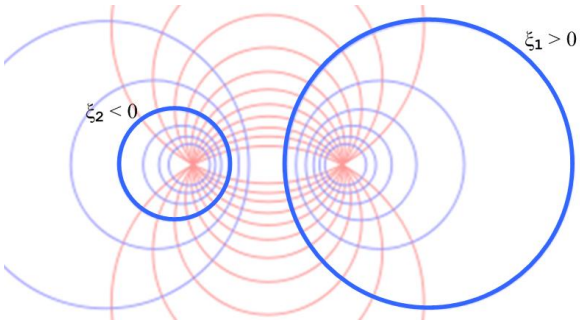
Practice on the bipolar coordinates Capacitor Problem PhL 2.18.14

The first question here is: how do you arrive at the claim that $\varphi = c\xi$ for the cylinders. I give a good answer to that now in bipolar doc. Most of the rest is "practice" for the pathway from φ to **E** to n to q. I spent a lot of hours then making the connection to φ_t and K of lines doc. This last is now clarified and written up in bipolar doc.

I arrive first at this picture:



which depicts the two cylinders.

Plan A. If this picture is viewed from "far away" , the cylinders become line charges? But I don't really have a justification for that claim. They are always offset from the focal point no matter how far away you go.

Plan B. Different argument: Suppose $\varphi(\xi)$ is the solution potential for the above capacitor in the region between the conductors. If we were to replace the right conductor cross section with the next smaller blue circle, ξ_1' say, we claim that the potential solution for this new capacitor problem is *the same* function $\varphi(\xi)$. We know that the function $\varphi(\xi)$ will be a constant on this smaller right blue circle since that entire circle has the value $\xi = \xi_1'$. We know that the Laplace equation is unique, so the same $\varphi(\xi)$ must solve both problems in the space between the conductors of each problem.

But I don't believe this argument. As you change one conductor, everything could shift around!

Plan C. Suppose I try my general method: that means I have to find the full Green's function. Do I know the Green's function for the above set of conductors? Heck no! I put a point charge in some weird location, it is probably a horrible function. So my general method would be very difficult to carry out.

Plan D. Suppose I make an *ansatz* that the solution potential has this form

$$\varphi(\xi,u) = \varphi(\xi) = c\,\xi$$

where c is some unknown constant that probably depends on ξ_1 and ξ_2 and a. At least I can then say that this potential is constant on the two blue cylinder surfaces, so it is at least a viable candidate solution.

$V_1 = \varphi(\xi_1) = c\xi_1$
$V_2 = \varphi(\xi_2) = c\xi_2$

$\Delta V = V_1 - V_2 = c(\xi_1 - \xi_2)$

Side question: what are the dimensions of φ? Recall that in 3D we have $-\nabla^2\varphi(\mathbf{x}) = \rho(\mathbf{x})/\varepsilon$. If φ does not depend on z, this becomes $-\nabla^2_{2D}\,\varphi(x,y) = \rho(\mathbf{x})/\varepsilon$. Suppose $\rho(\mathbf{x})$ describes a line charge in cylindrical coordinates at r = 0. Then $\rho(x) = \lambda\,\delta(r)/2\pi r$. The RHS has dim = C/m * 1/m * 1/m = C/m^3 correct. Now consider $-\nabla^2_{2D}\, f(x,y) = \delta(\mathbf{r})_{2D}$. We know that $f(x,y) = \ln(1/r)/2\pi$ .Then from (I.1.8) I get

$-\nabla^2\,\varphi(\mathbf{x}) = \rho(\mathbf{x})/\varepsilon. \quad \Rightarrow \quad \varphi(\mathbf{x}) = \int d^2x'\,[\ln(1/R)/2\pi]\,\rho(\mathbf{x}')/\varepsilon$ + homogeneous solutions

The Poisson Equation (I.1.8)

so then

$\varphi(\mathbf{x}) = \int d^2x'\,[\ln(1/R)/2\pi]\,\lambda\,\delta(r')/2\pi r'\,(1/\varepsilon) \qquad R = |\,\mathbf{x} - \mathbf{r}'\,|$

$= (\lambda/2\pi\varepsilon) \int d^2x'\,[\ln(1/R)/2\pi]\;\delta(r')/r'$

$= (\lambda/2\pi\varepsilon) \int d^2x'\,[\ln(1/R)/2\pi]\;\delta(r')/r'$

$= (\lambda/2\pi\varepsilon)\,2\pi \int r'dr'\,[\ln(1/R)/2\pi]\;\delta(r')/r'$

$= (\lambda/2\pi\varepsilon) \int dr'\,[\ln(1/R)]\;\delta(r')$

$= (\lambda/2\pi\varepsilon)\,[\ln(1/R)|^{r'=0}]$

$= (\lambda/2\pi\varepsilon)\,\ln(1/|\mathbf{x}|)$

So one might write

$\varphi(\mathbf{r}) = (\lambda/2\pi\varepsilon)\,\ln(1/r)$ // like 3D that says $\varphi(r) = (q/4\pi\varepsilon)(1/r)$.

This 2D result is in agreement with my Appendix L result that says

$\varphi_1(r) = (1/2\pi\varepsilon_1)\,q\,\ln(1/r) + (q/2\pi)\,\ln(1/R)\,(1/\varepsilon_0 - 1/\varepsilon_1)$ (L.2.6)'

where we can ignore the constant second term and we have q = λ.

Now that I finally have a real-world potential, what are its units?

$\varphi(\mathbf{r}) = (\lambda/2\pi\varepsilon) \ln(1/r)$ = C/m * m/F = C/F = volt

So in 2D, the potential still has dimensions of volts.

Therefore:

dim(c) = volts

What is the electric field implied by this potential?

$\mathbf{E}$ = - grad φ

But I showed earlier that in bipolar coordinates

$$[\text{grad } f](\mathbf{x}) = (1/h) \left[(\partial_\xi f)\, \hat{\boldsymbol{\xi}} + (\partial_u f)\, \hat{\mathbf{u}} \right]$$

so then

$$[\text{grad } \varphi](\mathbf{x}) = (1/h) \left[(\partial_\xi \varphi)\, \hat{\boldsymbol{\xi}} \right] = (c/h)\, \hat{\boldsymbol{\xi}}$$

$$\mathbf{E} = -\, (c/h)\, \hat{\boldsymbol{\xi}}$$

This is certainly a simple result. I don't know the sign of α yet, so all I can say is that this electric field when evaluated at $\xi = \xi_1$ is normal to the cylinder surface, which is what it must be in electrostatics.

Now since $\hat{\boldsymbol{\xi}}$ points "inward" at my ξ_1 sphere on the right, if I let E_n be the normal electric field pointing out, then the above says

$E_n(\xi_1,u) = c/h(\xi_1,u)$ dim OK, both sides are volts/m

Now, how does one obtain from this the surface charge density on the surface? This sounds like a possibly treacherous area. I am tempted to use my Section 1 lines doc formula which says

$n = \varepsilon E_n$ (1.1.47) dim OK

// dim $n = C/m^2$ $\dim(\varepsilon E_n)$ = farad/m * volts/m = C/m^2

This n is a 2D charge density. There is some possible sign ambiguity which I will ignore. This seems then to say

$n(\xi_1,u) = (c\varepsilon)\, 1/h(\xi_1,u)$ $h(\xi_1,u) = a/(\text{ch}\xi_1 - \text{cos}u)$

$n(\xi_1,u) = (c\varepsilon/a)\,(ch\xi_1 - cosu)$ // dim = Coul/m^2 both sides OK

This n has dimensions C/m^2 and it is the physical surface charge density on the surface of the right cylinder which has parameter ξ_1. It is a function of u. [I tried but am unable to describe this in terms of a 1D charge density.]

Now, how do we describe a patch of area on the conductor where this surface charge n exists? I think the distance can be dz in the z direction along the cylinder, and distance will be ds around the perimeter. My idea then is to use result

$$(ds)^2 = \Sigma_k\, h_k^2\, (dx_k)^2 = [a^2/(ch\xi - cosu)^2]\,[\,(d\xi)^2 + (du)^2] \qquad (3.4)$$

which tells me that if I don't vary ξ, I basically have

$(ds)^2 = h^2\,(du)^2$ $\qquad$ $ds = h\,du$

which I guess is what a scale factor is all about! So the charge on my patch is

$$dq = n(\xi_1,u)dz\,ds = n(\xi_1,u)dz\,h\,du = [(c\varepsilon)\,1/h(\xi_1,u)]\,dz\,h\,du = c\varepsilon\,dz\,du$$

Notice that dq is a constant times du so equal du's have equal dq's. But equal du's do not correspond to equal $d\theta$'s.

Then the total charge in a ring around our conductor is

$$Q = c\varepsilon\,dz \int du = 2\pi\varepsilon\,c\,dz$$

The charge per unit length on conductor C_1 is then

$q = 2\pi\varepsilon\,c$ $\qquad$ /Coul/m

At least I seem to be going *somewhere*, albeit at a snail's pace. Exploration in the canoe.

For the other conductor, I think the result is exactly but same but we pick up a minus sign because for a blue circle on the left, $\hat{\xi}$ points out instead of points in, so E_n has a change of sign. So here then is what I think I have found

$q_1 = 2\pi\varepsilon\,c = q$ // charge per unit length on conductor C_1 on the right
$q_2 = -2\pi\varepsilon\,c = -q$

Am I yet in a position to talk about capacitance?

$q = -C\Delta V$

$C = -q/\Delta V = -2\pi\varepsilon\, c\, /\, [c(\xi_1-\xi_2) = 2\pi\varepsilon/[\xi_1-\xi_2]$

Issues now to study:

(1) why is this result independent of ansatz constant c ?

(2) are the dimensions of C what I expect them to be? **DONE**

(3) does this result agree with lines doc where I refer to ξ_i as B_i ? Yes, **DONE**

Sisyphus: each question answered spawns 5 new questions.

Issue (2): Looking at $C = 2\pi\varepsilon/[\xi_1-\xi_2]$, dim(C) = farad/m since capacitance per meter.
But dim(ε) = farad/m as well, so dimensions are OK.

Issue (3): In (6.3.2) I claim that $K = 2(B_2-B_1)$ so then $K = 2(\xi_2-\xi_1)$. But why is this negative?

ouch! I have C_1 on the left in lines doc since that is how I always drew things with two conductors. So C_2 is on the right, and it is B_2 that is ξ_1 !! That is unfortunate.

But apart from that problem, we get agreement, so (3) is OK.

But I still don't know the constant c !! Well, I do know that

$\Delta V = V_1-V_2 = c(\xi_1-\xi_2)$

So if the potential between the two conductors is V > 0, then I can say

$c = V/(\xi_1-\xi_2)$

Then my ansatz potential is

$\varphi = [V/(\xi_1-\xi_2)]\, \xi$

Suppose I write this in the language of lines doc

$$\varphi = \frac{1}{4\pi\xi}\, q\, \varphi_t$$

Then I would argue that

$\varphi_t = 4\pi\varepsilon\, \varphi/q = (4\pi\varepsilon/q)\ [V/(\xi_1-\xi_2)]\, \xi$

$= 4\pi\varepsilon\, [\xi/(\xi_1-\xi_2)]\, (V/q)$

$= 4\pi\varepsilon\, [\xi/(\xi_1-\xi_2)]\, (1/C)$

$= 4\pi\varepsilon\, [\xi/(\xi_1-\xi_2)]\, (1/2\pi\varepsilon/[\xi_1-\xi_2])$

$= 4\pi\varepsilon\ [\xi/(\ \xi_1-\xi_2)]\ [(\xi_1-\xi_2)/2\pi\varepsilon)$

$= 4\pi\varepsilon\ [\xi]\ [1/2\pi\varepsilon)$

$= 2\ \xi$

$= 2\ \ln\left(\frac{s_1}{s_2}\right)$

and finally this agrees with lines doc.

$\varphi_t(\mathbf{x}) \approx \ln(s_2^2/s_1^2)$ // limiting form as point $\mathbf{x} = (x,y)$ moves far from the conductors

So it has taken me about 8 hours to obtain this factor of 2, and I am not really sure how I did it!