ and $f^{-1}(b) \in V$. Hence, $f(U)$ and $f(V)$ are disjoint open sets of Y containing a and b respectively.

To show that Hausdorff is hereditary, assume that X is Hausdorff and that $A \subset X$. Let $a \neq b \in A$. Then, $a \neq b \in X$ and there are disjoint open sets $U, V \subset X$ so that $a \in U$ and $b \in V$. Then, $U \cap A$ and $V \cap A$ are disjoint relatively open subsets of A containing a and b respectively. ■