

MATH 4181 001

Fall 1999

Problem Set 3 Solutions

1. (Exercise 6, page 46) Let a and b be distinct points in a metric space X . Prove that there are neighborhoods N_a and N_b of a and b respectively such that $N_a \cap N_b = \emptyset$.

Since a and b are distinct points, $d(a, b) = r > 0$. Let $N_a = B_d(a; r/3)$ and $N_b = B_d(b; r/3)$. Now, clearly, $N_a \cap N_b = \emptyset$.

2. Let $\{x_n\}$ be a sequence in the metric space X and assume that $\{x_n\}$ converges to $x \in X$. Prove that this limit is unique, i.e., if $\lim x_n = x$ and $\lim x_n = L$ then $x = L$.

We will use the result from above to prove this. Assume that $x \neq L$. Then there are disjoint neighborhoods N_x and N_L of x and L , respectively. In fact, if we let $r = d(x, L) \neq 0$ (by assumption), then we may take each neighborhood to be the ball of radius $r/3$ centered at each respective point. Now, since the sequence converges to x we know that for any $\epsilon > 0$ there is a positive integer N_ϵ so that for any $n > N_\epsilon$ $x_n \in B_d(x; \epsilon)$. This must be true for $\epsilon = r/3$. Thus, there is a positive integer N so that for all $n > N$, $x_n \in B(x; r/3)$. Applying the same analysis to the limit L of the sequence $\{x_n\}$ gives us a positive integer M so that for all $n > M$ $x_n \in B(L; r/3)$.

Let $P = \max\{M, N\}$. Now, if $n > P$ we must have that $x_n \in B(x; r/3)$ and $x_n \in B(L; r/3)$, or $x_n \in B(x; r/3) \cap B(L; r/3) = \emptyset$. This is impossible, so we must have $x = L$.

3. Show that a finite subset of a metric space has no limit points and is therefore a closed set.

Let $A = \{x_1, \dots, x_n\}$ be a finite set in the metric space (X, d) . Let $y \notin A$, then let $d_k = d(y, x_k)$. Since $y \notin A$ $d_k \neq 0$ for $k = 1, \dots, n$. Let $r = \min\{d_1, \dots, d_n\}$. Then for any $\epsilon < r$, $B(y; \epsilon)$ does not contain any elements of A , because they are all further away from y than ϵ . Thus, y is not a limit point of A . If $y \in A$ we use the same technique to construct a ball about y which contains no other points of A . Thus showing that y is not a limit point of A . Therefore, A has no limit points, $A' = \emptyset$. This then means that A contains all of its limit points, since $\emptyset \subset A$. Therefore by our theorem, A must be closed.

4. Let (X, d) be a metric space with the discrete metric. Prove

(a) Every subset of X is open.

Note that $B(x; 1/2) = \{x\}$ is an open set. Thus, each singleton set is an open set. Every set is a union of its elements and therefore is a union of open sets. Therefore every set is open.

(b) Every subset of X is closed.

For any set F , we know that $X \setminus F$ is open from above. Thus, by definition, F is closed.

(c) *No subset of X has a limit point.*

Using the same open set as the first part shows that each point has an open neighborhood that contains no other point of any other set. Thus, each point cannot be a limit point of any set. Thus, no subset of X has a limit point.