

Problem Set 1 Solutions

1. (Exercise 2, page 11) *Prove that if A has precisely n distinct elements and B has precisely m distinct elements, where m and n are positive integers, then $A \times B$ has precisely mn distinct elements.*

We can prove this by appealing to a counting argument and showing that the product set is in one-to-one correspondence with an $m \times n$ matrix, which we know has mn elements.

2. (Exercise 3, page 11) *Let A and B be sets, both of which have at least two distinct members. Prove that there is a subset $W \subset A \times B$ that is not the product of a subset of A with a subset of B .*

Let $A = B = \{a, b\}$. Then $A \times B$ consists of 4 elements: $\{(a, a), (a, b), (b, a), (b, b)\}$. The subset $\{(a, a), (b, b)\}$ is not the product of a subset of A and a subset of B , because the subset of A would have to contain both a and b , as would the subset of B . Then, the product of those two subsets would be all of $A \times B$.

3. (Exercise 1, page 14) *Let $f: A \rightarrow B$ be given. Prove the following:*

- (a) *For each subset $X \subset A$, $X \subset f^{-1}(f(X))$.*

Let $a \in X$. Then $f(a) \in f(X)$ by definition. Thus, $a \in \{x \in X \mid f(x) \in f(X)\}$. This is just the set $f^{-1}(f(X))$. Thus, $X \subset f^{-1}(f(X))$.

- (b) *For each subset $Y \subset B$, $f(f^{-1}(Y)) \subset Y$.*

Let $y \in f(f^{-1}(Y))$. Thus, there is an element $x \in X$ so that $y = f(x)$. What is more, $x \in f^{-1}(Y)$, which means that $f(x) \in Y$. Thus, $y \in Y$ and $f(f^{-1}(Y)) \subset Y$.

- (c) *If f is injective, then for each subset $X \subset A$,*

$$f^{-1}(f(X)) = X.$$

We have shown that $X \subset f^{-1}(f(X))$, so it only remains to show that $f^{-1}(f(X)) \subset X$. Let $y \in f(X)$. Then there is an $x \in X$ so that $f(x) = y$. Let $a \in f^{-1}(f(x))$, i.e., $f(a) = f(x) = y$. Since f is one-to-one $a = x \in X$. Thus, any element of $f^{-1}(f(X))$ is in X and hence $f^{-1}(f(X)) \subset X$. It then follows that $f^{-1}(f(X)) = X$.

- (d) *If f is surjective, then for each subset $Y \subset B$,*

$$f(f^{-1}(Y)) = Y.$$

Again, we only need to show that $Y \subset f(f^{-1}(Y))$. Let $y \in Y$. We need to find an element $x \in A$ so that $y = f(x)$. What is more, we would like to show that this x actually belongs to $f^{-1}(Y)$. Since f is surjective, there is an element $x \in A$ so that $y = f(x)$, by the definition of surjectivity. Also, by definition of the set, $x \in f^{-1}(Y)$ since $f(x) = y \in Y$. Thus, $y \in f(f^{-1}(Y))$ and we are done.

4. (Exercise 4, page 14) Let $f: A \rightarrow B$ be given.

(c) If Y is a subset of B then $f^{-1}(B \setminus Y) = A \setminus (f^{-1}(Y))$.

We have to show the two set containments: $f^{-1}(B \setminus Y) \subset A \setminus (f^{-1}(Y))$ and $A \setminus (f^{-1}(Y)) \subset f^{-1}(B \setminus Y)$.

Let $x \in f^{-1}(B \setminus Y)$. Then $f(x) \in B \setminus Y$ or $f(x) \notin Y$. This means that $x \notin f^{-1}(Y)$ or $x \in A \setminus (f^{-1}(Y))$. Thus, $f^{-1}(B \setminus Y) \subset A \setminus (f^{-1}(Y))$.

Let $x \in A \setminus (f^{-1}(Y))$. Thus, $x \notin f^{-1}(Y)$ and $f(x) \notin Y$. Thus, $f(x) \in B \setminus Y$, making $x \in f^{-1}(B \setminus Y)$. Thus, $f^{-1}(B \setminus Y) \subset A \setminus (f^{-1}(Y))$ and we are done.

(d) If $X \subset A$ and $Y \subset B$, then

$$f(X \cap f^{-1}(Y)) = f(X) \cap Y.$$

Again, we have two set containments to show.

Let $v \in f(X \cap f^{-1}(Y))$. Then we know that $v = f(u)$ where $u \in X \cap f^{-1}(Y)$. Thus, $u \in X$ and $u \in f^{-1}(Y)$. It then easily follows that $f(u) \in f(X)$ and $f(u) \in Y$. Thus, $v = f(u) \in f(X) \cap Y$. Therefore $f(X \cap f^{-1}(Y)) \subset f(X) \cap Y$.

Now, let $v \in f(X) \cap Y$. Then $v \in Y$ and $v \in f(X)$. Thus, there is a $u \in X$ so that $f(u) = v$. Now, $f(u) \in Y$, which places $u \in f^{-1}(Y)$. Thus, $u \in X \cap f^{-1}(Y)$ and this places $v \in f(X \cap f^{-1}(Y))$ and $f(X) \cap Y \subset f(X \cap f^{-1}(Y))$. Thus the two sets are equal.

5. (Exercise 7, page 15) Let X be a set and $A, B, C \subset X$. The function $\chi_A: X \rightarrow \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is called the characteristic function of A . Show

(a) $\chi_{A \cap B} = \chi_A \cdot \chi_B$, where $\chi_A \cdot \chi_B(x) = \chi_A(x)\chi_B(x)$.

If $x \in A \cap B$ then $x \in A$ and $x \in B$. Thus, $\chi_{A \cap B}(x) = 1$, $\chi_A(x) = 1$ and $\chi_B(x) = 1$. In this case, $\chi_{A \cap B} = \chi_A \cdot \chi_B$.

If $x \notin A \cap B$, then there are three cases: (1) $x \in A$ and $x \notin B$; (2) $x \notin A$ and $x \in B$; and (3) $x \notin A$ and $x \notin B$. The proof for cases (1) and (2) are similar.

(1) If $x \in A$ and $x \notin B$, then $\chi_A(x) = 1$ and $\chi_B(x) = 0$ and $\chi_{A \cap B}(x) = 0$. Thus, in this case $\chi_{A \cap B} = \chi_A \cdot \chi_B$.

(3) If $x \notin A$ and $x \notin B$, then $\chi_A(x) = 0$ and $\chi_B(x) = 0$ and $\chi_{A \cap B}(x) = 0$. Thus, in this case $\chi_{A \cap B} = \chi_A \cdot \chi_B$.

Therefore, $\chi_{A \cap B} = \chi_A \cdot \chi_B$.

(b) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$ and find a similar expression for $\chi_{A \cup B \cup C}$.

If $x \in A \cup B$, then $x \in A$, $x \in B$, or x lies in both. Assume that $x \in A$ and $x \notin B$. Then $\chi_{A \cup B}(x) = 1$ and $\chi_A(x) = 1$, $\chi_B(x) = 0$, and $\chi_{A \cap B}(x) = 0$. Thus,

in this case $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$. If $x \in B$ and $x \notin A$, then we have a similar computation. If $x \in A$ and $x \in B$, then $\chi_A(x) = \chi_B(x) = \chi_{A \cap B}(x) = \chi_{A \cup B}(x) = 1$. Then, $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x) = 1 + 1 - 1$. Therefore $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$.

What we do in this case is we see how many times each point is counted. The points in $A \cap B$ are counted twice, once in χ_A and once in χ_B . Thus, for $\chi_{A \cup B \cup C}$ the points in $A \cap B$, $A \cap C$ and $B \cap C$ are counted twice. When we pull them out though, we pull out the points in $A \cap B \cap C$ three times – one for each of the intersections. Thus,

$$\chi_{A \cup B \cup C} = \chi_A + \chi_B + \chi_C - \chi_{A \cap B} - \chi_{A \cap C} - \chi_{B \cap C} + \chi_{A \cap B \cap C}.$$

6. Prove that $\emptyset \times B = \emptyset$ for each set B .

Assume not, that is assume that there is a point in $\emptyset \times B$. Then this point is an ordered pair (x, y) where $x \in \emptyset$ and $y \in B$. However, the empty set contains no elements, thus $x \notin \emptyset$. Thus, there is no point in $\emptyset \times B$. Thus, $\emptyset \times B = \emptyset$.

7. Give an example to show that $f(A \cap B)$ may not equal $f(A) \cap f(B)$. Show that equality holds if f is injective.

Let $f: \mathbf{R} \rightarrow \mathbf{R}$ by $f(x) = x^2$. Let $A = [-2, 1]$ and $B = [-1, 2]$. Then, $f(A) = [0, 4]$ and $f(B) = [0, 4]$ and $f(A) \cap f(B) = [0, 4]$. Now, $A \cap B = [-1, 1]$ and $f(A \cap B) = [0, 1]$. Thus, in this case, $f(A \cap B) \neq f(A) \cap f(B)$.

Clearly, we have that $f(A \cap B) \subset f(A) \cap f(B)$ since it is contained in each. If f is injective, we need to show the opposite inclusion. Let $x \in f(A) \cap f(B)$. Then there is an $a \in A$ and $b \in B$ so that $f(a) = x$ and $f(b) = x$. Thus, $f(a) = f(b)$. Since f is injective, this implies that $a = b$. Hence, $a \in A \cap B$ and $x \in f(A \cap B)$. Thus, $f(A) \cap f(B) \subset f(A \cap B)$ and we have that $f(A) \cap f(B) = f(A \cap B)$.