

MATH 4181 001

Fall 1999

Problem Set 5 Solutions

1. Let $f: X \rightarrow Y$ be a function. Then f is an **open mapping** if for each open set $O \subset X$, $f(O)$ is open in Y .

(a) Give an example of a mapping that is continuous, but not open.

Let $X = Y = \mathbb{R}$. Let $\mathcal{T}_X$ denote the discrete topology on X and $\mathcal{T}_Y$ denote the usual, metric topology on Y . Define $f: (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ by $f(x) = x$. Then, f is a continuous mapping, because the inverse image of any set is open in $(X, \mathcal{T}_X)$. However, f is not open since $f(\{0\}) = \{0\}$ which is not open in $(Y, \mathcal{T}_Y)$.

(b) Give an example of a mapping that is open, but not continuous.

Use the same notation as in (a) and define $g: (Y, \mathcal{T}_Y) \rightarrow (X, \mathcal{T}_X)$ by $g(x) = x$. Then, g is open, since the image of any set is open in $(X, \mathcal{T}_X)$, but g is not continuous since, for example, the inverse image of the open set $\{0\}$ is not open in $(Y, \mathcal{T}_Y)$.

(c) Prove that a one-to-one, onto mapping $f: X \rightarrow Y$ is a homeomorphism if and only if f and f^{-1} are open mappings.

Let f be a homeomorphism. Then we know that f and f^{-1} are both continuous. We also know that $f = (f^{-1})^{-1}$, so that the statement that f is open is equivalent to saying that f^{-1} is continuous. Likewise, the statement that f^{-1} is open is equivalent to saying that f is continuous. Thus, f and f^{-1} are open mappings.

If we know that f is one-to-one and onto, then assuming that f and f^{-1} are open mappings makes f and f^{-1} both continuous. Thus, f is a homeomorphism.

2. Prove that a finite subset A of a Hausdorff space X has no limit points. Conclude that A must be closed.

Let $A = \{x_1, \dots, x_n\}$ and let p be a limit point of A . Then, for every open set, U , containing p we must have that $U \cap A \setminus \{p\} \neq \emptyset$. Since X is Hausdorff, there are open sets U_i containing p and V_i containing x_i so that $U_i \cap V_i = \emptyset$. Let $U = \bigcap_{i=1}^n U_i$ and let $V = \bigcup_{i=1}^n V_i$. Then,

- (i) $p \in U$;
- (ii) $A \subset V$;
- (iii) $U \cap V = \emptyset$.

Thus, we found a neighborhood of p which does not intersect A . Hence, p cannot be a limit point of A and A has no limit points.

Since $A' = \emptyset$, we have that $\overline{A} = A \cup A' = A$, making A closed.

3. If X is a space which is homeomorphic to a subspace A of a space Y , then X is said to be **embedded** in Y . Give an example of spaces A and B for which A can be embedded in B and B can be embedded in A , but A and B are not homeomorphic. (Simple examples can be found in $\mathbb{R}$.)

Let $A = (0, 1)$ and let $B = [0, 1]$. Clearly, A can be embedded in B using the identity function, $f(x) = x$. Define a map $g: B \rightarrow A$ by $g(x) = \frac{1}{2}x + \frac{1}{4}$. This maps B onto the interval $[\frac{1}{4}, \frac{3}{4}]$ embedding it into A .

Thus, we can embed A into B and B into A . However, we know that the two intervals $(0, 1)$ and $[0, 1]$ are not homeomorphic.

4. Prove that every countable subset of $\mathbb{R}$ is totally disconnected.

We shall prove in Problem 7a that the property of being totally disconnected is a topological invariant. Since we have shown that $\mathbb{Q}$ is totally disconnected. Since $\mathbb{Q}$ is countable it is homeomorphic to every countable subset of $\mathbb{R}$. Hence, every countable subset of $\mathbb{R}$ is totally disconnected.

5. Give examples of subsets A and B in $\mathbb{R}^2$ to illustrate each of the following. A drawing is sufficient.

- (a) A and B are connected, but $A \cap B$ is disconnected.

Let A be the segment on the x -axis, $A = \{(x, 0) \mid -1 \leq x \leq 1\}$. Let B denote the upper hemisphere of the unit circle: $B = \{(x, y) \mid x^2 + y^2 = 1 \text{ and } y \geq 1\}$. Then each of A and B is connected but the intersection is the two disjoint points $(-1, 0)$ and $(1, 0)$ which are disconnected.

- (b) A and B are connected, but $A \setminus B$ is disconnected.

Let A be the rectangle in the plane with vertices at $(2, 1)$, $(-2, 1)$, $(-2, -1)$ and $(2, -1)$. Let B be the square in the plane with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$ and $(1, -1)$. Then $A \setminus B$ is two disconnected squares.

- (c) A and B are disconnected, but $A \cup B$ is connected.

Take A to be the two vertical sides of the unit square: $A = \{(1, t) \mid -1 \leq t \leq 1\} \cup \{(-1, t) \mid -1 \leq t \leq 1\}$. Take B to be the two horizontal sides of the unit square: $B = \{t, 1\} \mid -1 \leq t \leq 1\} \cup \{t, -1\} \mid -1 \leq t \leq 1\}$. Then, each of A and B is disconnected, but $A \cup B$ is connected.

- (d) A and B are connected and $\overline{A} \cap \overline{B} \neq \emptyset$, but $A \cup B$ is disconnected.

Let $A = (0, 1)$ and $B = (1, 2)$. Then A and B are connected,

$$\overline{A} \cap \overline{B} = [0, 1] \cap [1, 2] = \{1\} \neq \emptyset,$$

but $A \cup B = (0, 1) \cup (1, 2)$ is disconnected.

6. **Definition:** A Hausdorff space X is **0-dimensional** if X has a basis $\mathcal{B}$ of sets which are simultaneously open and closed.

Prove that every 0-dimensional space is totally disconnected.

Let X be a 0-dimensional space. Let C be a component of X containing the point $x \in X$. Let $y \in C$. Since X is Hausdorff, there are disjoint open sets, U and V , containing x and y , respectively. Now, each of these open sets consists of a union of basic open sets, thus, we can separate x and y by disjoint basic open sets. Thus, there are sets U_x and V_y , disjoint and open and closed, since X is 0-dimensional. Then, C contains sets which are both open and closed, making C not connected. The only way in which this can be prevented is for $C = \{x\}$. Thus, each component consists of a single point and X is totally disconnected.

7. *Prove:*

- (a) *The property of being totally disconnected is a topological invariant but not a continuous invariant.*

Let $f: X \rightarrow Y$ be a homeomorphism and assume that X is totally disconnected. Let C be a connected component of Y . Then $f^{-1}(C)$ is a connected subset of X , being the continuous image of a connected set. If $x \in f^{-1}(C)$, then $f^{-1}(C)$ lies in the connected component of X containing x . Since X is totally disconnected, this component is x . Thus, $f^{-1}(C)$ is a single point. Since f is a homeomorphism, C consists of a single point. Thus, Y is totally disconnected.

Let $(X, \mathcal{T})$ denote the reals with the discrete topology. Then $(X, \mathcal{T})$ is totally disconnected. Let $(Y, \mathcal{S})$ denote the reals with the usual topology. Define $f: X \rightarrow Y$ by $f(x) = x$. This function is continuous, since the domain has the discrete topology, but the image space is connected, not totally disconnected.

- (b) *The property of being totally disconnected is hereditary.*

Let X be totally disconnected and let $A \subset X$. Let $C \subset A$ be the component of A containing $x \in A$. Let $y \in C$. Now, y is disconnected from x in X , since X is totally disconnected. The same disconnection in X will disconnect x and y in A . Hence, C cannot contain more than one point, and A is totally disconnected.