

Chapter 6

Product and Quotient Spaces

When we study topology, we do not differ from other areas of mathematics that much. We look at the different mathematical operations that are available to us and how they affect the mathematical structure we are studying. We also want to see how our mathematical structure affects these operations.

The standard operations are addition, subtraction, multiplication and division. In topology addition corresponds to the union of spaces, $A \cup B$, and subtraction corresponds to the set difference $A \setminus B$. Note that addition is commutative, but difference is not. How do multiplication and quotients carry to the area of topology. Actually, multiplication is quite simple to see, but quotients are not as easily introduced.

6.1 Finite Products

Let X and Y be sets. The *Cartesian product* of X and Y is denoted by $X \times Y$ and is defined to be the set of all ordered pairs of points whose first coordinate comes from X and whose second coordinate comes from Y :

$$X \times Y = \{(a, b) \mid a \in X \text{ and } b \in Y\}.$$

Now, if X and Y are topological spaces, then we would like to see what natural topology we should have on the product space, $X \times Y$. The topology on $X \times Y$ should be built from the topologies on X and Y , but how? If U is open in X and V is open in Y , then we would want $U \times V$ to be open in $X \times Y$. However, the collection of all sets of this form is not closed under unions, and hence cannot be a topology. It is not difficult to see, in the plane, that the union of two of these rectangles $(0, 1) \times (0, 1) \cup (2, 3) \times (2, 3)$ is not a set of the form $U \times V$, because $U \times V$ would have four pieces, not two.

Let $\mathcal{B} = \{U_\alpha \times V_\beta \mid U_\alpha \text{ is open in } X \text{ and } V_\beta \text{ is open in } Y\}$. We have just seen that $\mathcal{B}$ is not a topology. It is, however, a basis for a topology on $X \times Y$. We need to show that it satisfies the conditions of Theorem 4.8. We must show that the union of the members of $\mathcal{B}$ is $X \times Y$ and that for each $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is a member B_x of $\mathcal{B}$ such that $x \in B_x \subset B_1 \cap B_2$.

It should be clear that the union of the members of $\mathcal{B}$ is all of $X \times Y$. For the second condition, let $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ where U_i is open in X and V_i is open in Y . Now,

$$B_1 \cap B_2 = (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2),$$

so that $B_1 \times B_2 \in \mathcal{B}$. This is stronger than what is required in Theorem ???. Thus, $\mathcal{B}$ is a basis for a topology on $X \times Y$. The topology determined by $\mathcal{B}$ is called the *product topology* on $X \times Y$.

We can generalize this definition:

Definition 47 Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2), \dots, (X_n, \mathcal{T}_n)$ be a finite collection of nonempty topological spaces, and let X denote the Cartesian product

$$X = \prod_{i=1}^n X_i = X_1 \times X_2 \times X_3 \times \cdots \times X_n.$$

Let $\mathcal{B}$ be the family of all subsets of X of the form

$$O = \prod_{i=1}^n O_i = O_1 \times O_2 \times O_3 \times \cdots \times O_n$$

where each O_i is open in X_i . Then $\mathcal{B}$ is a basis for the **product topology** for X . The set X with the product topology is a **product space**. The spaces $X_1, X_2, \dots, X_n$ are called the **coordinate spaces** or **factor spaces** of X .

Each point $x \in X$ is of the form

$$x = (x_1, x_2, \dots, x_n), \quad x_i \in X_i.$$

Define a function $\pi_i: X \rightarrow X_i$ by $\pi_i(x) = x_i$. This is called the **projection map on the i th coordinate space** or the **i th projection map**.

Theorem 57 The projection maps $\pi_i: X \rightarrow X_i$ from the product space to the coordinate spaces are continuous.

PROOF: Let $\pi_i: X \rightarrow X_i$ and let $U_i \subset X_i$ be open. Then $\pi_i^{-1}(U_i) = X_1 \times \cdots \times X_{i-1} \times U_i \times X_{i+1} \times \cdots \times X_n$. This is a product of open sets, and is open. Thus, π_i is continuous. ■

Theorem 58 Let $f: Y \rightarrow X$ be a function from a topological space to a product space $X = \prod_{i=1}^n X_i$. Then f is continuous if and only if $\pi_i \circ f$ is continuous for each projection map.

What properties are preserved by products?

Theorem 59 *The product of a finite number of Hausdorff spaces is Hausdorff.*

PROOF: Let X_i , $i = 1, \dots, n$, be a Hausdorff space for each n . Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be distinct points in $\prod_{i=1}^n X_i$. Since they are distinct, there is at least one coordinate space, X_j , for which $a_j \neq b_j$. Since X_j is Hausdorff, there are disjoint open sets, U_j and V_j in X_j so that $a_j \in U_j$ and $b_j \in V_j$. Then

$$U = \pi_j^{-1}(U_j) \quad \text{and} \quad V = \pi_j^{-1}(V_j)$$

are disjoint open sets of X containing a and b , respectively. ■

Theorem 60 *The product of a finite number of connected spaces is connected.*

PROOF: The proof is most easily done by induction. We will prove it for the case of $n = 2$ and indicate the manner of the rest of the proof.

Assume that X_1 and X_2 are connected and let $x_1 \in X_1$. Then, $\{x_1\} \times X_2$ is a subspace of $X_1 \times X_2$ which is homeomorphic to X_2 using the function

$$(x_1, t) \mapsto t, \quad t \in X_2.$$

Since X_2 is connected, $\{x_1\} \times X_2$ is connected. Now a similar argument shows that $X_1 \times \{t\}$ is a connected set for each $t \in X_2$. Now, $\{x_1\} \times X_2$ and $X_1 \times \{t\}$ have the point (x_1, t) in common. Thus, the set

$$(\{x_1\} \times X_2) \cup \bigcup_{t \in X_2} (X_1 \times \{t\}) = X_1 \times X_2$$

is connected.

Now, by induction, suppose that $\prod_{i=1}^{n-1} X_i$ is connected and consider $\prod_{i=1}^n X_i$, where each X_i is connected. By this previous argument, $\prod_{i=1}^n X_i$ is connected since it is homeomorphic to the product of two connected spaces: $\prod_{i=1}^{n-1} X_i$ and X_n . Hence the finite product of connected spaces is connected. ■

Theorem 61 *The finite product of separable spaces is separable.*

Theorem 62 *The finite product of first countable spaces is first countable.*

Theorem 63 *The finite product of second countable spaces is second countable.*

Theorem 64 *If $(X_1, d_1), \dots, (X_n, d_n)$ are metric spaces, then the product topology on the product space is generated by the product metric:*

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \left(\sum_{i=1}^n d_i(x_i, y_i)^2 \right)^{1/2}.$$

In order to prove an analogous result for compactness, we need the following lemma.

Lemma 9 *In order that a space X be compact it is sufficient that there exists a basis $\mathcal{B}$ for X such that every open cover of X by members of $\mathcal{B}$ has a finite subcover.*

Theorem 65 *The finite product of compact spaces is compact.*

PROOF: Using an inductive argument as in our last theorem, it suffices to show that $X_1 \times X_2$ is compact if X_1 and X_2 are compact.

Let $\mathcal{B} = \{U \times V \mid U \text{ is open in } X_1 \text{ and } V \text{ is open in } X_2\}$ be the subbasis for the product topology. Let $\mathcal{O}$ be an open cover of $X_1 \times X_2$ composed of members of $\mathcal{B}$. By the above lemma, the compactness of $X_1 \times X_2$ will be proved if it can be shown that there is a finite subcover for $X_1 \times X_2$ derived from $\mathcal{O}$.

For $x \in X_1$, the subset $\{x\} \times X_2$ is compact and is therefore contained in the union of a finite number of members, say $U_1 \times V_1, U_2 \times V_2, \dots, U_m \times V_m$ of $\mathcal{O}$, each of which meets $\{x\} \times X_2$. Then

$$U_x = \bigcap_{i=1}^m U_i$$

is an open set in X_1 containing x . Note that

$$U_x \times X_2 = U_x \times \left(\bigcup_{i=1}^m V_i \right) = \bigcup_{i=1}^m (U_x \times V_i) \subset \bigcup_{i=1}^m (U_i \times V_i).$$

This last inclusion is due to the fact that $U_x \subset U_i$ for all i . Thus, for each $x \in X_1$ there is an open set U_x containing x for which the set $U_x \times X_2$ is contained in the union of a finite number of members of $\mathcal{O}$.

This family of open sets $\{U_x \mid x \in X_1\}$ is an open cover for X_1 . Since X_1 is compact, this cover admits a finite subcover $\{U_{x_i}\}_{i=1}^r$. Now,

$$X_1 \times X_2 = \left(\bigcup_{i=1}^r U_{x_i} \right) \times X_2 = \bigcup_{i=1}^r (U_{x_i} \times X_2)$$

is the union of a finite number of the sets of the form $U_{x_i} \times X_2$. Each of these sets is contained in the union of a finite number of members of $\mathcal{O}$. This forms a finite subcover for $X_1 \times X_2$ of the members of $\mathcal{O}$, making $X_1 \times X_2$ compact. ■

6.2 Arbitrary Products

We have previously defined a countable product of spaces as follows:

$$\prod_{i=1}^{\infty} X_i = \{(x_1, x_2, x_3, \dots) \mid x_i \in X_i \text{ for each } i = 1, 2, \dots\}.$$

We identify these with sequences of elements, each element coming from the appropriate factor space. We had a different way to define sequences when we studied calculus. If you will recall a sequence is a function

$$f: \mathbb{Z}^+ \rightarrow X = \bigcup_{i=1}^n X_i, \quad f(i) \in X_i.$$

We will use this idea to extend the definition of products to arbitrary indexing sets.

Definition 48 *Let I be an index set and $\{X_\alpha \mid \alpha \in I\}$ be a family of sets. The Cartesian product $X = \prod_{\alpha \in I} X_\alpha$ is the collection of all functions x with domain I having the property that the value x_α of x at α belongs to the set X_α :*

$$X = \prod_{\alpha \in I} X_\alpha = \left\{ x: I \rightarrow \bigcup_{\alpha \in I} X_\alpha \mid x(\alpha) = x_\alpha \in X_\alpha \text{ for each } \alpha \in I \right\}$$

For $\alpha \in I$ the function $\pi_\alpha: X \rightarrow X_\alpha$ defined by $\pi_\alpha(x) = x_\alpha$ is called the projection map of X on the α th coordinate set.

Note that this does not differ from our previous definition if the index set I is finite. We get exactly the same space.

We define the product topology on the space $X = \prod_{\alpha} X_\alpha$ using the projection maps. Let $\mathcal{S}$ consist of all sets of the form $\pi_\alpha^{-1}(U_\alpha)$ where U_α is open in X_α . This set forms a subbasis for the product topology. This means that the basis for the product topology for X consists of all finite intersections:

$$\bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_{\alpha_i}), \quad \alpha_i \in I, \quad 1 \leq i \leq n.$$

Such a basic open set may be expressed in the product form as

$$\bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_{\alpha_i}) = \prod_{\alpha \in I} V_\alpha$$

where $V_{\alpha_i} = U_{\alpha_i}$ for $i = 1, \dots, n$ and $V_\alpha = X_\alpha$ otherwise.

Proofs of the following theorems are similar to there finite analogues.

Theorem 66 *Let $f: Y \rightarrow X$ be a function from a space Y into a product space $X = \prod_{\alpha} X_\alpha$. Then f is continuous if and only if the composition $\pi_\alpha \circ f$ of f with each projection map is continuous.*

Theorem 67 *The product of any family of Hausdorff spaces is a Hausdorff space.*

Theorem 68 *The product of an arbitrary collection of connected spaces is connected.*

This proof is based on the previous proof, but is not exactly the same.

To prove the analogous theorem for compact spaces, we need some further results. The proofs are not trivial, but are not included here. The first is needed for the second.

Lemma 10 (The Alexander Subbasis Theorem) *In order that a space X be compact, it is necessary and sufficient that there exist a subbasis $\mathcal{S}$ for X such that every open cover of X by members of $\mathcal{S}$ has a finite subcover.*

Theorem 69 (The Tychonoff Theorem) *The product of an arbitrary family of compact spaces is compact.*

6.3 Quotient Spaces

Quotient spaces are sometimes called *identification spaces*, because they result from a *gluing*, or identification, process.

Definition 49 *Let X be a space and let $\sim$ be an equivalence relation on X . Let $X/\sim$ denote the set of all equivalence classes $[x] = \{y \in X \mid x \sim y\}$. $X/\sim$ is called the **quotient space** of X by the equivalence relation $\sim$. Define a function $q: X \rightarrow X/\sim$ by $q(x) = [x]$. This map q is called the **quotient map**. Define a set $A \subset X/\sim$ to be open if $q^{-1}(A)$ is open in X . This collection of open sets defines a topology on $X/\sim$ called the **quotient topology**.*

A particularly simple example of an equivalence relation giving rise to interesting quotient spaces is the following. Let $A \subset X$ and, for the sake of interest, let A have more than one member. Define a relation, $\sim_A$, on X by: $x \sim_A y$ if $x, y \in A$ or if $x = y \notin A$. Thus, every point of A is related to every other point of A , and if a point is not in A it is related only to itself. The quotient space is denoted by X/A and is called the **identification space obtained by identifying the members of A to a point**.

Example 44 The quotient space obtained by identifying the two endpoints 0 and 1 to a point is the circle, S^1 .

Example 45 The quotient space of D^2 obtained by identifying the boundary circle, S^1 , to a point is homeomorphic to a sphere in $\mathbb{R}^3$.

Example 46 Let $I^2 = [0, 1] \times [0, 1]$. The quotient space of I^2 obtained by identifying the pairs of points $(0, x_2)$ and $(1, 1 - x_2)$, $0 \leq x_2 \leq 1$, is homeomorphic to the Möbius strip.

Example 47 Let $X = I^2$ with equivalence relation $\sim$ defined as follows: (1) $(x_1, 0) \sim (x_1, 1)$ for each $x_1 \in [0, 1]$, and (2) $(0, x_2) \sim (1, x_2)$ for each $x_2 \in [0, 1]$. Then the quotient space $X/\sim$ is homeomorphic to a torus in $\mathbb{R}^3$.

Example 48 Let $\sim$ be an equivalence relation on $\mathbb{R}$ defined by $x \sim y$ if and only if $x - y \in \mathbb{Z}$. In other words, $x \sim x + n$ where $n \in \mathbb{Z}$. We can actually think of these equivalence classes as

$$[x] = \{x + n \mid x \in [0, 1] \text{ and } n \in \mathbb{Z}\}$$

We have reduced our consideration from $\mathbb{R}$ to $[0, 1]$. It is clear that $[0] = [1]$, so that $\mathbb{R}/\sim$ is homeomorphic to our first example, and $\mathbb{R}/\sim$ is topologically equivalent to the circle. The quotient map here is equivalent to $t \mapsto (\cos(t), \sin(t))$.

Example 49 Let $X = I^2$ and define $\sim$ on X by:

1. $(x_1, 0) \sim (x_1, 1)$ for $x_1 \in [0, 1]$, and
2. $(0, x_2) \sim (1, 1 - x_2)$ for $x_2 \in [0, 1]$.

This is slightly different from 46 in that in the last stage the circles are identified with reversed orientations. The quotient space is the *Klein bottle*.

Example 50 Define an equivalence relation on the circle, S^1 , by $z \sim -z$, where we consider the circle as the set of all complex numbers of modulus 1 (or the set of vectors in the plane with norm 1). Then the quotient space $S^1/\sim$ is topologically equivalent to S^1 .

Example 51 Let $X = I^2$ and define $\sim$ on X by:

1. $(x_1, 0) \sim (1 - x_1, 1)$ for $x_1 \in [0, 1]$, and
2. $(0, x_2) \sim (1, 1 - x_2)$ for $x_2 \in [0, 1]$.

This is equivalent to taking the unit disk and applying the previous equivalence relation to the bounding circle. The resulting space is called the *projective plane*, and cannot be reasonably drawn in three dimensions.

Example 52 Let X be a topological space and let A be the subset of $X \times I$ given by $A = X \times \{1\}$. The quotient space $T(X) = (X \times I)/A$ is called the *cone over X*.

Note that our definition of the quotient topology guarantees that the quotient map is continuous. This is extremely useful in later results.

Theorem 70 *Let $X/\sim$ be a quotient space with quotient map $q: X \rightarrow X/\sim$. Then a function $f: X/\sim \rightarrow Y$ is continuous if and only if the composite function $f \circ q: X \rightarrow Y$ is continuous.*

Quotient topologies are not as uncommon as you might expect. A lot of topologies may be considered as quotient topologies, especially considering the characterization in the above theorem, Theorem 70

Let X be a space and let $f: X \rightarrow Y$ be a function onto the set Y . Define a subset O in Y to be *open* if its inverse image $f^{-1}(O)$ is open in X . The family of open sets so defined on Y is called the **quotient topology determined by f** .

Note that this actually is a topology. Clearly, $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$, so that those conditions are satisfied. If $\{U_\alpha \mid \alpha \in I\}$ are open in Y , then

$$f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(U_\alpha)$$

is open in X , so arbitrary unions of open sets are open in Y . If $\{U_i\}$ are open in Y , then

$$f^{-1}\left(\bigcap_{i=1}^n U_i\right) = \bigcap_{i=1}^n f^{-1}(U_i)$$

is open in X , so the finite intersection of open sets is open.

Theorem 71 *Let X and Y be spaces and $f: X \rightarrow Y$ a continuous, onto function. If f is either open or closed, then Y has the quotient topology determined by f .*

PROOF: Let $\mathcal{T}$ denote the topology on Y and let $\mathcal{T}_f$ denote the quotient topology induced by the function f . If $U \in \mathcal{T}$, then, since f is continuous, $f^{-1}(U)$ is open. This puts $U \in \mathcal{T}_f$. Thus, $\mathcal{T} \subset \mathcal{T}_f$.

Suppose that f is an open function and let $U \in \mathcal{T}_f$. Then $f^{-1}(U)$ is open in X . Since f is an open function with respect to $\mathcal{T}$, $f(f^{-1}(U)) = U$ is an open set in the topology $\mathcal{T}$. Thus, $\mathcal{T}_f \subset \mathcal{T}$, and we are done. ■

Example 53 [Adjunction Spaces] Let X and Y be topological spaces and let $A \subset X$ be non-empty. Assume that X and Y are disjoint and that there is a continuous function $f: A \rightarrow Y$. Define a function $\phi: X \cup Y \rightarrow (X \setminus A) \cup Y$ by

$$\phi(z) = \begin{cases} f(z) & \text{if } z \in A \\ z & \text{if } z \in X \setminus A \\ z & \text{if } z \in Y \end{cases}$$

The topology on $X \cup Y$ is that a set is open if and only if its intersection with both X and Y are open. Clearly, ϕ is onto. When we put the quotient topology induced by ϕ on the space $(X \setminus A) \cup Y$, we will denote this space by $X \cup_f Y$ and call it the **adjunction space via f** .

- a) If Y is a single point, then attaching X to a single point by the function $f: A \rightarrow \{a\}$ is the same as shrinking A to a point.
- b) Let $I^2 = I \times I$ be the unit square in the plane. Let $A = \{(x, y) \mid x = 0 \text{ or } x = 1 \text{ and } 0 \leq y \leq 1\}$. Let $Y = I = [0, 1]$ and define $f: I^2 \rightarrow Y$ by $f(x, y) = y$. Then $I^2 \cup_f Y$ is a cylinder formed by identifying the two vertical edges of I^2 .
- c) Let X be a topological space and let $Y = \{p_0, p_1\}$ be a two point discrete space not in $X \times [-1, 1]$. Let $A = X \times (\{-1\} \cup \{1\})$. Define $f: A \rightarrow Y$ be defined by $f(x, -1) = p_0$ and $f(x, 1) = p_1$. Since Y has the discrete topology, f is continuous. The adjunction space $\Sigma(X) = X \times [-1, 1] \cup_f Y$ is called the *suspension* of X . The *equator* is the image of $X \times \{0\}$ in $\Sigma(X)$. The image of $X \times [0, 1]$ in $\Sigma(X)$ is homeomorphic to the cone over X . Thus, the suspension of X is two cones over X identified along the equator. As an example, $\Sigma(S^1)$ is homeomorphic to the 2-sphere, S^2 .