Absolute value theorems PhL 8.14.09

********** A. Theorems where a and b are complex numbers **********

Theorem 1: $|ab| = |a||b|$

Proof: Let $a = Ae^{i\alpha}$ and $b = Be^{i\beta}$. Then $|ab| = AB$, $|a| = A$, $|b| = B$, QED. You can think of a and b as vectors in the space C^n where n = 1. If we made n-dimensional vectors $\mathbf{A} = (a_1, a_2, ,,,,, a_n)$ and similarly for b, then this Theorem 1 is seen to be the famous Cauchy Schwarz Inequality $|\mathbf{A}\bullet\mathbf{B}| \le |\mathbf{A}|\,|\mathbf{B}|$ in C^n. So let's write this out as a variant of Theorem 1: [note that a_i and b_i are just arbitrary complex numbers]

Theorem 1A: $|a_1b_1 + a_2b_2| \le |a_1+a_2|\,|b_1+ b_2| \le (|a_1| + |a_2|)\,(|b_1| + |b_2|)$

$$| \Sigma_i a_ib_i | \le |\Sigma_i a_i|\,|\Sigma_i b_i| \le (\Sigma_i |a_i|)\,(\Sigma_i |b_i|)$$

Proof: The first ≤ sign on both lines is the CSI as just noted (and which I will prove in a moment). The second ≤ sign on each line comes from Theorem 2 (or Theorem 2C) stated below. According to Ahlfors, it is the comparison of the first with the third expressions which he calls **Cauchy's Inequality** on p 10. But others refer to the CSI itself as Cauchy's Inequality.

Theorem 1B: $|\langle x,y\rangle| \le \langle x,x\rangle\langle y,y\rangle$ x,y ∈ any Hilbert Space, including C^n

The **Cauchy-Schwarz Inequality** = **CSI** in my notes (apologies to the Russian B)

Proof: Stakgold page 107 has a short proof, but it is specific to R^n. Wiki gives a short proof which works for any Hilbert Space, as follows. (The trick is to pick λ in a smart way.) Remember that an inner product is a complex number and that $\langle a,b\rangle = \langle b,a\rangle^*$. This proof uses $\langle x,\lambda y\rangle = \bar{\lambda}\langle x,y\rangle$ which is the Stakgold/Math form, not the Quantum Mechanics form.

As the inequality is trivially true in the case $y = 0$, we may assume $\langle y, y\rangle$ is nonzero. Let λ be a complex number. Then,

$$0 \leq \|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y\rangle = \langle x, x\rangle - \bar{\lambda}\langle x, y\rangle - \lambda\langle y, x\rangle + |\lambda|^2\langle y, y\rangle.$$

The above expression is valid for any complex λ. Choosing

$$\lambda = \langle x, y\rangle \cdot \langle y, y\rangle^{-1}$$

we obtain

$$0 \leq \langle x, x\rangle - |\langle x, y\rangle|^2 \cdot \langle y, y\rangle^{-1}$$

which is true if and only if

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle$$

or equivalently:

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|,$$

which is the Cauchy–Schwarz inequality.

Here is some algebra to go with the above:

$(x,x) - \bar{\lambda}(x,y) - \lambda\,(y,x) + |\lambda|^2\,(y,y) =$

$= (x,x) - (y,x)(x,y)/(y,y) - (x,y)\,(y,x)/(y,y) + (x,y)\,(y,x)/\,(y,y)$ // two terms cancel
$= (x,x) - (y,x)(x,y)/(y,y) = (x,x) - |(x,y)|^2/(y,y) \geq 0$

So you conclude that $|(x,y)|^2 \leq (x,x)\,(y,y)$

Theorem 2: $|a + b| \leq |a| + |b|$

Proof # 1. Let $a = x+iy$ and $b = u + iv$. Then

$|a + b|^2 = |(x+u) + i(y+v)|^2 = (x+u)^2 + (y+v)^2$
$|a|^2 = x^2 + y^2$
$|b|^2 = u^2 + v^2$

So we need to show that, where all square roots are positive,

$$[(x+u)^2 + (y+v)^2]^{1/2} \leq [x^2 + y^2]^{1/2} + [u^2 + v^2]^{1/2} \qquad (5)$$

We will now construct a sequence of steps which, when run backwards, show the above is true. Square both sides:

$$(x+u)^2 + (y+v)^2 \leq x^2 + y^2 + u^2 + v^2 + 2\,[x^2 + y^2]^{1/2}[u^2 + v^2]^{1/2} \qquad (4)$$

$$2xu + 2yv \leq 2\,[x^2 + y^2]^{1/2}[u^2 + v^2]^{1/2}$$

$$xu + yv \leq [x^2 + y^2]^{1/2}[u^2 + v^2]^{1/2} \qquad (3)$$

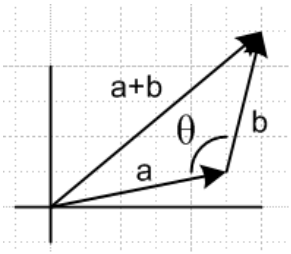
Square both sides

$$x^2u^2 + y^2v^2 \le [x^2 + y^2][u^2 + v^2] = x^2u^2 + y^2 v^2 + y^2u^2 + x^2 v^2 \qquad (2)$$

$$0 \le y^2u^2 + x^2 v^2 \qquad (1)$$

So start with (1) and go backwards and you end up with 5 and hence theorem is proved.

Proof #2. Think of a and b as 2D vectors in the complex plane:

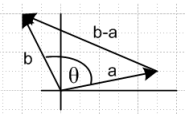


Then $|a + b| \le |a| + |b|$ because the length of side a+b is obviously at most the sum of the lengths of sides a and b and this happens when $\theta = \pi$. Hence this theorem is always called "the triangle inequality".

Comment: we know that $c^2 \equiv (\mathbf{a}+\mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a}\bullet\mathbf{b} = a^2 + b^2 - 2ab\cos\theta$ where now a and b are lengths of triangle sides, not complex numbers. Thus, the RHS here ranges from $(a-b)^2$ to $(a+b)^2$. So for sure this tells us $c^2 \le (a+b)^2$ or $c \le a+b$, which is our theorem's claim.

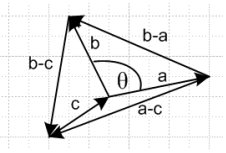
Theorem 2A: $|a - b| \le |a| + |b|$

Proof: Just take $b \rightarrow -b$ in Theorem 2, QED. Just a different triangle:



Theorem 2B: $|a-b| \le |a-c| + |b-c|$.

Proof: start with Theorem 2A which says $|A - B| \le |A| + |B|$. Then set A = a-c and B = b-c, QED.
Just another triangle:



Theorem 2C: $|a + b + c| \le |a| + |b| + |c|$ and so on $|\Sigma_i a_i| \le \Sigma_i |a_i|$

Proof: Let a+b = d and apply theorem 2 twice. Can install any signs you want on the LHS.

********** **B. Theorems where a and b are real numbers:** **********

Since real numbers are a subset of complex numbers, all versions of Theorem 1 and Theorem 2 given above are true for reals. But there are now more theorems for reals only that we can produce.

Theorem 3: $|a-b| \geq |a| - |b|$ a,b real

Proof: Again for $|b| \geq |a|$ the RHS is negative and LHS is positive, so theorem is blatantly true, so we only need consider $|a| > |b|$. In this realm, there are two cases to consider: a and b have the same sign, or they have opposite sign. This seems to involve 4 different cases, but the ab = ++ case is the same as the ab = - - case, since nothing on either side of our theorem changes under this double sign change. Similarly, the ab = -+ case is the same as the ab = +- case. So we can now examine our only two unique cases:

++ $a-b \geq a-b$ true

+- Here, let $B = -b > 0$. Then theorem says

$a+B \geq a - B$ which is obviously true

Thus our theorem is proved.

Theorem 4: $|a-b| \geq ||a| - |b||$ a,b real

Proof: If $|a| > |b|$, we showed this to be true in Theorem 3. If $|b| \geq |a|$, Theorem 3 would also tell us that

$|a-b| \geq |b| - |a|$, just by swapping $a \leftrightarrow b$. Thus, if $|b| \geq |a|$, we know that $|a-b| \geq ||a| - |b||$. Thus we have shown that $|a-b| \geq ||a| - |b||$ in both cases $|a| > |b|$ and $|b| \geq |a|$, QED.

This is a stronger theorem in a sense than Theorem.

Theorem 5: $|a-b| < \varepsilon \Rightarrow |a| - |b| < \varepsilon$ a,b real

Proof: By theorem 3 we have $|a-b| \geq |a| - |b|$. Adding the hypothesis of Theorem 5 we have

$|a| - |b| \leq |a-b| < \varepsilon$

Therefore $|a| - |b| < \varepsilon$, QED.

Here is some verification of some of our theorems from http://en.wikipedia.org/wiki/Absolute_value, these involve only reals:

The absolute value has the following four fundamental properties:

$\|a\| \geq 0$	(2)	Non-negativity
$\|a\| = 0 \iff a = 0$	(3)	Positive-definiteness
$\|ab\| = \|a\|\|b\|$	(4)	Multiplicativeness
$\|a + b\| \leq \|a\| + \|b\|$	(5)	Subadditivity

Other important properties of the absolute value include:

$\|-a\| = \|a\|$	(6)	Symmetry
$\|a - b\| = 0 \iff a = b$	(7)	Identity of indiscernibles (equivalent to positive-definiteness)
$\|a - b\| \leq \|a - c\| + \|c - b\|$	(8)	Triangle inequality (equivalent to subadditivity)
$\|a/b\| = \|a\|/\|b\|$ (if $b \neq 0$)	(9)	Preservation of division (equivalent to multiplicativeness)
$\|a - b\| \geq \|\|a\| - \|b\|\|$	(10)	(equivalent to subadditivity)