VECTOR IDENTITIES

PhL 11.18.09

---

1. STRAIGHT

$[\mathbf{A},\mathbf{B},\mathbf{C}] \equiv \mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet (\mathbf{C} \times \mathbf{A}) = \varepsilon_{ijk} A_i B_j C_k = \det [\ (\mathbf{A})(\mathbf{B})(\mathbf{C})\ ]$

= + cyclic = - anticylic = - swap (0 if two same) = ABCsin(B,C) = volume of piped **A**,**B**,**C**

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C}$ $\qquad$ $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \text{cyclic} = 0$

$\mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{A} - A^2\mathbf{C}$ $\qquad$ $\mathbf{A} \times (\mathbf{A} \times \mathbf{A}) = 0$ $\qquad$ $\mathbf{A} \times \mathbf{A} = 0$

$(\mathbf{A} \times \mathbf{B}) \bullet (\mathbf{C} \times \mathbf{D}) = (\mathbf{A}\bullet\mathbf{C})\,(\mathbf{B}\bullet\mathbf{D}) - (\mathbf{A}\bullet\mathbf{D})\,(\mathbf{B}\bullet\mathbf{C})$

$(\mathbf{A} \times \mathbf{B}) \bullet (\mathbf{A} \times \mathbf{D}) = A^2\,(\mathbf{B}\bullet\mathbf{D}) - (\mathbf{A}\bullet\mathbf{D})\,(\mathbf{B}\bullet\mathbf{A})$

$(\mathbf{A} \times \mathbf{B}) \bullet (\mathbf{A} \times \mathbf{B}) = A^2B^2 - (\mathbf{A}\bullet\mathbf{B})^2$

$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{A},\mathbf{B},\mathbf{D}]\,\mathbf{C} - [\mathbf{A},\mathbf{B},\mathbf{C}]\,\mathbf{D}$

$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{A} \times \mathbf{D}) = [\mathbf{A},\mathbf{B},\mathbf{D}]\,\mathbf{A}$

$\mathbf{A} \times [\ \mathbf{B} \times (\mathbf{C} \times \mathbf{D})\ ] = \mathbf{B}\,[\mathbf{A},\mathbf{C},\mathbf{D}] - (\mathbf{A}\bullet\mathbf{B})\,(\mathbf{C} \times \mathbf{D})$

$\mathbf{A} \times [\ \mathbf{A} \times (\mathbf{C} \times \mathbf{D})\ ] = \mathbf{A}\,[\mathbf{A},\mathbf{C},\mathbf{D}] - A^2(\mathbf{C} \times \mathbf{D})$

$\mathbf{A} \times [\ \mathbf{A} \times (\mathbf{A} \times \mathbf{D})\ ] = -\,A^2(\mathbf{A} \times \mathbf{D})$

$\mathbf{A} \bullet [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})\ ] = (\mathbf{A}\bullet\mathbf{C})(\mathbf{B}\bullet\mathbf{D}) - (\mathbf{A}\bullet\mathbf{D})(\mathbf{B}\bullet\mathbf{C})$

---

2. GRADIENT AND LAPLACIAN

$\nabla(\varphi v) = \varphi\nabla\psi + \psi\nabla\varphi$

$\nabla^2(\varphi\psi) = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\,\nabla\varphi \bullet \nabla\psi$

$\nabla(\mathbf{A}\bullet\mathbf{B}) = (\mathbf{A}\bullet\nabla)\mathbf{B} + (\mathbf{B}\bullet\nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$ $\qquad$ p 153 MM

---

3. DIVERGENCE

$\nabla\bullet(\varphi\mathbf{A}) = \mathbf{A} \bullet \nabla\varphi + \varphi\,(\nabla\bullet\mathbf{A}) \quad \Rightarrow \quad \nabla\bullet(x_i\mathbf{A}) = \mathbf{A}_i + x_i\,(\nabla\bullet\mathbf{A})$

$\nabla\bullet(\nabla\varphi) = \nabla^2\varphi$

$\nabla\bullet(\nabla\text{x}\mathbf{A}) = 0$

$\nabla\bullet(\mathbf{A}\text{x}\nabla) = 0 \quad \Rightarrow \nabla\bullet(\mathbf{r} \times \nabla) = 0$

$\nabla\bullet(\varphi\nabla\psi) = \nabla\varphi\bullet\nabla\psi + \varphi\,\nabla^2\psi$

$\nabla\bullet(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet (\nabla \times \mathbf{A}) - \mathbf{A} \bullet (\nabla \times \mathbf{B})$

$\nabla\bullet(\,\varphi\nabla \times \mathbf{A}) = \nabla\varphi \bullet (\nabla \times \mathbf{A})$

---

3. CURL

$\nabla \times (\varphi \mathbf{A}) = \nabla\varphi \times \mathbf{A} + \varphi (\nabla \times \mathbf{A})$
$\nabla \times (\nabla\varphi) = 0$
$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \bullet A) - \nabla^2 \mathbf{A}$
$\nabla \times (\varphi \nabla\psi) = \nabla\varphi \times \nabla\psi$

$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} (\nabla \bullet \mathbf{B}) - \mathbf{B} (\nabla \bullet \mathbf{A}) + (\mathbf{B} \bullet \nabla)\mathbf{A} - (\mathbf{A} \bullet \nabla)\mathbf{B}$

$\nabla \times (\mathbf{A} \times \nabla) = ??$ $\quad \nabla \times (\mathbf{r} \times \nabla) = \mathbf{r}\, \nabla^2 - \nabla (1 + r\, \partial_{\mathbf{r}})$
$\nabla \times (\hat{\mathbf{r}} \times \nabla f(\theta,\varphi)) = \nabla^2 f\, \hat{\mathbf{r}} - (1/r)\, \nabla f$

$A \bullet (\nabla \times \mathbf{B}) = (\mathbf{A} \times \nabla) \bullet B$

---

4. RADIAL

$\nabla(f(r)) = f'(r)\, \hat{\mathbf{r}}$ $\quad \Rightarrow \quad$ $\nabla(r) = \hat{\mathbf{r}}$ $\quad \nabla(1/r) = -\hat{\mathbf{r}}/r^2$
$\nabla(e^{ikr}/r) = \hat{\mathbf{r}}(ik/r - 1/r^2)\, e^{ikr}$

$\nabla \times [f(r)\mathbf{r}] = 0$ $\quad \Rightarrow \quad$ $\nabla \times \hat{\mathbf{r}} = 0$ $\quad \nabla \times \mathbf{r} = 0$

$\nabla \bullet [f(r)\mathbf{r}] = 3\, f(r) + r\, f'(r)$ $\quad \Rightarrow \quad$ $\nabla \bullet \mathbf{r} = 3$ $\quad \nabla \bullet \hat{\mathbf{r}} = 2/r$

$\nabla \bullet (\mathbf{r} \times \nabla) = 0$

$\nabla^2(1/r) = -4\pi\, \delta^3(\mathbf{r})$ $\quad \nabla^2(\mathbf{r}) = 0$ $\quad \nabla^2(e^{ikr}/r) = -k^2\, (e^{ikr}/r)$

$\nabla^2[\, f(\mathbf{r})\, \mathbf{r}\,] = (\nabla^2 f)\, \mathbf{r} + 2\, \nabla f$

Derivations (started 8.10.12)

$\mathbf{A} \bullet [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})\,] = A_i \varepsilon_{ijk} B_j (\mathbf{C} \times \mathbf{D})_k = A_i \varepsilon_{ijk} B_j \varepsilon_{kab} C_a D_b$

$= \varepsilon_{ijk} \varepsilon_{kab} A_i B_j C_a D_b$

$= \varepsilon_{ijk} \varepsilon_{abk} A_i B_j C_a D_b$

$= (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja})\, A_i B_j C_a D_b$

$= (\mathbf{A} \bullet \mathbf{C})(\mathbf{B} \bullet \mathbf{D}) - (\mathbf{A} \bullet \mathbf{D})(\mathbf{B} \bullet \mathbf{C})$