

MATH 4181 001

Fall 1999

Problem Set 4 Solutions

1. (Exercise 4, page 74) *Let $(X, \mathcal{T})$ be a topological space. Prove that $\emptyset$ and X are closed sets, that a finite union of closed sets is a closed set, and that an arbitrary intersection of closed sets is a closed set.*

Let $(X, \mathcal{T})$ be a topological space. To show that a set is closed we must show that its complement is open. Since $X \setminus X = \emptyset$ is open, X must be closed. Likewise, since $X \setminus \emptyset = X$ is open, the empty set is closed.

Let $\{F_i\}_{i=1}^n$ be a finite collection of closed sets. To show that their union is closed, we must show that the complement of the union is open.

$$X \setminus \left(\bigcup_{i=1}^n F_i \right) = \bigcap_{i=1}^n (X \setminus F_i)$$

and each of the sets $X \setminus F_i$ is open since F_i is closed. Since the finite intersection of open sets is open, we have then that $X \setminus (\bigcup_{i=1}^n F_i)$ is open, hence making $\bigcup_{i=1}^n F_i$ closed.

Let $\{F_\alpha \mid \alpha \in I\}$ be a collection of closed sets. To show that the arbitrary intersection is closed, we must show that its complement is open.

$$X \setminus \left(\bigcap_{\alpha \in I} F_\alpha \right) = \bigcup_{\alpha \in I} (X \setminus F_\alpha)$$

and each of the sets $X \setminus F_\alpha$ is open since F_α is closed. Since the arbitrary union of open sets is open, we have then that $X \setminus (\bigcap_{\alpha \in I} F_\alpha)$ is open, hence making $\bigcap_{\alpha \in I} F_\alpha$ closed.

2. (Exercise 6, page 75) *Prove that in a discrete topological space, each subset is simultaneously open and closed.*

We already know that each subset is open, by the definition of the discrete topology. Let $A \subset X$. Then $X \setminus A$ is a subset of X and hence open. Thus, A is closed. Therefore, every subset of a discrete topological space is simultaneously open and closed.

3. *Show that a topological space $(X, \mathcal{T})$ is discrete if and only if each set consisting of only one point is open.*

If $(X, \mathcal{T})$ is discrete then every subset is open. Hence the singleton sets are open.

If $(X, \mathcal{T})$ has the property that the singleton sets are open, then each subset of X being a union of its elements is a union of open sets and is thus open. Thus, every subset is open, making $(X, \mathcal{T})$ a discrete topological space.

4. Let X be a set and let $\mathcal{T}'$ the finite complement topology for X .

(a) Show that $(X, \mathcal{T}')$ is discrete if and only if X is a finite set.

If X is a finite set, then the complement of a singleton set is finite. Thus, the singleton sets are open. By the previous problem that gives X the discrete topology. If the finite complement topology and the discrete topology coincide on the space X , then the singleton set $\{a\}$ must be open (by the discrete topology). Since it is open, its complement must be finite (by the finite complement topology). But $X = \{x\} \cup (X \setminus \{x\})$. Each of the two sets on the right are finite, making X a finite set.

(b) Show that if A is an infinite subset of X , then every point of X is a limit point of A .

Let A be an infinite set and let $x \in X$. If U is any open set containing x , then $X \setminus U$ is a finite set. If $A \subset X \setminus U$, then we would have a finite set with an infinite subset — which is impossible. Thus, $A \cap U \neq \emptyset$. This means that x is a limit point of A . Since $x \in X$ was arbitrary, each point of X is a limit point of A .

5. Let X be a set. The countable complement topology, or co-countable topology, $\mathcal{T}''$ for X consists of X , $\emptyset$ and all subsets O of X for which $X \setminus O$ is a countable set.

(a) Show that $\mathcal{T}''$ is a topology on X .

Clearly, $\emptyset$ and X are in $\mathcal{T}''$, by definition.

Let $\{O_\alpha \mid \alpha \in I\}$ be a collection of open sets. Then, $X \setminus O_\alpha$ is countable in X . To see that $\bigcup_{\alpha \in I} O_\alpha$ is open, we need look at $X \setminus \bigcup_{\alpha \in I} O_\alpha$.

$$X \setminus \bigcup_{\alpha \in I} O_\alpha = \bigcap_{\alpha \in I} (X \setminus O_\alpha) \subset X \setminus O_\alpha.$$

Since each complement is countable, and the subset of a countable set is countable, we have that $X \setminus \bigcup_{\alpha \in I} O_\alpha$ is countable and $\bigcup_{\alpha \in I} O_\alpha$ is open.

Let $\{O_i\}_{i=1}^n$ be a finite collection of open sets. Then, $X \setminus O_i$ is countable in X . To see that $\bigcap_{i=1}^n O_i$ is open, we need look at $X \setminus \bigcap_{i=1}^n O_i$.

$$X \setminus \bigcap_{i=1}^n O_i = \bigcup_{i=1}^n (X \setminus O_i) \subset X \setminus O_n.$$

Since each complement is countable, and the union of a finite number of countable sets is countable, we have that $X \setminus \bigcap_{i=1}^n O_i$ is countable and $\bigcap_{i=1}^n O_i$ is open.

(b) For the space $(X, \mathcal{T}'')$, show that a countable set A of X has a derived set $A' = \emptyset$ and that an uncountable set B has $B' = X$.

Let A be countable and let $x \in X$. If $x \notin A$, then $X \setminus A$ is an open set containing x and $A \cap X \setminus A = \emptyset$.

If $x \in A$, then $(X \setminus A) \cup \{x\}$ is an open set containing a and $(A \setminus \{x\}) \cap (X \setminus A \cup \{x\}) = \emptyset$.

In neither case is x a limit point of A . Hence, no points are limit points of A , or $A' = \emptyset$.

If B is uncountable, let $x \in X$ be any point and $U \subset X$ be any open set with $x \in U$. Since $X \setminus U$ is countable, we cannot have $B \subset X \setminus U$. This means that $U \cap B \neq \emptyset$. Thus, each point is a limit point for B and $B' = X$.

- (c) Show that the intersection of any countable family of members of $\mathcal{T}''$ is a member of $\mathcal{T}''$.

Let $\{O_i\}_{i=1}^{\infty}$ be a countable collection of open sets. Then, $X \setminus O_i$ is countable in X . To see that $\bigcap_{i=1}^{\infty} O_i$ is open, we need look at $X \setminus \bigcap_{i=1}^{\infty} O_i$.

$$X \setminus \bigcap_{i=1}^{\infty} O_i = \bigcap_{i=1}^{\infty} (X \setminus O_i) \subset X \setminus O_1.$$

Since each complement is countable, and the union of a countable number of countable sets is countable, we have that $X \setminus \bigcap_{i=1}^{\infty} O_i$ is countable and $\bigcap_{i=1}^{\infty} O_i$ is open.

6. Let $X = \{a, b\}$ be a two-element set and let $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}\}$. Show that $\mathcal{T}$ is a topology on X and identify the limit points of each subset of X . (This space is called Sierpinski space.)

Clearly, $\mathcal{T}$ contains the empty set and the whole space. It is easy to see that it is closed under unions and intersections. Thus, it is a topology on X .

For the limit points we have

	Limit Points
$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$
$\{b\}$	$\emptyset$
$\{a, b\}$	$\{b\}$

7. *How many different topologies are there for a set with three members?*

Since there are 8 subsets of this space, there are at most 2^8 possibilities for topologies. Fortunately, there are not that many.

It is probably best just to list the collections that form a topology on this set $X = \{a, b, c\}$.

$$\begin{array}{ll}
 \mathcal{T}_1 = \{\emptyset, X\} & \mathcal{T}_2 = \{\emptyset, \{a\}, X\} \\
 \mathcal{T}_3 = \{\emptyset, \{b\}, X\} & \mathcal{T}_4 = \{\emptyset, \{c\}, X\} \\
 \mathcal{T}_5 = \{\emptyset, \{a, b\}, X\} & \mathcal{T}_6 = \{\emptyset, \{a, c\}, X\} \\
 \mathcal{T}_7 = \{\emptyset, \{b, c\}, X\} & \mathcal{T}_8 = \{\emptyset, \{a\}, \{a, b\}, X\} \\
 \mathcal{T}_9 = \{\emptyset, \{a\}, \{a, c\}, X\} & \mathcal{T}_{10} = \{\emptyset, \{a\}, \{b, c\}, X\} \\
 \mathcal{T}_{11} = \{\emptyset, \{b\}, \{a, b\}, X\} & \mathcal{T}_{12} = \{\emptyset, \{b\}, \{a, c\}, X\} \\
 \mathcal{T}_{13} = \{\emptyset, \{b\}, \{b, c\}, X\} & \mathcal{T}_{14} = \{\emptyset, \{c\}, \{a, b\}, X\} \\
 \mathcal{T}_{15} = \{\emptyset, \{c\}, \{a, c\}, X\} & \mathcal{T}_{16} = \{\emptyset, \{c\}, \{b, c\}, X\} \\
 \mathcal{T}_{17} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} & \mathcal{T}_{18} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\} \\
 \mathcal{T}_{19} = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\} & \mathcal{T}_{20} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\} \\
 \mathcal{T}_{21} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\} & \mathcal{T}_{22} = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\} \\
 \mathcal{T}_{23} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\} & \mathcal{T}_{24} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\} \\
 \mathcal{T}_{25} = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\} & \mathcal{T}_{26} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\} \\
 \mathcal{T}_{27} = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\} & \mathcal{T}_{28} = \{\emptyset, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\} \\
 \mathcal{T}_{29} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} &
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