

Inverse Vector Operators

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Abstract In different branches of physics, we frequently deal with vector del operator ($\vec{\nabla}$). This del operator is generally used to find curl or divergence of a vector function or gradient of a scalar function. In many important cases, we need to know the parent vector whose curl or divergence is known or require to find the parent scalar function whose gradient is known. But the task is not very easy, especially in case of finding vector potential whose curl is known. Here, ‘inverse curl’, ‘inverse divergence’ and ‘inverse gradient’ operators are defined to solve those problems easily.

Keywords del operator, curl, gradient, divergence, solenoidal vector, vector potentials, curvilinear co-ordinate system.

1 Introduction

In physical science, usefulness of the concepts of vector and scalar potentials is indisputable. They can be used to replace more conventional concepts of magnetic and electric fields. In fact, in modern era of science, these concepts of potentials became so useful and popular that their presence can be seen in virtually all disciplines related somehow to ‘electromagnetism’! The operator which relates those potentials to the conventional fields is the so-called vector ‘del’ operator ($\vec{\nabla}$). This operator plays a very important role in a wide range of physics besides being indispensable in electromagnetism. As for example, it is now very common to see the operator in fluid dynamics, quantum mechanics, statistical mechanics, classical mechanics, theory of relativity, thermodynamics, etc. There are three basic operations, namely, curl, divergence and gradient, which can be performed by the operator to relate different quantities of importance. For the ease of discussion, let us divide the functions in two categories, (a) potential functions (this can be vector or scalar) and (b) field functions (again can be vector or scalar). Above mentioned operations are performed on potential functions to obtain corresponding field functions. While these operations are easy to perform, a back operation to get a potential function from a given field function is not always trivial. Since in some cases, it is convenient to replace one type of functions with others, one should be able to interchange these two types of functions comfortably. Unfortunately, a potential function is not unique for a given field function and there is no general and unique procedure to obtain a potential function from a given field function. In fact, due to this non-uniqueness, one may wonder whether it is at all possible to find any general procedure for this purpose. It is exactly the concern addressed in this article. Here it is shown that, it is possible to find some general procedures in the form of inverse vector operators, which can be applied easily to the field functions to obtain the corresponding potential functions.

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All the inverse operators are defined here in the orthogonal curvilinear co-ordinate system [1]. Let u_1, u_2, u_3 be the three mutually perpendicular families of co-ordinate surfaces and $\hat{e}_1, \hat{e}_2, \hat{e}_3$ as unit vectors normal to respective surfaces.

2 Inverse Curl Operator

At first we mention some symbols, used later.

- a. $(\vec{\nabla} \times)^{-1}$ indicates inverse curl operator.
- b. $(\int^+ + \int^-)(\partial u_1)$ does integration w.r.t. u_1 , while other two variables of the integrand are to be treated fixed. If the integrand is the 3rd component (along $\hat{e}_3$) of a vector, then $\int^+(\partial u_1)$ acts only on the part of the integrand having the variable u_3 . Similarly $\int^-(\partial u_1)$ acts only on the part of the integrand does not contain the variable u_3 . For example, let $\phi(u_1, u_2, u_3) = \Psi_1(u_1, u_2) + \Psi_2(u_1, u_2, u_3)$ be the component of a vector along $\hat{e}_3$, then, $(p_1 \int^+ + p_2 \int^-)\phi \partial u_1 = p_1 \int^+ \Psi_2 \partial u_1 + p_2 \int^- \Psi_1 \partial u_1$, with p_1 and p_2 being any two numbers (u_2, u_3 are to be treated as constants).

Now take two vectors $\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$ and $\vec{B} = B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3$, such that,

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad (1)$$

Using inverse curl operator, this vector potential $\vec{A}$ can be expressed as,

$$\vec{A} = (\vec{\nabla} \times)^{-1} \vec{B} \quad (2)$$

From eqn (1) it is obvious that,

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \\ \Rightarrow \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 B_1) + \frac{\partial}{\partial u_2} (h_3 h_1 B_2) + \frac{\partial}{\partial u_3} (h_1 h_2 B_3) \right] &= 0 \\ \Rightarrow \frac{\partial}{\partial u_1} [c_1(u_1^+) + c_1(u_1^-)] + \frac{\partial}{\partial u_2} [c_2(u_2^+) + c_2(u_2^-)] + \frac{\partial}{\partial u_3} [c_3(u_3^+) + c_3(u_3^-)] &= 0 \\ \Rightarrow \frac{\partial}{\partial u_1} c_1(u_1^+) + \frac{\partial}{\partial u_2} c_2(u_2^+) + \frac{\partial}{\partial u_3} c_3(u_3^+) &= 0 \end{aligned} \quad (3)$$

where $c_1(u_1^+)$ and $c_1(u_1^-)$ are two parts of the term $h_2 h_3 B_1$, containing and not containing u_1 respectively, such that, $h_2 h_3 B_1 = c_1(u_1^+) + c_1(u_1^-)$. Other terms can be defined similarly. From eqn (1) we get,

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \quad (4)$$

$$\begin{aligned}
\Rightarrow B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3 &= \frac{\hat{e}_1}{h_2 h_3} \left[\frac{\partial}{\partial u_2} (h_3 A_3) - \frac{\partial}{\partial u_3} (h_2 A_2) \right] \\
&+ \frac{\hat{e}_2}{h_3 h_1} \left[\frac{\partial}{\partial u_3} (h_1 A_1) - \frac{\partial}{\partial u_1} (h_3 A_3) \right] \\
&+ \frac{\hat{e}_3}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (h_2 A_2) - \frac{\partial}{\partial u_2} (h_1 A_1) \right] \quad (5)
\end{aligned}$$

We already have defined components of $\vec{B}$ in terms of c 's. Now, we have to choose $h_1 A_1$, $h_2 A_2$ and $h_3 A_3$ in terms of c 's in such a way that, *r.h.s.* of the eqn (5) becomes same as *l.h.s.*

Let (by inspection),

$$h_1 A_1 = k_1 \int c_2(u_2^+) \partial u_3 + k_2 \int c_3(u_3^+) \partial u_2 + k_3 \int c_2(u_2^-) \partial u_3 + k_4 \int c_3(u_3^-) \partial u_2$$

Similarly,

$$h_2 A_2 = k_5 \int c_3(u_3^+) \partial u_1 + k_6 \int c_1(u_1^+) \partial u_3 + k_7 \int c_3(u_3^-) \partial u_1 + k_8 \int c_1(u_1^-) \partial u_3$$

$$\text{and, } h_3 A_3 = k_9 \int c_1(u_1^+) \partial u_2 + k_{10} \int c_2(u_2^+) \partial u_1 + k_{11} \int c_1(u_1^-) \partial u_2 + k_{12} \int c_2(u_2^-) \partial u_1$$

where, $k_1, k_2, \dots, k_{12}$ are some constants to be determined.

(Note that, c_1 terms are not included for $h_1 A_1$, since, $h_1 A_1$ does not appear in the expression of 1st component B_1 in eqn (5). Also note that, terms in the expression of $h_1 A_1$ are in the integral form w.r.t. variables u_2 and u_3 but not u_1 , as in eqn (5), $h_1 A_1$ is never differentiated w.r.t. u_1 .)

Now, see the coefficient of $\hat{e}_3$ (expression of B_3) on the *r.h.s.* of eqn (5),

$$\begin{aligned}
&\frac{1}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (h_2 A_2) - \frac{\partial}{\partial u_2} (h_1 A_1) \right] \\
&= \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial u_1} \{ k_5 \int c_3(u_3^+) \partial u_1 + k_6 \int c_1(u_1^+) \partial u_3 + k_7 \int c_3(u_3^-) \partial u_1 + k_8 \int c_1(u_1^-) \partial u_3 \} \right. \\
&\quad \left. - \frac{\partial}{\partial u_2} \{ k_1 \int c_2(u_2^+) \partial u_3 + k_2 \int c_3(u_3^+) \partial u_2 + k_3 \int c_2(u_2^-) \partial u_3 + k_4 \int c_3(u_3^-) \partial u_2 \} \right] \\
&= \frac{1}{h_1 h_2} [k_5 c_3(u_3^+) + k_6 \int \frac{\partial}{\partial u_1} c_1(u_1^+) \partial u_3 + k_7 c_3(u_3^-) + k_8 \cdot 0 - k_1 \int \frac{\partial}{\partial u_2} c_2(u_2^+) \partial u_3 \\
&\quad - k_2 c_3(u_3^+) - k_3 \cdot 0 - k_4 c_3(u_3^-)] \\
&= \frac{1}{h_1 h_2} [2k_5 c_3(u_3^+) - k_1 \int \{ \frac{\partial}{\partial u_1} c_1(u_1^+) + \frac{\partial}{\partial u_2} c_2(u_2^+) \} \partial u_3 + 2k_7 c_3(u_3^-)] \\
&\quad \text{[Taking } k_5 = -k_2, k_7 = -k_4 \text{ and } k_6 = -k_1] \\
&= \frac{1}{h_1 h_2} [2k_5 c_3(u_3^+) - k_1 \int \{ -\frac{\partial}{\partial u_3} c_3(u_3^+) \} \partial u_3 + 2k_7 c_3(u_3^-)] \quad \text{[Using eqn (3)]} \\
&= \frac{1}{h_1 h_2} [2k_5 c_3(u_3^+) + k_1 c_3(u_3^+) + 2k_7 c_3(u_3^-)] \\
&= \frac{1}{h_1 h_2} [3k_5 c_3(u_3^+) + 2k_7 c_3(u_3^-)] \quad \text{[Taking } k_1 = k_5]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{h_1 h_2} [3 \cdot \frac{1}{3} c_3(u_3^+) + 2 \cdot \frac{1}{2} c_3(u_3^-)] && [\text{Considering } k_5 = 1/3 \text{ and } k_7 = 1/2] \\
&= \frac{1}{h_1 h_2} [c_3(u_3^+) + c_3(u_3^-)] \\
&= \frac{1}{h_1 h_2} (h_1 h_2 B_3) \\
&= B_3
\end{aligned}$$

Similarly we can show that, if,

$$\begin{aligned}
&k_1 = k_5 = k_9 = -k_2 = -k_6 = -k_{10} = 1/3 \\
&\text{and, } k_3 = k_7 = k_{11} = -k_4 = -k_8 = -k_{12} = 1/2, \text{ then,} \\
&\frac{1}{h_2 h_3} [\frac{\partial}{\partial u_2} (h_3 A_3) - \frac{\partial}{\partial u_3} (h_2 A_2)] = B_1 \text{ and } \frac{1}{h_1 h_3} [\frac{\partial}{\partial u_3} (h_1 A_1) - \frac{\partial}{\partial u_1} (h_3 A_3)] = B_2
\end{aligned}$$

So, using these values of $k_1, k_2, \dots, k_{12}$, we get,

$$A_1 = \frac{1}{3h_1} [\int c_2(u_2^+) \partial u_3 - \int c_3(u_3^+) \partial u_2] + \frac{1}{2h_1} [\int c_2(u_2^-) \partial u_3 - \int c_3(u_3^-) \partial u_2]$$

$$A_2 = \frac{1}{3h_2} [\int c_3(u_3^+) \partial u_1 - \int c_1(u_1^+) \partial u_3] + \frac{1}{2h_2} [\int c_3(u_3^-) \partial u_1 - \int c_1(u_1^-) \partial u_3]$$

$$A_3 = \frac{1}{3h_3} [\int c_1(u_1^+) \partial u_2 - \int c_2(u_2^+) \partial u_1] + \frac{1}{2h_3} [\int c_1(u_1^-) \partial u_2 - \int c_2(u_2^-) \partial u_1]$$

Now from eqn (2), we get,

$$(\vec{\nabla} \times)^{-1} \vec{B} = \vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$$

$$\begin{aligned}
&= \frac{\hat{e}_1}{3h_1} [\int c_2(u_2^+) \partial u_3 - \int c_3(u_3^+) \partial u_2] + \frac{\hat{e}_1}{2h_1} [\int c_2(u_2^-) \partial u_3 - \int c_3(u_3^-) \partial u_2] \\
&\quad + \frac{\hat{e}_2}{3h_2} [\int c_3(u_3^+) \partial u_1 - \int c_1(u_1^+) \partial u_3] + \frac{\hat{e}_2}{2h_2} [\int c_3(u_3^-) \partial u_1 - \int c_1(u_1^-) \partial u_3] \\
&\quad + \frac{\hat{e}_3}{3h_3} [\int c_1(u_1^+) \partial u_2 - \int c_2(u_2^+) \partial u_1] + \frac{\hat{e}_3}{2h_3} [\int c_1(u_1^-) \partial u_2 - \int c_2(u_2^-) \partial u_1] \\
&= -\frac{\hat{e}_1}{3h_1} \begin{vmatrix} f(\partial u_2) & f(\partial u_3) \\ c_2(u_2^+) & c_3(u_3^+) \end{vmatrix} + \frac{\hat{e}_2}{3h_2} \begin{vmatrix} f(\partial u_1) & f(\partial u_3) \\ c_1(u_1^+) & c_3(u_3^+) \end{vmatrix} - \frac{\hat{e}_3}{3h_3} \begin{vmatrix} f(\partial u_1) & f(\partial u_2) \\ c_1(u_1^+) & c_2(u_2^+) \end{vmatrix} \\
&\quad - \frac{\hat{e}_1}{2h_1} \begin{vmatrix} f(\partial u_2) & f(\partial u_3) \\ c_2(u_2^-) & c_3(u_3^-) \end{vmatrix} + \frac{\hat{e}_2}{2h_2} \begin{vmatrix} f(\partial u_1) & f(\partial u_3) \\ c_1(u_1^-) & c_3(u_3^-) \end{vmatrix} - \frac{\hat{e}_3}{2h_3} \begin{vmatrix} f(\partial u_1) & f(\partial u_2) \\ c_1(u_1^-) & c_2(u_2^-) \end{vmatrix} \\
&= -\frac{1}{3} \begin{vmatrix} \frac{\hat{e}_1}{h_1} & \frac{\hat{e}_2}{h_2} & \frac{\hat{e}_3}{h_3} \\ f(\partial u_1) & f(\partial u_2) & f(\partial u_3) \\ c_1(u_1^+) & c_2(u_2^+) & c_3(u_3^+) \end{vmatrix} - \frac{1}{2} \begin{vmatrix} \frac{\hat{e}_1}{h_1} & \frac{\hat{e}_2}{h_2} & \frac{\hat{e}_3}{h_3} \\ f(\partial u_1) & f(\partial u_2) & f(\partial u_3) \\ c_1(u_1^-) & c_2(u_2^-) & c_3(u_3^-) \end{vmatrix}
\end{aligned}$$

$$\begin{aligned}
&= - \begin{vmatrix} \frac{\hat{e}_1}{h_1} & \frac{\hat{e}_2}{h_2} & \frac{\hat{e}_3}{h_3} \\ f(\partial u_1) & f(\partial u_2) & f(\partial u_3) \\ \frac{1}{3}c_1(u_1^+) + \frac{1}{2}c_1(u_1^-) & \frac{1}{3}c_2(u_2^+) + \frac{1}{2}c_2(u_2^-) & \frac{1}{3}c_3(u_3^+) + \frac{1}{2}c_3(u_3^-) \end{vmatrix} \\
&= - \begin{vmatrix} \frac{\hat{e}_1}{h_1} & \frac{\hat{e}_2}{h_2} & \frac{\hat{e}_3}{h_3} \\ (\frac{1}{3}f^+ + \frac{1}{2}f^-)(\partial u_1) & (\frac{1}{3}f^+ + \frac{1}{2}f^-)(\partial u_2) & (\frac{1}{3}f^+ + \frac{1}{2}f^-)(\partial u_3) \\ h_2h_3B_1 & h_3h_1B_2 & h_1h_2B_3 \end{vmatrix}
\end{aligned}$$

This is the desired expression for our inverse curl operator.

Since, $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$, so, it should be noted that, for a given solenoidal vector, corresponding vector potential is not unique, which can be in general expressed as, -

$$\vec{A} = (\vec{\nabla} \times)^{-1} \vec{B} + \vec{\nabla}\phi$$

where ϕ is an arbitrary scalar function.

2.1 An Example in Cartesian System

One example in Cartesian system would clarify the procedure of getting vector potential for a solenoidal vector. In this system, $\hat{e}_1, \hat{e}_2, \hat{e}_3, u_1, u_2$ and u_3 are respectively replaced by $\hat{i}, \hat{j}, \hat{k}, x, y$, and z . Here $h_1 = h_2 = h_3 = 1$.

Now take a solenoidal vector $\vec{B} = (xyz + y^2)\hat{i} + (xz + y)\hat{j} - z(1 + yz/2)\hat{k}$. (note $\vec{\nabla} \cdot \vec{B} = 0$)

So, vector potentials of $\vec{B}$ are,-

$$\vec{A} = (\vec{\nabla} \times)^{-1} \vec{B} + \vec{\nabla}\phi$$

$$\begin{aligned}
&= - \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (\frac{1}{3}f^+ + \frac{1}{2}f^-)(\partial x) & (\frac{1}{3}f^+ + \frac{1}{2}f^-)(\partial y) & (\frac{1}{3}f^+ + \frac{1}{2}f^-)(\partial z) \\ xyz + y^2 & xz + y & -z - yz^2/2 \end{vmatrix} + \vec{\nabla}\phi \\
&= -\hat{i}[\frac{1}{3}f(-z - yz^2/2)\partial y + \frac{1}{2}f(0)\partial y - \frac{1}{3}f(y)\partial z - \frac{1}{2}f(xz)\partial z] \\
&\quad - \hat{j}[\frac{1}{3}f(xyz)\partial z + \frac{1}{2}f(y^2)\partial z - \frac{1}{3}f(-z - yz^2/2)\partial x - \frac{1}{2}f(0)\partial x] \\
&\quad - \hat{k}[\frac{1}{3}f(y)\partial x + \frac{1}{2}f(xz)\partial x - \frac{1}{3}f(xyz)\partial y - \frac{1}{2}f(y^2)\partial y] + \vec{\nabla}\phi \\
&= \hat{i}[(zy + z^2y^2/4)/3 + (yz/3 + xz^2/4)] - \hat{j}[(zx + yxz^2/2)/3 + (xyz^2/6 + y^2z/2)] \\
&\quad + \hat{k}[-(yx/3 + x^2z/4) + xy^2z/6 + y^3/6] + \vec{\nabla}\phi
\end{aligned}$$

2.1.1 Verification of the Result

$$\begin{aligned}
\vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (zy + z^2y^2/4)/3 & -(zx + xyz^2/2)/3 & -(yx/3 + x^2z/4) \\ +(yz/3 + xz^2/4) & -(xyz^2/6 + y^2z/2) & +(xy^2z/6 + y^3/6) \end{vmatrix} + \vec{\nabla} \times (\vec{\nabla}\phi) \\
&= \hat{i}(xyz + y^2) + \hat{j}(xz + y) - \hat{k}z(1 + yz/2) + 0 \\
&= \vec{B}
\end{aligned}$$

3 Inverse Divergence Operator

Here we indicate this operator by $(\vec{\nabla} \cdot)^{-1}$. Now if $\vec{A}$ and ϕ be a vector and a scalar function respectively, such that $\phi = \vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} [\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3)]$, then $\vec{A}$ can be expressed as $\vec{A} = (\vec{\nabla} \cdot)^{-1} \phi$.

We can now define the operator as (by inspection),

$$(\vec{\nabla} \cdot)^{-1} = k_1 \frac{\hat{e}_1}{h_2 h_3} \int h_1 h_2 h_3 (\partial u_1) + k_2 \frac{\hat{e}_2}{h_3 h_1} \int h_1 h_2 h_3 (\partial u_2) + k_3 \frac{\hat{e}_3}{h_1 h_2} \int h_1 h_2 h_3 (\partial u_3)$$

where, $k_1 + k_2 + k_3 = 1$.

3.1 Verification

For a given ϕ , $\vec{A} = (\vec{\nabla} \cdot)^{-1} \phi$

$$= k_1 \frac{\hat{e}_1}{h_2 h_3} \int h_1 h_2 h_3 \phi (\partial u_1) + k_2 \frac{\hat{e}_2}{h_3 h_1} \int h_1 h_2 h_3 \phi (\partial u_2) + k_3 \frac{\hat{e}_3}{h_1 h_2} \int h_1 h_2 h_3 \phi (\partial u_3)$$

Now note that, divergence of the obtained vector gives the scalar function ϕ ,

$$\begin{aligned}
\vec{\nabla} \cdot \vec{A} &= \frac{1}{h_1 h_2 h_3} [k_1 h_1 h_2 h_3 \phi + k_2 h_1 h_2 h_3 \phi + k_3 h_1 h_2 h_3 \phi] \\
&= (k_1 + k_2 + k_3) \phi = \phi
\end{aligned}$$

Since, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$, so, it should be noted that for a given scalar function, corresponding potential function is not unique, which can be in general expressed as,

$$\vec{A} = (\vec{\nabla} \cdot)^{-1} \phi + \vec{\nabla} \times \vec{B}$$

where $\vec{B}$ is any vector.

4 Inverse Gradient Operator

We indicate this operator by $(\vec{\nabla})^{-1}$.

Let $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$ and $\phi(u_1, u_2, u_3)$ be a vector and a scalar function, such that $\vec{A} = \vec{\nabla}\phi$. Since, the vector field represented by $\vec{A}$ is conservative, thus, line integral of $\vec{A}$ is path independent. Choose (a,b,c) as initial point, such that $\phi(a, b, c)$ exists.

$$\begin{aligned}
 \text{Now } \phi(u_1, u_2, u_3) &= \int_{(a,b,c)}^{(u_1, u_2, u_3)} d\phi + c_0 \quad [c_0 \text{ is integration constant}] \\
 &= \int_{(a,b,c)}^{(u_1, u_2, u_3)} (\vec{\nabla}\phi) \cdot \vec{dr} + c_0 \\
 &= \int_{(a,b,c)}^{(u_1, u_2, u_3)} \vec{A} \cdot \vec{dr} + c_0 \\
 &= \int_{(a,b,c)}^{(a,b,u_3)} A_3 h_3 du_3 + \int_{(a,b,u_3)}^{(a,u_2,u_3)} A_2 h_2 du_2 + \int_{(a,u_2,u_3)}^{(u_1,u_2,u_3)} A_1 h_1 du_1 + c_0
 \end{aligned}$$

So,
 $\phi = (\vec{\nabla})^{-1}\vec{A}$

$$\begin{aligned}
 &= \int_a^{u_1} A_1 h_1 du_1 + \int_b^{u_2} A_2 h_2 du_2 + \int_c^{u_3} A_3 h_3 du_3 + c_0 \\
 &\quad u_1 = a \quad \quad \quad u_1 = a \\
 &\quad \quad \quad u_2 = b
 \end{aligned}$$

Here in the first integration u_1 is variable but other two (u_2, u_3) are fixed, and so on for other two integrations.

Note: The original work was done in 2002 and published in 2006 in Science and Culture [2]. This write-up is almost same as published one.

References

- [1] Any text on vector del operator and curvilinear co-ordinate system. For example: by George B. Arfken and Hans J. Weber, *Mathematical Methods For Physicists*, Academic Press: 5th edition (2000); 1st and 2nd chapters.
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