A Math Problem PhL 12.10.13

Overview: I had a simple trig function $f(x) = a\cos^2(x) + b\sin^2(x-c)$ which I could see had four places in the range $(-\pi.\pi)$ where the slope is zero. But my usual calculus method for finding these zero-slope points was only showing up the maxima and not the minima. The paradox was reconciled when I realized that the $w = (1/2)\tan^{-1}(z)$ solution really needed $N\pi/2$ added to it due to the multiple branches, and then this filled in the missing zero slope values. I then learned how Maple displays these extra solutions using its inscrutable _Z notation, and I updated my user's guide with this new info.

1. Setting the Stage

Consider this function. where a and b are positive real, c is real as well,

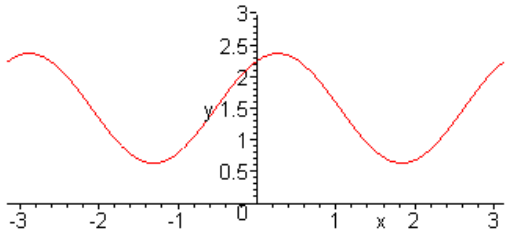
$f(x) = a\cos^2(x) + b\sin^2(x-c)$.

Here is what this function looks like

```
f := a*cos(x)^2 + b*sin(x-c)^2;
```

$$f := a\cos(x)^2 + b\sin(x-c)^2$$

```
a := 2: b := 1: c := -Pi/6:
plot(f,x=-Pi..Pi,y=0..3);
```



With the range $-\pi$ to π there are two maxima and two minima. We wish to find these locations, so we take the derivative of f(x)

```
fd1 := diff(f,x);
```

$$fd1 := -2a\cos(x)\sin(x) + 2b\sin(x-c)\cos(x-c)$$

which we write as

$f'(x) = -a\sin(2x)+b\sin(2x-2c)$

To find the locations of zero slope we write4

$-a\sin(2x)+b\sin(2x-2c) = 0$

or

$a\sin(2x) = b\sin(2x-2c)$

or

asin(2x) = bsin(2x)cos(2c)-bcos(2x)sin(2c)

or

[a-bcos(2c)] sin(2x)= -bsin(2c)cos(2x) huh??????

or

tan(2x) = [-bsin(2c)]/[a-bcos(2c)]

We are now going to make some name changes. First, replace x by θ since we are talking about angles

tan(2θ) = [-bsin(2c)]/[a-bcos(2c)]

Second, let x = num and y = den, so we then have

tan(2θ) = y/x

In our example above we have y and x both positive, since

```
y := -b*sin(2*c);
```

$$y := \frac{1}{2}\sqrt{3}$$

```
x := a - b*cos(2*c);
```

$$x := \frac{3}{2}$$

2. Solving a certain equation

We seek the solution θ to this last equation. Note that y and x could each have either sign. Now finally let φ = 2θ so this equation then reads

tan(φ) = y/x

which is our "canonical form" which we studied in the Earth Sun appendix, from which I quote :

Consider these two equations,

$x = \cos\varphi$
$y = \sin\varphi$

where φ runs through the range $(0,2\pi)$ so that $\mathbf{r} = (x,y)$ moves in a circle. Given x and y, how do we find φ? Consider this picture,

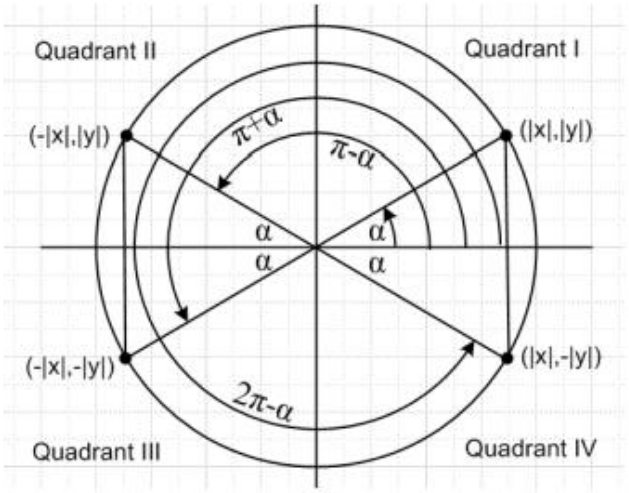


Fig A.1

From the signs of x and y, we can certainly determine the quadrant of (x,y). So here is how we find φ :

If (x,y) is in Quadrant I, then $\varphi = \alpha$ where $\alpha \equiv \tan^{-1}(|y|/|x|)$
If (x,y) is in Quadrant II, then $\varphi = \pi - \alpha$
If (x,y) is in Quadrant III, then $\varphi = \pi + \alpha$
If (x,y) is in Quadrant IV, then $\varphi = 2\pi - \alpha$. (A.1)

This shows that there are *four* different solutions of $\tan(\varphi) = y/x$ depending on the quadrant defined by

x>0	y>0	Quadrant I	$\varphi = \alpha \equiv \tan^{-1}(|y|/|x|)$	solution #1
x<0	y>0	Quadrant II	$\varphi = \pi-\alpha$	solution #2
x>0	y<0	Quadrant III	$\varphi = \pi+\alpha$	solution #3
x<0	y<0	Quadrant IV	$\varphi = 2\pi-\alpha$	solution #4

Let us *suppose* that x > 0 and y > 0 (as in our example). Then we are Quadrant I and there is only one solution which is the above solution #1. That solution is then $\varphi = \alpha$. Translating back to our problem we find

$$2\theta_s = \tan^{-1}(y/x)$$

This then is the ONLY solution and it is in quadrant I for 2θ. Perhaps the answer is $\theta_s = 20°$.

[This is the only solution *on the principle branch* of the function $w = \tan^{-1}(z)$. But as Spiegel p 18 shows, there are other branches, and in fact if $w = \tan^{-1}(z)$ is a solution on the principle branch, then other

solutions are w + Nπ where N is any integer. Thus, if $2\theta_s$ is a solution, so is $2\theta_s + N\pi$. This can be restated: if θ_s is a solution, then so is $\theta_s + N(\pi/2)$. This then gives us not one but four solutions where θ_s takes the range (-π,π).]

Now go back to our starting function

$$f(\theta) = a\cos^2(\theta) + b\sin^2(\theta - c) \quad .$$

This function is periodic with period π so we have

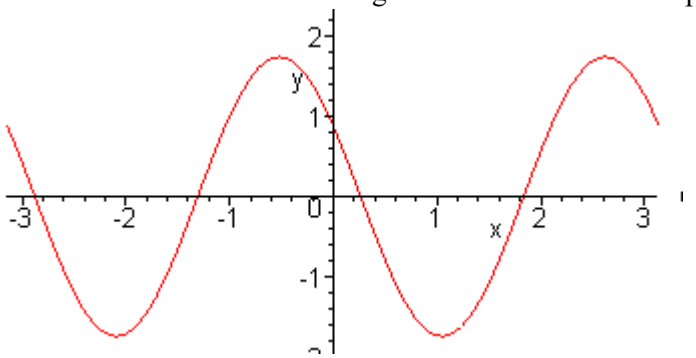
$$f(\theta) = f(\theta \pm \pi)$$

Therefore, if $\theta = \theta_s$ is a place where f(θ) has zero slope, f(θ) also has zero slope at $\theta_s \pm \pi$ which is in agreement with our plot above. Thus, this function has zero slope at $20° \pm \pi$.

[This fact is true, but we already have the conclusion we want without using this last fact .]

3. My confusion and why I am treating this Math Problem

My solution method has "discovered" the two points where f(x) is a maximum, but it seems to be overlooking the two places where f(x) is minimum! What have I done wrong? Let's now plot the derivative and see if it really hits 0 four times in (-π,π). I have plotted both fd1 and fd1m (manual) wherein I did the sin2x = 2sinxcosx thing in each term. Both these plots are the same and here it is:



You clearly see the four zero crossings. So why does my method seem to miss two of these crossings? For my example numbers we have

$$\tan(2\theta) = y/x \quad = [\sqrt{3}/2]/[3/2] = 1/\sqrt{3}$$

Now, the function tax(ξ) is periodic with period π, as shown Spiegel page 14. Thus

$$\tan(\xi) = \tan(\xi \pm \pi)$$

That means in the above we have

$$\tan(2\theta) = \tan(2\theta \pm \pi)$$

Thus, if $2\theta_s$ is a solution of tan(2θ) = y/x, then so is $2\theta_s \pm \pi$.
Thus, if θ_s is a solution of tan(2θ) = y/x, then so is $\theta_s \pm \pi/2$.

So this explains the mystery, finally. I now go back and add some comments in red.

4. How Maple deals with this fact:

Several interesting facts here, and I want to get this down for future reference.

(a) Maple is *unable* to solve the slope=0 equation before I go in and rewrite it manually:

```
f := a*cos(x)^2 + b*sin(x-c)^2;
```

$$f := a\cos(x)^2 + b\sin(-x+c)^2$$

```
fd1 := diff(f,x);
```

$$fd1 := -2a\cos(x)\sin(x) - 2b\sin(-x+c)\cos(-x+c)$$

```
sols := solve(fd1=0,x);
```

$$sols :=$$

but it does not set the env variable that Help talks about

```
_SolutionsMayBeLost;
```

$$_SolutionsMayBeLost$$

Maple is saying that "there are no solutions" in its behavior, but that is wrong. It just cannot find the solution because it does not make use of sin(2x) = 2sin(x)cos(x). However, we can *make* it use this identity in the following manner, and then it CAN find a solution

```
fd1m := combine(fd1,trig);
```

$$fd1m := -a\sin(2x) - b\sin(-2x+2c)$$

```
sols := solve(fd1m=0,x);
```

$$sols := \frac{1}{2}\arctan\left(\frac{b\sin(2c)}{-a+b\cos(2c)}\right)$$

(b) Maple knows about the π/2 fact, but does not say so unless you ask:

```
_EnvAllSolutions := true:
sols := solve(fd1m=0,x);
```

$$sols := \frac{1}{2}\arctan\left(\frac{b\sin(2c)}{-a+b\cos(2c)}\right) + \frac{1}{2}\pi_Z$$

This last mysterious notation means (π/2) times any integer in the field Z of real numbers. I finally found something on this notation in the official Maple manual

Transcendental Equations: In general, the **solve** command returns one solution to transcendental equations.

```
> equation1 := sin(x) = cos(x) :

> solve(equation1)
```

$$\frac{1}{4}\pi$$

To produce all solutions, use the **allsolutions** option.

```
> solve(equation1, allsolutions = true)
```

$$\frac{1}{4}\pi + \pi_Z1\sim$$

Maple uses variables of the form **_ZN~**, where **N** is a positive integer, to represent arbitrary integers. The tilde (~) indicates that it is a quantity with an assumption. For information about names with assumptions, see *Assumptions on Variables (page 146)*.

So probably if I had the current version of Maple, I would see $\frac{1}{2}\pi_Z1$ instead of $\frac{1}{2}\pi_Z$, and this then allows for multiple possible integers. This is a different use than in the RootOf stuff. I just added all this _Z stuff to my user guide. Hurray!