The Inverse Gradient PhL 3.12.14

Summary of Result: You are given this problem to solve for $\varphi(x,y)$

$$\nabla_{2D}\,\varphi(x,y) = \mathbf{A}(x,y)\ .$$

Here are two possible ways to write the solution

$$\varphi(x,y) - \varphi(x_1,y_1) = \int_{x_1}^{x} A_x(x,y_1)dx + \int_{y_1}^{y} A_y(x,y)dy$$

$$= \quad f(x) \quad + \quad g(x,y)$$

$$\varphi(x,y) - \varphi(x_1,y_1) = \int_{y_1}^{y} A_y(x_1,y)dy + \int_{x_1}^{x} A_x(x,y)dx$$

$$= \quad F(y) \quad + \quad G(x,y)$$

In graphics terms, you imagine you have a plot3D of a function $\varphi(x,y)$ as a height over a plane x,y and you know 2D gradient at every point x,y, and that gradient is called $\mathbf{A}(x,y)$. Then just from this gradient, you can figure out the function of the surface $\varphi(x,y)$, but you have to pick an arbitrary point x_1,x_2 in order to express the result. And $\varphi(x_1,y_1)$ is then selected as an arbitrary constant of integration. Knowing the gradient of course does not tell you the absolute DC level of a function. There is of course a 3D version of this result implied by the discussion below.

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This arose today in the context of "low freq limit of my theory.doc" concerning transmission lines doc and some repair work I was doing there. I had these equations

$$\partial_y E_z(x,y) = -j\omega B_x(x,y)$$
$$\partial_x E_z(x,y) = j\omega B_y(x,y)$$

and was wondering how to "solve for E_z". I realized after a while this said $\nabla_{2D}\varphi = \mathbf{A}$ and so the more general problem is this inverse gradient problem. Here are my notes moved here from the just mentioned doc, since this result is just a general calculus issue having nothing really to do with lines doc.

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Here I first thought of this as a system of two PDE's !

Now of course we hit exactly on a huge PL Knowledge Hole which is PDE's. Let's give it a dumb try (I did this somewhere, maybe Sneddon; no, in my own PDE notes!)

$$E_z(x,y) = -j\omega \int^{y} dy\, B_x(x,y) + f(x) \quad = F(x,y) + f(x)$$

$$E_z(x,y) = -j\omega \int^{x} dx\, B_y(x,y) + g(y) \qquad = G(x,y) + g(y)$$

Then it must be that

$$F(x,y) + f(x) = G(x,y) + g(y)$$

so

$$F(x,y) - G(x,y) = g(y) - f(x)$$

which seems rather amazing -- that the difference between these two messy functions is the difference of two functions each of one variable. It seems unlikely. My little 2-equation system is too simple for Polyanin, he doesn't do first order anything. Well, he has a separate first order handbook, but it doesn't have any "systems".

After that distraction, I got on with the problem in question:

Well how about a simpler approach. Suppose

$$\nabla\varphi = \mathbf{A} \qquad\qquad \mathbf{A} = \text{specified}$$

Then write

$$\varphi(\mathbf{r}) - \varphi(\mathbf{r}_1) = \int \nabla\varphi \bullet d\mathbf{r} = \int \mathbf{A} \bullet d\mathbf{r} \quad = \text{line integral in 3D space from } \mathbf{r}_1 \text{ to point } \mathbf{r}$$

$$= \int_{x1}^{x} A_x(x,y,z)dx + \int_{y1}^{y} A_y(x,y,z)dy + \int_{z1}^{z} A_z(x,y,z)dz \qquad \text{[wrong, fixed below]}$$

This obvious solution suggested by a PDF where a certain web author writes (pdf is nearby)

4 Inverse Gradient Operator

We indicate this operator by $(\vec{\nabla})^{-1}$.

Let $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$ and $\phi(u_1, u_2, u_3)$ be a vector and a scalar function, such that $\vec{A} = \vec{\nabla}\phi$. Since, the vector field represented by $\vec{A}$ is conservative, thus, line integral of $\vec{A}$ is path independent. Choose (a,b,c) as initial point, such that $\phi(a, b, c)$ exists.

$$\text{Now } \phi(u_1, u_2, u_3) = \int_{(a,b,c)}^{(u_1,u_2,u_3)} d\phi \quad + c_0 \qquad [c_0 \text{ is integration constant}]$$

$$= \int_{(a,b,c)}^{(u_1,u_2,u_3)} (\vec{\nabla}\phi) \cdot \vec{dr} \quad + c_0$$

$$= \int_{(a,b,c)}^{(u_1,u_2,u_3)} \vec{A} \cdot \vec{dr} \quad + c_0$$

$$= \int_{(a,b,c)}^{(a,b,u_3)} A_3 h_3 du_3 + \int_{(a,b,u_3)}^{(a,u_2,u_3)} A_2 h_2 du_2 + \int_{(a,u_2,u_3)}^{(u_1,u_2,u_3)} A_1 h_1 du_1 + c_0$$

So,

$\phi = (\vec{\nabla})^{-1}\vec{A}$

$$= \int_a^{u_1} A_1 h_1 \partial u_1 + \int_b^{u_2} A_2 h_2 \partial u_2 + \int_c^{u_3} A_3 h_3 \partial u_3 + c_0$$

$u_1 = a$ $\qquad$ $u_1 = a$, $u_2 = b$

Here in the first integration u_1 is variable but other two (u_2, u_3) are fixed, and so on for other two integrations.

Now I will attempt to do this "my way" and see if I get the same result. The above author has used an appropriate curvilinear coordinates scale factor h_i for each "leg" of his taxi-cab path, but I am just interested now in Cartesians so I ignore the h_i factors.

$\nabla_{2D}\varphi = \mathbf{A}$ $\qquad$ $\mathbf{A}$ = specified $\qquad$ solve for φ

So write

$\varphi(x,y) - \varphi(x_1,y_1) = \int \nabla\varphi \bullet d\mathbf{r} = \int \mathbf{A} \bullet d\mathbf{r}$ = line integral in 2D space from $\mathbf{r}_1$ to point $\mathbf{r}$

$$= \int_{x_1}^{x} A_x(x,y)dx + \int_{y_1}^{y} A_y(x,y)dy \qquad \text{[wrong still, need to see path]}$$

But then I get

$$\partial_x\varphi = \partial_x \left[\int_{x_1}^{x} A_x(x,y)dx \right] + \partial_x \left[\int_{y_1}^{y} A_y(x,y)dy \right] \qquad \text{[wrong]}$$

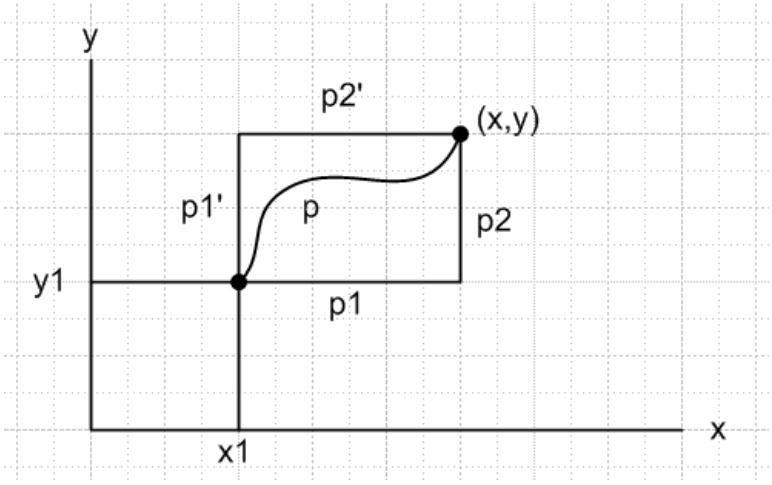
As usual, I am confused. Why doesn't this second term contribute. [Every question spawns 5 more questions.]

I think I should clear this up right here to have it in one place, so digressing on a spawned question now.

The claim is that when you write for $\nabla_{2D}\varphi = \mathbf{A}$

$$\varphi(x,y) - \varphi(x_1,y_1) = \int \nabla_{2D}\varphi \bullet d\mathbf{r} = \int \mathbf{A} \bullet d\mathbf{r} = \text{line integral in 2D space from } \mathbf{r}_1 \text{ to point } \mathbf{r}$$

the integral is the same for all possible paths of integration. Therefore, suppose you take the following carefully selected path written here as p1 + p2 (as opposed to some arbitrary path p)



Then:

$$\varphi(x,y) - \varphi(x_1,y_1) = \int \mathbf{A} \bullet d\mathbf{r} = \int_{p1} \mathbf{A} \bullet d\mathbf{r} + \int_{p2} \mathbf{A} \bullet d\mathbf{r}$$

$$= \int_{x_1}^{x} A_x(x,y_1)dx + \int_{y_1}^{y} A_y(x,y)dy$$

$$= \quad f(x) \quad + \quad g(x,y)$$

We could instead take the pair of primed paths. Then we get

$$\varphi(x,y) - \varphi(x_1,y_1) = \int \mathbf{A} \bullet d\mathbf{r} = \int_{p1'} \mathbf{A} \bullet d\mathbf{r} + \int_{p2'} \mathbf{A} \bullet d\mathbf{r}$$

$$= \int_{y_1}^{y} A_y(x_1,y)dy + \int_{x_1}^{x} A_x(x,y)dx$$

$$= \quad F(y) \quad + \quad G(x,y)$$

Staring at things, we find that

$$g(x_1,y) = F(y)$$
$$G(x,y_1) = f(x)$$

but I don't see how this adds anything.

Consider:

$$A_x(x,y) = \partial_x\varphi(x,y) = \partial_x f(x) + \partial_x g(x,y) = A_x(x,y_1) + \int_{y_1}^{y} \partial_x A_y(x,y)dy$$

$$A_y(x,y) = \partial_y\varphi(x,y) = \partial_y g(x,y) = A_y(x,y)$$

$$A_x(x,y) = \partial_x\varphi(x,y) = \partial_x G(x,y) = A_x(x,y)$$

$$A_y(x,y) = \partial_y\varphi(x,y) = \partial_y F(y) + \partial_y G(x,y) = A_y(x_1,y) + \int_{x_1}^{x} \partial_y A_x(x,y)dx$$

Again, probably OK, but not exactly relevant to what the web author is saying.

I don't see how this gives the author's result which I will first quote in 3D and change to 2D. Here is the quote:

$$= \int_{(a,b,c)}^{(a,b,u_3)} A_3 h_3 du_3 + \int_{(a,b,u_3)}^{(a,u_2,u_3)} A_2 h_2 du_2 + \int_{(a,u_2,u_3)}^{(u_1,u_2,u_3)} A_1 h_1 du_1 + c_0$$

So,

$$\phi = (\vec{\nabla})^{-1}\vec{A}$$

$$= \int_{a}^{u_1} A_1 h_1 \partial u_1 + \int_{b}^{u_2} A_2 h_2 \partial u_2 + \int_{c}^{u_3} A_3 h_3 \partial u_3 + c_0$$

$$u_1 = a \qquad \begin{matrix} u_1 = a \\ u_2 = b \end{matrix}$$

and here is my 2D version of the quote with the scale factors set to 1,

$$\varphi(u_1,u_2) = \int_{(a,b)}^{(a,u_2)} A_2 du_2 + \int_{(a,u_2)}^{(u_1,u_2)} A_1 du_1 + c_0$$

$$= \int_{(a,u_2)}^{(u_1,u_2)} A_1 du_1 + \int_{(a,b)}^{(a,u_2)} A_2 du_2 + c_0$$

$$= \int_{a}^{u_1} A_1(u_1,u_2)\, du_1 + \int_{b}^{u_2} A_2(a,u_2) du_2 + c_0$$

or

$$\varphi(x,y) = \int_{x_1}^{x} A_1(x,y)\, dx + \int_{b}^{y} A_2(x_1,y) dy + c_0 \qquad a = x_1 \quad x = u_1 \quad y = u_2$$

which compare to my 2nd result

$$\varphi(x,y) = \int_{x_1}^{x} A_x(x,y) dx + \int_{y_1}^{y} A_y(x_1,y) dy + \varphi(x_1,y_1)$$

$$= \quad G(x,y) \quad + \quad F(y)$$

So OK, we are saying the same thing.

Conclusion: You are given this problem to solve for $\varphi(x,y)$

$$\nabla_{2D}\, \varphi(x,y) = \mathbf{A}(x,y)$$

Here are two possible ways to write the solution

$$\varphi(x,y) - \varphi(x_1,y_1) = \int_{x_1}^{x} A_x(x,y_1) dx + \int_{y_1}^{y} A_y(x,y) dy$$

$$= \quad f(x) \quad + \quad g(x,y)$$

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In graphics terms, you imagine you have a plot3D of a function $\varphi(x,y)$ as a height over a plane x,y and you know 2D gradient at every point x,y, and that gradient is called $\mathbf{A}(x,y)$. Then just from this gradient, you can figure out the function of the surface $\varphi(x,y)$, but you have to pick an arbitrary point x_1,x_2 in order to express the result.