Two Homogeneous Linear Equations in 3 unknowns. PhL 12.9.04

This sounds completely trivial, but I have spent several hours unable to draw a clean set of conclusions! So let's start by writing down our two equations (where constants can take ANY values)

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a x + by  + c z  = 0
a'x + b'y + c'z  = 0
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You can think of (a,b,c) and (a',b',c') as the normals to the two planes implied by these equations. However, if (a,b,c) = (0,0,0), then there is no plane associated with the first equation, it just says 0=0 so we just throw it out and then we have 1 equation in 3 unknowns. So now we start our case analysis:

(Or just read conclusions at the end!)

Case 1. Assume that **(a',b',c') = (0,0,0)** so we have 1 equation in 3 unknowns.

Case 1A: If (a,b,c) = (0,0,0) as well, then both equations say 0=0 and we then have NO equations at all, so we cannot even talk about what the "solutions" look like. I guess any (x,y,z) solves NO equations.

Case 1B: If (a,b,c) = (a,0,0), then we have that x=0 as our only constraint. The solution is then (0,y,z) for any y,z, and we have geo = 2. Similarly if (a,b,c) = (0,b,0) or if (a,b,c) = (0,0,c).

Case 1C. If (a,b,c) = (a,b,0), then we have ax+by=0 so we can solve for y = kx , and then the solution has the form (x,kx,z) for any x and any z. We can take our basis as (1,k,0) and (0,0,1) which are clearly linearly independent, so again we have geo = 2. Same for cases (0,b,c) and (a,0,c)

Case 1D. If (a,b,c) = (a,b,c), then we have ax+by+cz = 0. This is a plane with n = (a,b,c) and any point on this plane satisfies our equation set. Our ray is then perhaps (x,y,-(ax+by)/c). One basis vector can then be (0,y,-by/c) and the other (x,0,-ax/c). These are lin indep and again we have geo = 2.

Theorem: If you have 1 linear homo equation in 3 unknowns, and the equation is not just 0=0, then the solution of the equation has two independent basis vectors.

Case 2. Assume that **(a',b',c') = (a',0,0)**. This says that x=0.

Case 2A. If (a,b,c) = (a,b,c) or (0,b,c), then we have by+cz = 0 and we solve for z = ky. Ray is then (0,y,ky) and our basis vector is perhaps (0,1,k) and that is all there is. So geo = 1 ! k = -b/c.

Case 2B. If (a,b,c) = (a,0,c) or (0,0,c) then we have cz = 0 so z = 0. Solution is then (0,y,0) and geo = 1. Same idea for (a,b,c) = (a,b,0) or (0,b,0).

Case 2C: If (a,b,c) = (a,0,0) or (0,0,0), then second equation 0 = 0 with x=0, so ray is (0,y,z) and geo = 2.

Case 2D: If (a,b,c) = (0,b,0) then y = 0 and ray is (0,0,z) so geo = 1.

Case 3: Assume that **(a',b',c') = (a',b',0)** which says a'x + b'y = 0 so y = kx or x = Ky.

Case 3A: If (a,b,c) = (a,b,c), then ax +b(kx) +cz = 0 so z = k'x and ray = (x,kx,k'x) or (1,k,k') and then we have geo = 1.

Case 3B: If (a,b,c) = (a,b,0), then ax +b(kx) = 0 and x = 0, so ray is (0,0,z) and again geo = 1.

Case 3C: If (a,b,c) = (0,b,c) then b(kx)+cz = 0 so z = k'x and ray is then (x,kx,k'x) and geo = 1.
Same idea for (a,0,c).

Case 3D: If (a,b,c) = (a,0,0) then x=0 so ray is (0,0,z) and geo = 1.
Same idea for (0,b,0)

Case 3E: If (a,b,c) = (0,0,c) then z=0 and ray is (x,kx,0) and geo = 1

Case 4: Assume that **(a',b',c') = (a',b',c')**.

Case 4A: If (a,b,c) = (a,b,c), then solve other equation for z = z(x,y) and plug that in here to get
ax + by + cz(x,y) = 0 and find that y = kx, so resulting ray is (x,kx, k'x) so (1,k,k') and we have geo = 1.

Case 4B: If (a,b,c) = (a,b,0), then solve other equation for y = y(x,z) and plug in here to get
ax + by(x,z) = 0. Solve this for z = k'x. Solution is then (x, y(x,k'x), k'x) = (x,k"x,k'x), geo = 1.

Case 4C: If (a,b,c) = (a,0,0), then get x=0 from this equation. Other then says b'y+c'z = 0 so get z = ky, and solution is then (0,y,ky) or (0,1,k) and geo = 1.

Table of Results: Gives the geometric multiplicity of the solution ray set.

		1	2	3	4
(a'b'c')	=	000	a'00	a'b'0	a'b'c'
abc =					
	000	3	2	2	2
	a00	2	2	1	1
	ab0	2	1	1	1
	abc	2	1	1	1
	0bc	2	1	1	1
	a0c	2	1	1	1
	00c	2	1	1	1
	0b0	2	1	1	1

Conclusions:

Again write down our two equations,

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a x + by  + c z  = 0
a'x + b'y + c'z  = 0
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You can think of (a,b,c) and (a',b',c') as the normals to the two planes implied by these equations. However, if (a,b,c) = (0,0,0), then there is no plane associated with the first equation, it just says 0=0 so we just throw it out and then we have 1 equation in 3 unknowns.

Case (a). If the second equation is a multiple of the first, then we really only have one equation. In this case, we are dealing with a sub-problem: 1 equation in 3 unknowns. Assuming that the first equation does not just say 0 = 0, then this is treated in Case 1 above and we found that geo = 2 in all subcases. This seems very reasonable: you have one plane and (x,y,z) must lie on that plane, so geo = 2.

Case (b). If one of the equations says 0 = 0, then again, we have 1 equation in 3 unknowns, and as in case (a) above, we get geo = 2. There is one plane and it is 2D.

Case (c). If neither equation is 0=0 and neither equation is a multiple of the other, then you are talking about the intersection of two planes which is a line, and you expect geo = 1. This is what we found in all the cases above.

We can summarize these cases by saying: if the two equations are linearly independent, then geo = 1, otherwise geo = 2. If either equation is 0=0, we include that in the linearly dependent situation. If both equations are 0=0, then in fact we have geo = 3 because there are then no planes at all!