About e PhL 7.21.08

Euler is credited for "finding out about e". Here are some meanderings.

1. About e^x being its own derivative.

Plan A. Let's start with arbitrary exponentials and logs. People certainly knew what was meant by $f(x) = a^x$ for x = integer values, and could plot same (a real), and could see an interpolated function hitting these values. When x = ½, they know the value, and x = n + ½ as well. And n + 1/3, etc. So I don't think there was anything mysterious about $f(x) = a^x$ being a continuous function of real argument x. The inverse function could be simply defined as the log, so we might say $x = \log_a f(x) = f^{-1}(x) = f_inv(x)$. The graph would show a unique inverse solution.

What happens for differentiation? They knew that a^x has a slope at each point. So:

$$f'(x) = (a^{x+dx} - a^x)/dx = a^x[\, a^{dx} - 1]/dx = f(x)\, [\, a^{dx} - 1]/dx$$

What next? What do we do with a^{dx} here? We don't yet have a Taylor series expansion for a^x because we don't know the derivative. We can see a 0/0 situation in our limit here since $a^0 = 1$. We might try a linear fit

$$a^{dx} = A + B\, dx \qquad \text{// for small dx}$$

Then certainly must have A = 1 to get the 0/0, so our fit is then $a^{dx} = 1 + B_a\, dx$. How do we determine this constant B which we imagine depends on a? This would seem to be a dead end.

Plan B. Following Thomas, let's start in a different place. *Define* $\ln(x) \equiv \int_1^x (1/x')\, dx'$. So we cannot do such a definition without calculus. This is simple to plot and approximate if you know nothing. But then we know that $d\ln(x)/dx = 1/x$ and we can then compute all derivatives of ln(x). So far "ln" is just a function name like "f" in f(x). A plot of y = lnx is shown page 383. We can see that ln(x) = 1 at some value of x which lies between 2 and 3. So denote the solution to ln(x) = 1 by "e", a number which we know is about 2.7. So then we have ln(e) = 1 as the definition of e. Fine. We also know from the definition that ln(1) = 0.

What can we say more about the function ln(x) ?

(1) What about ln(x/a) where a is some number?

$$\ln(x/a) = \int_1^{x/a} (1/x')\, dx'$$

Let s = ax' so that ds = adx', then we have

$$\ln(x/a) = \int_a^x (1/s)\, ds = \int_a^1 (1/s)\, ds + \int_1^x (1/s)\, ds = -\ln(a) + \ln(x)$$

This tells us various things:

$\ln(1/x) = -\ln(x)$ $\quad$ $\ln(xyz...) = \ln(x) + \ln(y) + \ln(z) + ...$ $\quad$ $\ln(x^n) = n \ln(x)$

So we know quite a lot now about our function ln(x). We can analytically continue the last result from the integers n to the continuum z and say this:

$\ln(x^z) = z \ln(x)$ and then $\ln(e^z) = z \ln(e) = z$ or $\ln(e^x) = x$

This last result says that $f(x) = e^x$ must be the *inverse* function of ln(x), that is,

$\ln_inv(x) = e^x$ so that $\ln [\ln_inv(x)] = \ln(e^x) = x$

The usual way we express this last result is this:

$y = e^x \quad \leftrightarrow \quad \ln(y) = x \quad => \quad y = e^{\ln y}$ or $x = e^{\ln x}$

Differentiate both sides of x = ln(y) and use the chain rule to get

$1 = d \ln(y)/dx = d \ln(y)/dy \; dy/dx = 1/y \; dy/dx \quad => \quad dy/dx = y$

which proves that the function $y(x) = e^x$ is its own derivative, which is what we set out to prove in the first place.

Now consider $g(x) = a^x$. We know that ln(g) = x ln(a) from above, so x = ln(g)/ln(a). Thus we have shown that

$g(x) = a^x \quad => \quad g_inv(x) = \ln(g(x))/\ln(a) \equiv \log_a(x)$

Then $g_inv [g(x)] = g_inv [a^x] = \ln(a^x)/\ln(a) = x \ln(a)/\ln(a) = x$

So we have now defined a new function $\log_a(x)$ as follows:

$\log_a(x) \equiv [1/\ln(a)] \ln(x)$

and we know that

$g = a^x \quad \leftrightarrow \quad x = \log_a(g)$

and finally we identify

$\log_e(x) = \ln(x)$ as a special case.

Now let's go back to our problem in Plan A where we were concerned about

$$f'(x) = (a^{x+dx} - a^x)/dx = a^x[\, a^{dx} - 1]/dx = f(x)\,[\, a^{dx} - 1]/dx$$

Now use $f(x) = a^x = e^{x \ln a}$ and then

$$f'(x) = \ln a\, e^{x \ln a} \text{ by chain rule } = \ln(a)\, f(x)$$

Thus we conclude that $a^{dx} \approx a^0 + dx\,[\, \ln(a)\, a^0] = 1 + dx \ln(a)$, Taylor expansion, so

$$\lim [\, a^{dx} - 1]/dx = \ln(a) = B_a$$

so we have now found the value of our little constant.

Finally, we know all the derivatives of e^x so we can make a Taylor series expansion

$$e^x = \Sigma_{n=0,\infty}\ x^n/n! * 1 \qquad \text{since } f^{(n)}(e^x)|^{x=0} = 1$$

<u>Summary:</u>

1) define $\ln(x) \equiv \int_1^x dt/t$

2) define e as the number that makes $\ln(e) = 1$, find that e = 2.7...

3) show these properties of ln(x) based on the above definition:

$$\ln(xyz...) = \ln(x) + \ln(y) + \ln(z) + ...$$

$$\ln(x^y) = y \ln(x)$$

4) special case of last says that $\ln(e^x) = x$.

5) This shows that if $y = e^x$ then the inverse is $x = \ln(y)$.

6) Can therefore write $y = e^x = e^{\ln(y)}$ or $x = e^{\ln(x)}$

7) Take $x = \ln(y)$ and differentiate d/dx to get $1 = d(\ln y)/dy * dy/dx = (1/y)\, dy/dx$ so $dy/dx = y$.
Therefore, $d(e^x)/dx = e^x$ so e^x is its own derivative.

8) If $g = a^x$ then $x = \log_a(g)$ where we define $\log_a(z) \equiv \ln(z)/\ln(a)$.

9) conclude that $\lim_{dx \to 0} [\, a^{dx} - 1]/dx = \ln(a)$

10) expansion is $e^x = \Sigma_{n=0,\infty}\ x^n/n!$

2. About $e^{ix} = \cos(x) + i \sin(x)$

Thomas uses geometric trig definitions and trig identities and the definition of a derivative to proof that dcos = -sin and dsin = cos. So if f(x) = sin(x), we know how to derive the power series expansion since we know that

$f(x) = \sin(x)$
$f'(x) = \cos(x)$
$f''(x) = -\sin(x)$
$f^{(n)}(x) = (-1)^{n/2} \sin(x)$ n = 0,2,4,,, $f^{(n)}(0) = 0$
$f^{(n)}(x) = (-1)^{(n-1)/2} \cos(x)$ n = 1,3,5... $f^{(n)}(0) = (-1)^{(n-1)/2}$ etc etc

Therefore, we can write the usual Taylor series expansions for sin(x) and cos(x). But we can also expand e^{ix} using our formula in 10) above. If we compare the power series on both sides, we find the result we are studying here, so that can be the proof! We then get these special cases:

$e^{i\pi} = \cos(\pi) = -1$

$e^{i\pi/2} = i$

And of course once we have the formula $e^{ix} = \cos(x) + i \sin(x)$, we can make the complex plane unit circle construction and represent $z = e^{i\theta}$ as a phasor having $x = \cos(\theta)$ and $y = \sin(\theta)$ with z = x+iy.

Notice also that

$ix = \ln [\cos(x) + i\sin(x)]$

All this early work was done by Euler around 1750 !!! The key starting point is to first have calculus, and then to define ln(x) by the integral of 1/x. Then all else follows from this definition. He was the first to realize this.

Notes Added:

1. There is only one function, e^x, which is its own derivative. To show this, f ' = f says dx = df/f and x = lnf, QED. This is true regardless of whether x is real or complex.

2. Define g = cos + i sin. It follows that g'(x) = ig(x). Therefore, $g = e^{ix}$ since by chain rule, g' = ig.

3. The derivative of a^x ? $f = a^x = (e^{\alpha})^x = e^{\alpha x}$ so f ' = α f. But $a = e^{\alpha}$ says α = lna, so f' = lna a^x . It is almost its own derivative, just a scale factor which is 1 when a = e.

Facts about e

1. Consider $\lim_{x\to\infty} (x+a)^x$. Expand the RHS by binomial expansion:

$$(x+a)^x = x^x + x\, x^{x-1}a + x(x-1)/2\, x^{x-2}a^2 + x(x-1)(x-2)/3!\, x^{x-3}\, a^3 + ...$$

$$\approx x^x + x^x\, a + 1/2\, x^x a^2 + 1/3!\, x^x\, a^3 + ...$$

$$= x^x(1 + a + a^2/2 + a^3/3! + ...) = x^x\, e^a$$

But write $(x+a)^x = x^x\,(1 + a/x)^x$. Thus we have shown these results which have Stirling use:

(1a) $\lim_{x\to\infty} (x+a)^x = x^x\, e^a$

(1b) $\lim_{x\to\infty} (1+a/x)^x = e^a$

(1c) $\lim_{x\to\infty} (1+1/x)^x = e$ // a traditional result