

MATH 4181 001

Fall 1999

## Problem Set 6 Solutions

1. *Identify the two path components of the topologist's sine curve. Show that one of the components is closed and the other is not.*

The two path components of the topologist's sine curve are the component along the  $y$ -axis,  $A = \{(0, y) \mid -1 \leq y \leq 1\}$ , and the  $\sin(\pi/x)$  curve component:  $B = \{(x, \sin(\pi/x)) \mid 0 < x \leq 1\}$ . We proved in class that there is no path joining any point of  $A$  to any point of  $B$ . The component  $A$  is closed, but  $B$  does not contain its limit points, which consist of  $A$ , and hence  $B$  is not closed.

2. *Prove that path connectedness is a topological invariant. Can you prove that it is a continuous invariant?*

Let  $f: X \rightarrow Y$  be a homeomorphism and assume that  $X$  is path connected. Let  $a, b \in Y$  be any two distinct points. Since  $f$  is onto, there are points  $a', b' \in X$  so that  $f(a') = a$  and  $f(b') = b$ . Since  $X$  is path connected, there is a path,  $p: I \rightarrow X$ , so that  $p$  is continuous,  $p(0) = a'$  and  $p(1) = b'$ . Let  $q = f \circ p$ . Then  $q$  is continuous, being the composition to two continuous functions;  $q(0) = f(a') = a$  and  $q(1) = f(b') = b$ . Thus, any two points of  $Y$  are connected by a path, meaning that  $Y$  is path connected.

All we required in the above proof is that  $f$  was onto and continuous. Thus, the continuous image of a path connected set is path connected, using the same proof.

3. *Prove that local connectedness is a topological invariant, but that the continuous image of a locally connected set need not be locally connected.*

Let  $f: X \rightarrow Y$  be a homeomorphism and assume that  $X$  is locally connected. Let  $p \in Y$  contained in  $U \subset Y$  an open set. We need to show that there is an open, connected set containing  $p$  and contained in  $U$ . Let  $x = f^{-1}(p)$ . Then  $x \in f^{-1}(U)$  which is open in  $X$ . Since  $X$  is locally connected, there is an open connected set,  $C$ , containing  $x$  and contained in  $f^{-1}(U)$ . We then know that  $f(C)$  is connected and  $p = f(x) \in f(C)$ . Since  $f$  is a homeomorphism,  $f$  is an open map and  $f(C)$  is open. Hence,  $Y$  is locally connected.

Note that this time, we did need the fact that  $f$  is a homeomorphism to guarantee that  $f(C)$  was open. Thus, we should not expect the continuous image of a locally connected set to be locally connected, because not every continuous map is open. As an example, take the identity function that maps the rationals with the discrete topology to the rationals with the usual topology. We showed in class that the rationals with the usual topology is not locally connected. However, any discrete space is locally connected, specifically because each point is an open set.

4. *Prove that every locally path connected space is locally connected.*

Let  $X$  be a locally path connected space and let  $x \in X$  be contained in an open set  $U$ . Now,  $U$  contains an open path connected set containing  $x$ , and any path connected

set is connected. Thus,  $U$  contains an open connected set containing  $X$ . Hence,  $X$  is locally connected.

5. *Prove that every finite subset of a topological space is compact.*

Let  $A = \{a_1, a_2, \dots, a_n\}$  be a finite set in  $X$ . Let  $\mathcal{O}$  be an open cover of  $A$ . Each  $a_i$  must be contained in at least one open set  $O_i$  from the open cover  $\mathcal{O}$ . Hence, we only need the sets  $\{O_1, O_2, \dots, O_n\}$  to cover  $A$ . This makes  $A$  compact.