

Problem Set 2 Solutions

1. (Exercise 1, page 34) Let (X, d) be a metric space and let $k > 0$ be any positive real number. Let $d_k(x, y) = k \cdot d(x, y)$ for any two points $x, y \in X$. Show that (X, d_k) is a metric space.

We have to check the three conditions for a metric space.

(1) $d_k(x, y) = k \cdot d(x, y) \geq k \cdot 0 = 0$ since $k > 0$, and if $d_k(x, y) = 0$, then $k \cdot d(x, y) = 0$ which implies that $d(x, y) = 0$. This means that $x = y$ since d is a metric.

(2) $d_k(x, y) = k \cdot d(x, y) = k \cdot d(y, x) = d_k(y, x)$.

(3) Let $x, y, z \in X$,

$$\begin{aligned} d_k(x, z) &= k \cdot d(x, z) \leq k \cdot (d(x, y) + d(y, z)) \\ &= kd(x, y) + kd(y, z) \\ &= d_k(x, y) + d_k(y, z) \end{aligned}$$

Thus, d_k is a metric.

2. (Exercise 8, page 35) Let $\mathbb{Z}$ be the set of integers. Let p be a positive prime integer. Given distinct integers $m, n \in \mathbb{Z}$ there is a unique integer $t = t(m, n)$ such that $m - n = p^t \cdot k$, where k is an integer not divisible by p . Define a function $d: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$ by

$$d(m, n) = \begin{cases} 0 & \text{if } m = n \\ \frac{1}{p^t} & \text{if } m \neq n \end{cases}$$

Prove that $(\mathbb{Z}, d)$ is a metric space. Let $p = 3$. What is the set of elements $n \in \mathbb{Z}$ such that $d(0, n) < 1$? What is the set of elements $m \in \mathbb{Z}$ such that $d(0, m) < \frac{1}{3}$?

First, we need to prove that it is a metric space.

(1) From its definition $d(m, n) = 0$ or $d(m, n) = 1/p^t > 0$. Thus, $d(m, n) \geq 0$. Again, according to the definition, the only time that $d(m, n) = 0$ is if $m = n$. Thus, d satisfies the first condition on being a metric.

(2) If $d(m, n) = 0$ then $m = n$ and $d(n, m)$. Now, if $d(m, n) \neq 0$, then $m - n = p^t k$, $k \neq 0$. Thus, $n - m = p^t \cdot (-k)$ and hence $d(m, n) = 1/p^t = d(n, m)$.

(3) If $m, n, r \in \mathbb{Z}$, then let $m - n = p^t k$, $n - r = p^s \ell$. Hence,

$$\begin{aligned} m - r &= (m - n) + (n - r) \\ &= p^t k + p^s \ell \end{aligned}$$

Without loss of generality we may assume that $s < t$

$$= p^s (p^{t-s} k + \ell) = p^s \alpha$$

and p does not divide α .

Therefore,

$$\begin{aligned} d(m, r) &= \frac{1}{p^s} \\ &\leq \frac{1}{p^t} + \frac{1}{p^s} \\ &= d(m, n) + d(n, r) \end{aligned}$$

The argument is similar if $t < s$.

Thus, d is a metric on $\mathbb{Z}$.

Now, let $p = 3$. Now, if $m, n \in \mathbb{Z}$, then write $m - n = 3^t k$ and if $m \neq n$ then $d(m, n) = 1/3^t$. Let $A = \{n \in \mathbb{Z} \mid d(n, 0) < 1\}$. Now, this means that we write $n = 3^t k$ and we want $1/3^t < 1$. Thus, we want $t > 0$. Thus, $A = \{n \in \mathbb{Z} \mid 3|n\}$ which are the multiples of 3.

Let $B = \{m \in \mathbb{Z} \mid d(m, 0) < 1/3\}$. With our analysis above that means that we need to find all numbers $m = 3^t k$ so that $t > 1$. Thus, we must have that $t \geq 2$ and B consists of all of the multiples of 9.

3. (Exercise 1, page 39) Let X be the set of continuous functions $f: [a, b] \rightarrow \mathbb{R}$. Let d^* be the metric on X given by

$$d^*(f, g) = \int_a^b |f(t) - g(t)| dt,$$

for $f, g \in X$. For each element $f \in X$ set

$$I(f) = \int_a^b f(t) dt.$$

Prove that this function $I: (X, d^*) \rightarrow (\mathbb{R}, d)$ is continuous.

Let $\epsilon > 0$ be given. We need to find $\delta > 0$ so that if $d^*(f, g) < \delta$ then $d(I(f), I(g)) < \epsilon$. From a theorem in calculus, we know that $|\int f(x) dx| \leq \int |f(x)| dx$. Thus, we have

$$\begin{aligned} d(I(f), I(g)) &= \left| \int_a^b f(x) - g(x) dx \right| \\ &\leq \int_a^b |f(x) - g(x)| dx \\ &= d^*(f, g) \end{aligned}$$

Thus, if we take $\delta = \epsilon$, then $d(I(f), I(g)) \leq d^*(f, g) < \delta = \epsilon$. Thus, I is continuous.

4. For $P = (-2, 1)$ and $Q = (3, 4)$ in $\mathbb{R}^2$, compute the distance from P to Q in each of the following metrics:

(a) *usual*;

$$d(P, Q) = \sqrt{(3 - (-2))^2 + (4 - 1)^2} = \sqrt{34}.$$

(b) *taxicab*;

$$d'(P, Q) = |3 - (-2)| + |4 - 1| = 8.$$

(c) *maximum*;

$$d''(P, Q) = \max\{|3 - (-2)|, |4 - 1|\} = 5.$$

(d) *discrete*.

Since $P \neq Q$, $d(P, Q) = 1$.

5. Let $B = \{P = (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$. Compute the diameter of B in each of the following metrics:

(a) *usual*;

The diameter is the least upper bound of the distances apart of two points in this set. In the usual metric, we know that the greatest distance will be diametrically opposed points. This gives a diameter of 2.

(b) *taxicab*;

Let $P = (x, y), Q = (a, b) \in B$. Then $|x| + |y| \leq 1$ and $|a| + |b| \leq 1$. Now,

$$\begin{aligned} d'(P, Q) &= |x - a| + |y - b| \\ &\leq |x| + |a| + |y| + |b| \\ &\leq (|x| + |y|) + (|a| + |b|) = 2 \end{aligned}$$

Thus, the least upper bound of these numbers is 2. The diameter of the set is 2.

(c) *maximum*;

Let $P = (x, y), Q = (a, b) \in B$. Then $\max\{|x|, |y|\} \leq 1$ and $\max\{|a|, |b|\} \leq 1$. Now,

$$\begin{aligned} d''(P, Q) &= \max\{|x - a|, |y - b|\} \\ &\leq \max\{|x| + |a|, |y| + |b|\} \\ &\leq 2 \end{aligned}$$

Thus, the least upper bound of these numbers is 2. The diameter of the set is 2.

(d) *discrete*.

In the discrete metric the greatest distance apart two points can be is 1. Thus the diameter must be 1.