Atoms and Problems for Cartesian Coordinates PhL 7.13.10

I consider here two Green's Function geometries: (1) point charge inside an infinite rectangular waveguide; (2) point charge between two infinite metal planes. For the waveguide, I also show a Dirichlet Problem solution.

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0. Overview (5 pages, written 8.22.10)

In the opening text I write two "atomic forms" for Cartesian coordinates. The first has two oscillatory coordinates, the second has only one. Here they are:

1 $[\sin(k_x x), \cos(k_x x)], [\sin(k_y y), \cos(k_y y)], \quad [\exp(\kappa_z z), \exp(-\kappa_z z)]$ $\quad k_z$ = imaginary = $i\kappa_z$

2 $[\sin(k_x x), \cos(k_x x)], [\exp(\kappa_y y), \exp(-\kappa_y y)], [\exp(\kappa_z z), \exp(-\kappa_z z)]$ $\quad k_y$ and k_z = imaginary

A major theme of this doc is that you cannot really do anything with a form like this second form since it has only one oscillatory coordinate. This point is summarized in the closing Comment.

Example 1 was originally just a **waveguide** problem, but it diversified into other things as well:

In **Section A** I treat the **interior Green's function** for an infinite rectangular waveguide whose cross section is located in the (x,y) first quadrant with dimensions (a_x,a_y). The Green's charge is at $(x_1,y_1,0)$. I

assume the obvious Smythian form with two transverse oscillatory coordinates and I use the pillbox method to find the coefficients. Here is the Smythian form, the solved-for coefficients, and the final answer

$$u(x,y,z) = \Sigma_{\mathbf{nx,ny}} A_{\mathbf{nx,ny}} \sin(k_{\mathbf{x}}x) \sin(k_{\mathbf{y}}y) \exp(-\kappa_{\mathbf{z}}|z|)$$
$$A_{\mathbf{nx,ny}} = 2\pi q \ (1/a_{\mathbf{x}}a_{\mathbf{y}}\kappa_{\mathbf{z}}) \sin(k_{\mathbf{x}}x_1) \sin(k_{\mathbf{y}}y_1)$$
$$u(x,y,z) = (2\pi q/a_{\mathbf{x}}a_{\mathbf{y}}) \Sigma_{\mathbf{nx=1}}^{\infty}\Sigma_{\mathbf{ny=1}}^{\infty} \sin(k_{\mathbf{x}}x_1)\sin(k_{\mathbf{y}}y_1) \sin(k_{\mathbf{x}}x) \sin(k_{\mathbf{y}}y) \exp(-\kappa_{\mathbf{z}}|z|)/\kappa_{\mathbf{z}}$$
$$k_{\mathbf{x}} = n_{\mathbf{x}}(\pi/a_{\mathbf{x}}) \quad k_{\mathbf{y}} = n_{\mathbf{y}}(\pi/a_{\mathbf{y}}) \quad \kappa_{\mathbf{z}} = \sqrt{k_{\mathbf{x}}^2 + k_{\mathbf{y}}^2}$$

In **Section B** I try to take a **limit** of the waveguide result as $a_{\mathbf{x}}$ and $a_{\mathbf{y}} \to \infty$ to obtain a 1/R expansion in terms of Cartesian atoms, as I did in other coordinate systems. The limit does not seem possible to do (since it forces V=0 on the x=0 and y=0 planes!) .

In **Section C** I write the 1/R expansion in terms of a triple integral where the integrand is clearly NOT a Cartesian atom. I then evaluate this integral several ways. Here are some results of this section: (the first is verified) ($R = |\mathbf{r} - \mathbf{r_1}|$)

$$1/R = (1/2\pi^2) \int d^3k \exp[\, i\mathbf{k}\bullet(\mathbf{r} - \mathbf{r_1}) \,] / k^2$$
$$1/R = (2/\pi)^2 \int_{\mathbf{0}}^{\infty} dk_{\mathbf{x}} \int_{\mathbf{0}}^{\infty} dk_{\mathbf{y}} \cos(k_{\mathbf{x}}[x-x_1]) \cos(k_{\mathbf{y}}[y-y_1]) K_0(|z-z_1|/\sqrt{k_{\mathbf{x}}^2 + k_{\mathbf{y}}^2})$$

In **Section D** I do the **Waveguide Dirichlet** problem and get this result:

$$u(x,y,z) = \Sigma_{\mathbf{nx,ny=1}}^{\infty} A_{\mathbf{nx,ny}} \sin(k_{\mathbf{x}}x) \sin(k_{\mathbf{y}}y) \exp(-\kappa_{\mathbf{z}}|z|)$$
$$A_{\mathbf{nx,ny}} = (1/a_{\mathbf{x}}a_{\mathbf{y}}) \int_{\mathbf{0}}^{\mathbf{a_x}} dx \int_{\mathbf{0}}^{\mathbf{a_y}} dy \sin(k_{\mathbf{x}}x) \sin(k_{\mathbf{y}}y) f(x,y)$$

where f(x,y) is the potential prescribed on the z = 0 plane inside the waveguide, 0 on the walls.

In **Section E** I comment that both the Green's Function and Dirichlet problems have **exterior** solutions, and I outline a superposition of four quadrants scheme to solve the exterior Dirichlet problem.

Example 2 is the **Green's Function for two parallel plates in Cartesian coordinates**. The text opens with a comment that my original goal was to show how you get in a jam if you don't have two oscillatory coordinates perp to your pillbox condition. This is illustrated in Attempt D below. I then made several "attempts" to solve this problem. The point charge is at x=y=0 and at z = b where 0 < b < d=S.

In **Attempt A** I tried a seemingly reasonable form, getting expo decay in x and y (so this is osc only in z)

$$u_+(x,y,z) = \Sigma_{\mathbf{nz}} \int_{\mathbf{0}}^{\infty} d\kappa_{\mathbf{x}} A_{\mathbf{nz},\kappa\mathbf{x}} \exp(-\kappa_{\mathbf{x}}|x|) \exp(-\kappa_{\mathbf{y}}|y|) \sin(k_{\mathbf{z}}z) \qquad z > b$$

$$u_-(x,y,z) = \Sigma_{\mathbf{nz}} \int_{\mathbf{0}}^{\infty} d\kappa_{\mathbf{x}} B_{\mathbf{nz},\kappa\mathbf{x}} \exp(-\kappa_{\mathbf{x}}|x|) \exp(-\kappa_{\mathbf{y}}|y|) \sin(k_{\mathbf{z}}z) \qquad z < b$$

But matching on the z = b plane forces A = B, so u_+ and u_- are exactly the same and then the pillbox condition $\partial_{\mathbf{z}}u_-(x,y,z) - \partial_{\mathbf{z}}u_+(x,y,z) = 4\pi q\, \delta(x)\delta(y)$ has a vanishing LHS and we are dead in the water. The conclusion is that this Smythian form does not cut the mustard, even though it "looks good".

In **Attempt B** I go back to oscillatory in the x and y directions. I try this:

$$u(x,y,z) = \int_0^\infty dk_x \int_0^\infty dk_y \cos(k_x x)\cos(k_y y)\,[\, A_{kx,ky}\sinh(\kappa_z z) + B_{kx,ky}\cosh(\kappa_z z)\,]$$

But when I require u(z=0) = 0 I get B = 0, and then u(z=d) = 0 forces A = 0 as well! So this "form" is also no good, but it is closer!

Now let's momentarily skip Attempt B1 and move on, then we'll come back to B1.

In **Attempt C** I try but fail to move the point charge away from x = y = 0. Stillborn.

In **Attempt D** I put the point charge at $x = x_1$ instead of x = 0, and I try this *triple* form:

$$u_1(x,y,z) = \Sigma_{nz} \int_0^\infty d\kappa_x\,[A\sinh(\kappa_x x) + B\cosh(\kappa_x x)]\ \exp(-\kappa_y |y|)\sin(k_z z) \qquad -x_1 < x < x_1$$

$$u_2(x,y,z) = \Sigma_{nz} \int_0^\infty d\kappa_x\ C\exp(-\kappa_x x)\exp(-\kappa_y |y|)\sin(k_z z) \qquad x > x_1$$

$$u_3(x,y,z) = \Sigma_{nz} \int_0^\infty d\kappa_x\ D\exp(\kappa_x x)\exp(-\kappa_y |y|)\sin(k_z z) \qquad x < -x_1$$

which you see is back to single-oscillatory in z. I flail around trying to match V and its slope at various boundaries in space to grind down to only a single coefficient $A = A_{nz}(\kappa_x,\kappa_y)$. I then try to do the pillbox in the x direction (rather than the usual z direction) and I get this condition

$$2\,\Sigma_{nz} \int_0^\infty d\kappa_x\ \kappa_x A\,[\cosh(\kappa_x x_1) - \tanh^2(\kappa_x x_1)]\exp(-\kappa_y |y|)\sin(k_z z) = 4\pi q\,\delta(y)\delta(z-b)$$

I can use up the orthogonality in z, but then I am stuck with this result

$$2\pi \int_0^\infty d\kappa_x\ \kappa_x A\,[\cosh(\kappa_x x_1) - \tanh^2(\kappa_x x_1)]\exp(-\kappa_y |y|) = 4\pi q\,\delta(y)\sin(k_z b)$$

and then we are completely stuck. All this stuff is probably riddled with errors, but I am just showing how you get *stuck* in the end if you don't have two oscillatory directions perp to your pillbox! I go on to show that if you try the Dirichlet problem instead of the Green's one, you get this same issue where you cannot "invert" to find the coefficient because two of the coordinates are expo/radial.

Now we come back to **Attempt B1** which I added later. Attempt B1 is like Attempt B, but I allow for two sets of coefficients, here unprimed and primed: (the pillbox perp direction is z)

$$u(x,y,z) = \int_0^\infty dk_x \int_0^\infty dk_y \cos(k_x x)\cos(k_y y)\,[\, A_{kx,ky}\sinh(\kappa_z z) + B_{kx,ky}\cosh(\kappa_z z)\,] \qquad z < b$$

$$u(x,y,z) = \int_0^\infty dk_x \int_0^\infty dk_y \cos(k_x x)\cos(k_y y)\,[\, A'_{kx,ky}\sinh(\kappa_z z) + B'_{kx,ky}\cosh(\kappa_z z)\,] \qquad z > b$$

I could have written down all the matching conditions and obtained the four coefficients. But by this time, I knew from other sources the correct answer, which I now just quote:

$$u(x,y,z) = (1/\pi^2) \int_0^\infty \int_0^\infty dk_x dk_y\ \cos(k_x x)\cos(k_y y)\sinh[\kappa_z(d - z_>)]\sinh(\kappa_z z_<)/[\,\kappa_z \sinh(\kappa_z d)]$$

$z_> = \max(z,b)$ $z_< = \min(z,b)$ $0 < b < d$ (between the plates) $\kappa_z = \sqrt{k_x^2 + k_y^2}$

You can see that this implies certain values for A,B,A' and B'. I found this solution in the Rothwell/Cloud book, and I have notes on that solution in the same doc quoted above, " question on Fourier + parallel plates Cartesian.doc" in same folder as this doc.

So with Rothwell's help, I was finally able to solve the Green's function for parallel plates *in Cartesian coordinates*. But when I was first writing the doc you are reading, I did not have that solution, so I went on to study the cylindrical coordinates solution of the same problem.

Example 3 is the **Green's Function for two parallel plates in *cylindrical* coordinates**. For the Smythian form I selected this: (which pre-matches at $\rho = \rho_1$)

$$u_i(z,\rho,\varphi) = \Sigma_{nz}\ \sin(k_z z)\ \Sigma_m\ A_{m,nz}\ K_m(k_z\rho_1)\ I_m(k_z\rho)\cos(m\varphi) \qquad k_z = n_z(\pi/a_z)$$
$$u_o(z,\rho,\varphi) = \Sigma_{nz}\ \sin(k_z z)\ \Sigma_m\ A_{m,nz}\ K_m(k_z\rho)\ I_m(k_z\rho_1)\cos(m\varphi)$$

The two oscillatories here are z and φ (both I and K are radial/expo). I put the point charge at $(z_1,\rho_1,0)$ and then apply a pillbox at $\rho = \rho_1$ to find, eventually, that

$$A_{m,nz} = (4q/S)\,\varepsilon_m \sin(n_z\theta_{z1}) \qquad \theta_{z1} \equiv \pi(z_1/S) \qquad S = \text{plate separation}$$

Installing this coefficient, the solution comes out being (where $k_z = n_z(\pi/S)$)

$$u(z,\rho,\varphi) = (4q/S)\ \Sigma_{nz=1}^{\infty}\ \sin(k_z z_>)\sin(k_z z_<)\ \Sigma_{m=0}^{\infty}\ \varepsilon_m\ K_m(k_z\rho_>)\ I_m(k_z\rho_<)\ \cos(m[\varphi-\varphi_1]) \qquad (\#)$$

where I have reinstalled φ_1 just to show that we have full $\mathbf{r}_1 \leftrightarrow \mathbf{r}$ symmetry. If we insist that the point charge be exactly on the z axis, we must take the limit $\rho_1 \to 0$. The result becomes (o = outside)

$$u_o(z,\rho,\varphi) = (4q/S)\ \Sigma_{nz=1}^{\infty}\sin(k_z z)\sin(k_z z_1)\ K_0(k_z\rho) \qquad k_z = n_z(\pi/S) \qquad (*)$$

Our only discomfort now is that $K_0(k_z\rho) \to -\ln(k_z\rho/2) - .5772$ as $\rho \to 0$. This says that the solution to this Green's problem *seems* to diverge logarithmically at every point on the z axis between the plates. I would guess this somehow goes away in the sum.

Next, I consider the nature of (*) when $z = z_1$ and $\rho > 0$. Then (*) takes this form

$$u_o(z,\rho,\varphi) = (1/2)(4q/S)\ \Sigma_{k=1}^{\infty}\ [\,1 - \cos(k2a_1)]\ K_0(kx) \qquad (**)$$

where $x = (\pi/S)\rho$ and $a_1 = (\pi/S)z_1$

Amazingly, I was able to find a closed form expression for the infinite sum $\Sigma_{k=1}^{\infty}\cos(bk)\,K_0(kx)$ in PBM which I can apply to the two terms in (**) above. Doing this, and then taking the small arg limit of K_0 as shown above, I find this result:

$$u_o(z,\rho,\varphi) = q/\rho + \text{terms that vanish as } \rho \to 0$$

which lends some credence to our result (*) above. I was later able to find solution (#) above in blue Jackson (problem 3.17), where I had to "fold over" his negative m sum part. Later in problem 3.20 Jackson verifies my limiting result (*) above as well. So I know things are right here!

Finally, I have a concluding Comment concerning the need for having two oscillatory coordinates if you expect to solve a problem by obtaining a coefficient by double orthogonality. The notion of ill-defined expansions comes up in this comment.

_____end of summary_____

See ODEs / "essay on separable coordinates.doc" for background. When we do our separation of the Laplace equation Lu=0 in n dimensions, n-1 separation constants appear (and here, n = 3, so n-1=2).

In Cartesians, the three dimensions are equivalent. The separated equations have this form

$\partial_x^2 X(y) = k_x^2 X(x)$ $\qquad$ $u(x,y,z) = X(x)\, Y(y)\, Z(z)$

$\partial_y^2 Y(y) = k_y^2 Y(y)$

$\partial_z^2 Z(z) = k_z^2 Z(z)$

with $k_x^2 + k_y^2 + k_z^2 = 0$ in order to have Lu=0, so we say there are only 2 independent separation constants, and the third is fixed by those two. Because Laplace is elliptical, a solution must have a *mix* of oscillatory and radial 1D eigenfunctions. There are then only two possibilities, which we can summarize in this schematic manner: [the second form is very rare for "functional reasons"]

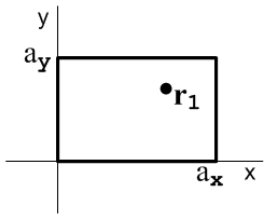
1 $[\sin(k_x x), \cos(k_x x)], [\sin(k_y y), \cos(k_y y)], [\exp(\kappa_z z), \exp(-\kappa_z z)]$ $\qquad$ k_z = imaginary = $i\kappa_z$

2 $[\sin(k_x x), \cos(k_x x)], [\exp(\kappa_y y), \exp(-\kappa_y y)], [\exp(\kappa_z z), \exp(-\kappa_z z)]$ $\qquad$ k_y and k_z = imaginary

1. Example 1: The infinite rectangular waveguide Green's and Dirichlet problems in Cartesians

A. The Waveguide Green's Function Problem

Consider a z-axis-aligned rectangular grounded metal tube ("waveguide") a_x x a_y of infinite length in z with a Green's charge at $\mathbf{r}_1$ somewhere in the z=0 plane inside the tube. We choose our Cartesian origin at the lower left corner of the waveguide



Away from the tube and point charge, we would expect the first case above might apply for the atoms of a Smythian form for the potential

atom = $\sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z|)$ $\qquad$ $k_x a_x = n_x \pi$ $\qquad$ => $\qquad$ $k_x = n_x(\pi/a_x)$

and similarly in y, so our quantum numbers are then n_x and n_y. The Smythian form is then

$$u_+(x,y,z) = \Sigma_{nx,ny}\, A_{nx,ny} \sin(k_x x) \sin(k_y y) \exp(-\kappa_z z) \qquad z > 0$$
$$u_-(x,y,z) = \Sigma_{nx,ny}\, A_{nx,ny} \sin(k_x x) \sin(k_y y) \exp(+\kappa_z z) \qquad z < 0$$

or

$$u(x,y,z) = \Sigma_{nx,ny}\, A_{nx,ny} \sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z|)$$

where $\kappa_z(n_x, n_y) = \sqrt{k_x^2 + k_y^2}$. This form matches things at z = 0 away from the point charge, and is of course symmetric z→ -z. Outside the waveguide we have u ≡ 0 everywhere. The fact that u = 0 on the entire planes x = 0 and y = 0 is just fine. We would next apply our pillbox condition

$$\partial_z u_-(x,y,z) - \partial_z u_+(x,y,z) = 4\pi q\, \delta(x-x_1)\delta(y-y_1)$$

$$\partial_z u_-(x,y,z) = \Sigma_{nx,ny}\, A_{nx,ny} \sin(k_x x) \sin(k_y y)\, \kappa_z \exp(\kappa_z z)$$
$$\partial_z u_+(x,y,z) = -\Sigma_{nx,ny}\, A_{nx,ny} \sin(k_x x) \sin(k_y y)\, \kappa_z \exp(-\kappa_z z)$$

and we get

$$\Sigma_{nx,ny} \sin(k_x x) \sin(k_y y)\, \kappa_z \{\ 2A_{nx,ny}\ \} = 4\pi q\, \delta(x-x_1)\delta(y-y_1)$$

The rest of the problem is trivial, we apply orthogonality twice in the two oscillatory dimensions and we will then obtain an expression for $A_{nx,ny}$ and the problem is solved. [This is the "functional reason" mentioned above that the second atomic form above is not used very often.] Let's do it. I will steal some results from the next section below (which I did first) (I converted it from z to x)

$$\delta(x-x_1) = (\pi/a_x)\, \delta(\theta_x-\theta_{x1}) \qquad k_x = n_x(\pi/a_x)$$
$$\sin(k_x x) = \sin(n_x\theta_x) \qquad \theta_x \equiv \pi(x/a_x) \qquad x \text{ in } (0,a_x) \qquad \theta_x \text{ in } (0,\pi)$$

and same for y, so we then have

$$\Sigma_{nx,ny} \sin(n_x\theta_x) \sin(n_y\theta_y)\, \kappa_z \{\ 2A_{nx,ny}\ \} = 4\pi q\ (\pi/a_x)\,(\pi/a_y)\, \delta(\theta_x-\theta_{x1})\, \delta(\theta_y-\theta_{y1}) \qquad (*)$$

and we shall make use of this transforms.doc result (knowing we don't have $n_x = 0$)

$$\int_0^{2\pi} d\theta_x \sin(n_x\theta_x) \sin(n_x'\theta_x) = \pi\, \delta_{nx,nx'}$$

and similarly for y. We then get

$$\pi^2\, \kappa_z \{\ 2\, A_{nx,ny}\ \} = 4\pi q\ (\pi/a_x)\,(\pi/a_y) \sin(k_x x_1) \sin(k_y y_1)$$

$$A_{nx,ny} = 2\pi q\ (1/a_x a_y \kappa_z) \sin(k_x x_1) \sin(k_y y_1)$$

and there is our coefficient. Notice that each orthog puts π on the left, and a sine on the right. Here then is our solution potential inside our little waveguide:

$$u(x,y,z) = \Sigma_{nx,ny} A_{nx,ny} \sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z|)$$

$$= \Sigma_{nx,ny} [2\pi q \ (1/a_x a_y \kappa_z) \sin(k_x x_1) \sin(k_y y_1)] \sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z|)$$

$$= (2\pi q/a_x a_y) \ \Sigma_{nx,ny} \sin(k_x x_1) \sin(k_y y_1) \sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z|)/\kappa_z$$

so we summarize the result for the interior Green's function for a rectangular infinite tube:

$$u(x,y,z) = (2\pi q/a_x a_y) \Sigma_{nx=1}^{\infty} \Sigma_{ny=1}^{\infty} \sin(k_x x_1) \sin(k_y y_1) \sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z|)/ \kappa_z$$

where $k_x = n_x(\pi/a_x)$

$k_y = n_y(\pi/a_y)$

$\kappa_z = \sqrt{k_x^2 + k_y^2}$ **** Result ****

I have not looked for validation of the above result. I am a bit concerned with the overall constant. If we go right to the z axis a small distance from the charge, we get

$$u(x,y,z) = (2\pi q/a_x a_y) \Sigma_{nx=1}^{\infty} \Sigma_{ny=1}^{\infty} \sin^2(k_x x_1) \sin^2(k_y y_1) \exp(-\kappa_z |z|)/ \kappa_z$$

but the two sums are coupled through κ_z so I don't know how to evaluate anything here. GR7 p 36 gives some single-sum series involving trig functions, but no double sums. Nor does a quick scan of PBM v1 show any double sums. Double sum tabulations are as rare as double integral ones I guess.

B. Try to find 1/R Expansion in Cartesians by taking a limit of the Waveguide result

The Green's function for a point charge with no BC's is this:

$$1/R = q / \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} \qquad \kappa_z = \sqrt{k_x^2 + k_y^2}$$

Now suppose we try to take the limit of our u(x,y,z) result above as a_x and $a_y \to \infty$. Based on the old idea that $\Sigma_n \Delta k \ f(k + n\Delta k) \to \int dk \ f(k)$, we might think of $\Delta k_x = (\pi/a_x)$ so that

$$\Sigma_{nx,ny} (\pi/a_x) (\pi/a_y) \ f(n_x(\pi/a_x), n_y(\pi/a_y)) \ \to \ \int dk_x \int dk_y \ f(k_x, k_y)$$

So write from above

$$u(x,y,z) = (2\pi q/a_x a_y) \Sigma_{nx=1}^{\infty} \Sigma_{ny=1}^{\infty} \sin(k_x x_1) \sin(k_y y_1) \sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z|)/ \kappa_z$$

$$= (2q/\pi) \Sigma_{nx=1}^{\infty} \Sigma_{ny=1}^{\infty} (\pi/a_x) (\pi/a_y) \sin(k_x x_1) \sin(k_y y_1) \sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z|)/ \kappa_z$$

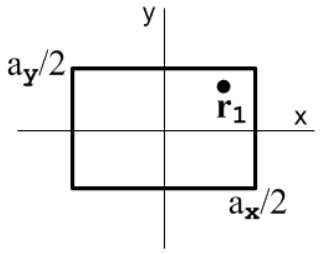
$$\rightarrow \quad (2q/\pi) \int_0^\infty dk_x \int_0^\infty dk_y \sin(k_x x_1)\sin(k_y y_1) \sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z|)/ \kappa_z$$

and then to get the correct symmetry under $\mathbf{r} \leftrightarrow \mathbf{r}_1$ we adjust the final |z| and we get then this *conjecture* for what 1/R might look like:

$$\text{"1/R"} =?= \quad (2q/\pi) \int_0^\infty dk_x \int_0^\infty dk_y \sin(k_x x_1) \sin(k_y y_1) \sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z-z_1|)/ \kappa_z$$

Our limit is of course no good because we still have waveguide walls in the x = 0 and y = 0 planes. The true 1/R would not of course vanish on these planes as the above 1/R does. But I think we might regard the above expression as the solution to **a different problem:** the Green's function for a point charge located in the "quarter space" bounded by two grounded planes at x = 0 and y = 0.

We could start this problem over again with the x,y origin at the center of the waveguide.



We can convert our previous solution to this new origin by taking $x \rightarrow x - a_x/2$ etc, so that

$$u(x,y,z) = (2\pi q/a_x a_y) \Sigma_{nx=1}^{\infty} \Sigma_{ny=1}^{\infty} \sin(k_x x_1)\sin(k_y y_1) \sin(k_x[x - a_x/2]) \sin(k_y[y - a_y/2]) \exp(-\kappa_z |z|)/ \kappa_z$$

Although it is not obvious, you can probably replace $\sin(k_x x_1)$ by $\sin(k_x[x_1 - a_x/2])$ and similarly for y and thereby recover the symmetry of this Green's function, without altering the above sum. But let's not pursue that and just consider the above form. We would like to take our large a_x and a_y limit in an attempt to get an expression for 1/R. But something like $\sin(k_x[x - a_x/2])$ *has no limit* as $a_x \rightarrow \infty$, so this approach is simply not going to work! Our form above has "built in" certain boundary conditions which mess things up in our desired limit. The potential near the origin, for example, is very sensitive to a_x even when the walls are miles away, whereas for the real 1/R this is not the case.

So the upshot here is that I am unable to take a limit of the waveguide problem to obtain a 1/R expansion in terms of Cartesian atoms.

C. The correct 1/R expansion in Cartesians

M&F do give an expansion for 1/R which is crudely similar to my attempts above, but is a triple integral:

$$\frac{1}{R} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \left\{ \frac{e^{i[k_x(x-x_0)+k_y(y-y_0)+k_z(z-z_0)]}}{[k_x^2 + k_y^2 + k_z^2]} \right\} \quad (10.3.6)$$

or

$$1/R = (1/2\pi^2)\int d^3k \exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,] / k^2$$

If we apply ∇^2 to our integrand we get

$$\nabla^2 \{ \exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,]/k^2 \} = -\exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)]$$

so indeed, the integrand is NOT a Cartesian atomic form! But the triple integral of this thing IS a Laplace solution away for **r** away from $\mathbf{r}_1$, and in fact (as shown in "question on Fourier + parallel plates Cartesian.doc" in the current folder), we have $\nabla^2(1/R) = -4\pi q\, \delta^3(\mathbf{r}-\mathbf{r}_1)$ so this 1/R is then exactly the free-space Green's function in Cartesians! The above expansion for 1/R differs from what we find in sphericals or cylindrical; for those two systems, and I think all others, the 1/R expansion is always a "double sum" or "sum + integral", whereas here we have a "triple integral". I suspect this is due to the fact that the surfaces of constant parameter are unbounded for all three coordinates in Cartesians.

We can attempt doing the integral in several ways. The most obvious is this

$$(1/2\pi^2)\int d^3k \exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,] / k^2 = (1/2\pi^2) \int_0^{\infty} dk \int d\Omega\, e^{ikR\cos\theta}$$

$$= (1/2\pi^2)\, 2\pi \int_0^{\infty} dk \int_{-1}^{1} dz\, e^{ikRz} = (1/2\pi^2)\, 4\pi \int_0^{\infty} dk \int_0^{1} dz \cos(kRz)$$

$$= (2/\pi) \int_0^{\infty} dk\ \sin(kR)/(kR) = (2/\pi)\,(1/R) \int_0^{\infty} d(kR) \sin(kR)/(kR) = (2/\pi)\,(1/R) \int_0^{\infty} dx\, \sin(x)/x$$

$$= (2/\pi)\,(1/R)\,(\pi/2) = 1/R \qquad \text{// using Schaum p 96 15.33}$$

But it might be interesting to do the k_z integral to get something that resembles the double integral failed forms discussed above. The innermost integral is then

$$I \equiv \int_{-\infty}^{\infty} dk_z \exp(ik_z[z-z_1])/\sqrt{k_z^2 + \kappa_z^2} \qquad \text{where } \kappa_z = \sqrt{k_x^2 + k_y^2}$$

$$= 2 \int_0^{\infty} dk_z \cos(k_z[z-z_1]) /\sqrt{k_z^2 + \kappa_z^2}$$

We then call upon this result (GR7 p 435)

2. $$\int_0^\infty \frac{\cos(ax)\,dx}{\sqrt{\beta^2+x^2}} = K_0(a\beta) \qquad [a>0, \quad \mathrm{Re}\,\beta > 0]$$

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with $\alpha = |z\text{-}z_1|$ and $\beta = \kappa_z$ to find that

$$I = 2\, K_0(|z\text{-}z_1|/\sqrt{k_x^2 + k_y^2}\,)$$

so we get this expansion then for 1/R:

$$1/R = (1/2\pi^2) \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; \exp(ik_x[x\text{-}x_1]) \exp(ik_y[y\text{-}y_1]) \; I$$

$$= (1/\pi^2) \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \quad \exp(ik_x[x\text{-}x_1]) \exp(ik_y[y\text{-}y_1]) \, K_0(|z\text{-}z_1|/\sqrt{k_x^2 + k_y^2}\,)$$

$$= (1/\pi^2) \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \quad \exp\{\, ik_x[x\text{-}x_1] + ik_y[y\text{-}y_1] \,\} \, K_0(|z\text{-}z_1|/\sqrt{k_x^2 + k_y^2}\,)$$

$$= (1/\pi^2) \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \quad \cos\{k_x[x\text{-}x_1] + k_y[y\text{-}y_1] \,\} \, K_0(|z\text{-}z_1|/\sqrt{k_x^2 + k_y^2}\,)$$

$$= (1/\pi^2) \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \quad \{\, \cos(k_x[x\text{-}x_1]) \cos(k_y[y\text{-}y_1]) - \sin(k_x[x\text{-}x_1]) \sin(k_y[y\text{-}y_1]) \,\}$$
$$* \; K_0(|z\text{-}z_1|/\sqrt{k_x^2 + k_y^2}\,)$$

$$= (1/\pi^2) \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; \cos(k_x[x\text{-}x_1]) \cos(k_y[y\text{-}y_1]) \, K_0(|z\text{-}z_1|/\sqrt{k_x^2 + k_y^2}\,)$$

$$= (4/\pi^2) \int_{0}^{\infty} dk_x \int_{0}^{\infty} dk_y \; \cos(k_x[x\text{-}x_1]) \cos(k_y[y\text{-}y_1]) \, K_0(|z\text{-}z_1|/\sqrt{k_x^2 + k_y^2}\,)$$

So this, then, is our 1/R in Cartesians expressed as a double integral. As already noted, the integrand is not a Cartesian atom. The result is manifestly symmetric under $\mathbf{r}_1 \leftrightarrow \mathbf{r}$. If you were to look up the required integral perhaps in Bateman ET II, I'll bet you could actually do the two integrals shown here and get this result

$$= 1/\sqrt{(x\text{-}x_1)^2 + (y\text{-}y_1)^2 + (z\text{-}z_1)^2}$$

D. The Waveguide Dirichlet Problem

A **Dirichlet version** of our problem above would have the same atomic form

$$u(x,y,z) = \Sigma_{n_x,n_y} A_{n_x,n_y} \sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z|)$$

with potential f(x,y) specified on the z = 0 plane within the tube, and then we would be faced with

$$\Sigma_{nx,ny}\, A_{nx,ny} \sin(k_x x) \sin(k_y y) = f(x,y)$$

where again we would apply double orthogonality and that would determine the $A_{nx,ny}$. We can now pretty much just read off the coefficient: (we assume f(x,y) = 0 when θ_x lies in $\pi,2\pi$ etc , which is to say, outside our metal waveguide)

$$\pi^2 A_{nx,ny} = \int_0^{\pi} d\theta_x \sin(n_x\theta_x) \int_0^{\pi} d\theta_y \sin(n_y\theta_y)\ f(x,y)$$

$$= (\pi/a_x)(\pi/a_y) \int_0^{a_x} dx \int_0^{a_y} dy\ \sin(k_x x) \sin(k_y y)\ f(x,y)$$

So here is our Dirichlet problem result:

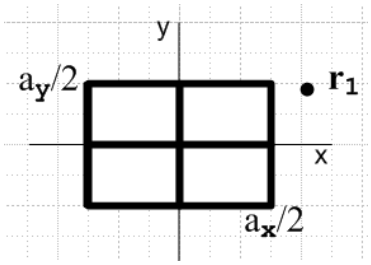
$$u(x,y,z) = \Sigma_{nx,ny=1}^{\infty}\ A_{nx,ny} \sin(k_x x) \sin(k_y y)\ \exp(-\kappa_z |z|)$$ **** Result ****

∂ where $A_{nx,ny} = (1/a_x a_y) \int_0^{a_x} dx \int_0^{a_y} dy\ \sin(k_x x) \sin(k_y y)\ f(x,y)$

Comment: You often see the Green's function problem solved by replacing δ functions with completeness sums, so you avoid orthogonality. That is fine, but that method does not work for the Dirichlet problem just done above, where you must use orthogonality.

E. One could imagine "exterior versions" of both these problems.

Let's consider just the Dirichlet problem. I think it can be solved by superposing solutions found for each of the four quadrants. This time we really need to put the origin in the center of our tube. Our Smythian form will be this



$$u(x,y,z) = \int_0^{\infty} dk_x \int_0^{\infty} dk_y\ A_{kx,ky} \sin(k_x[x-a_x/2]) \sin(k_y[y-a_y/2]) \exp(-\sqrt{k_x^2 + k_y^2}\,|z|)$$

where the sine functions forces u = 0 on the tube. We no longer have quantization of k_x because our sines are only tied down at one end (the other wiggles at ∞). We think of the tube as 4 sub tubes, one in each x-y quadrant. Our quadrant I ranges are $(a_x/2,\infty)$ for x and $(a_y/2,\infty)$ for y. We would then find a suitable Fourier Sine Integral Transform in x for this range, perhaps this way: Start with

$$\int_0^\infty dz \sin(kz) \sin(k'z) = (\pi/2)\delta(k-k') \qquad \text{// orthogonality}$$

Then write $z = x - a_x/2$ to get

$$\int_{a_x/2}^\infty dx \sin(k_x[x-a_x/2]) \sin(k_x'[x-a_x/2]) = (\pi/2)\delta(k_x-k_x') \qquad \text{// orthogonality}$$

I can pretty much read off the Dirichlet result. First we would have this on the z = 0 plane

$$f(x,y) = \int_0^\infty dk_x \int_0^\infty dk_y A_{kx,ky} \sin(k_x[x-a_x/2]) \sin(k_y[y-a_y/2]) \qquad \text{// = } u(x,y,z=0)$$

Then the double orthog gives

$$\int_{a_x/2}^\infty dx \sin(k_x[x-a_x/2]) \int_{a_y/2}^\infty dy \sin(k_y[y-a_y/2]) f(x,y)$$

$$= (\pi/2)(\pi/2) A_{kx,ky}$$

So our Dirichlet solution would be

$$u_I(x,y,z) = \int_0^\infty dk_x \int_0^\infty dk_y A_{kx,ky} \sin(k_x[x-a_x/2]) \sin(k_y[y-a_y/2]) \exp(-\sqrt{k_x^2 + k_y^2}\,|z|)$$

where $$A_{kx,ky,I} = (2/\pi)^2 \int_{a_x/2}^\infty dx \int_{a_y/2}^\infty dy \sin(k_x[x-a_x/2]) \sin(k_y[y-a_y/2]) f(x,y)$$

where the I indicates a first quadrant solution. Now we make a similar solution for the other three quadrants. For each quadrant's solution, we assume that the potential is 0 in all three other quadrants. When we superpose, we shall find that we still have u = 0 in the central tube, and our result will be the same as the above except we have

$$A_{kx,ky} = (2/\pi)^2 \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \sin(k_x[x-a_x/2]) \sin(k_y[y-a_y/2]) f(x,y)$$

with the understanding that the prescribed f(x,y) must be 0 in the tube.

I have no verification for this result and method. If I happen to run into it, I will report the situation here. Certainly we will have continuous k parameters.

2. Example 2: Infinite Parallel Plate Green or Dirichlet problems *attempted* in Cartesians

Emboldened by our successful (we think) solution above in Example 1, we now consider an infinite parallel plate capacitor with one plate on the z=0 plane, the other at $z = a_z$, and a point charge q in

between the plates on the z axis at z = b. Both plates are at V = 0. We are thinking of this as a square plate aligned with the x,y axes that is enlarged to be infinite.

Comment: When I started on this example, I was looking for an example that would demonstrate the problems encountered in applying the pillbox condition in a case where at least one of the perp coordinates was non-oscillatory. This example would have a pillbox condition something like this

$$\partial_z u_-(x,y,z) - \partial_z u_+(x,y,z) = 4\pi q\, \delta(x)\delta(y)$$

and the LHS would then be some sum/integral of the assumed Smythian form which involved an expo function in at least one perp dimension (x or y, or both). Then, since we did not have two sets of complete functions, we could not solve the above equation for the coefficients because at least one of the orthogonalities would be missing, or to say it another way, one of the δ completenesses would be missing. My first three attempts below, A,B and C, all failed before I could *get* to this problematic pillbox situation, but I think attempt D was successful in this regard, ie, successful in showing the problem.

Attempt A. Here we want oscillation in z, and decay in x,y, so we might *try* our second form above, where only one dimension is oscillatory and the other two are expo,

$$\text{atom} = \exp(-\kappa_x|x|)\exp(-\kappa_y|y|)\sin(k_z z) \qquad \text{where } \kappa_y = \sqrt{k_z^2 - \kappa_x^2}$$

$$k_z = n_z(\pi/a_z) \qquad n_z = 1,2,3...$$

Now there is only one quantum number n_z which determines k_z. (There is only one SL problem, and it is in z). We try to have two "regions", one above and one below the Green's point charge, and we allow κ_x then to have a free range of values in $(0,\infty)$ for its "spectrum",

$$u_+(x,y,z) = \Sigma_{nz} \int_0^\infty d\kappa_x\, A_{nz,\kappa x} \exp(-\kappa_x|x|)\exp(-\kappa_y|y|)\sin(k_z z) \qquad z > b$$

$$u_-(x,y,z) = \Sigma_{nz} \int_0^\infty d\kappa_x\, B_{nz,\kappa x} \exp(-\kappa_x|x|)\exp(-\kappa_y|y|)\sin(k_z z) \qquad z < b$$

But the potential must match on the z = b plane everywhere away from the point charge, and that really forces A = B in this proposed Smythian form. In this case $u_+ \equiv u_-$ and we have

$$u(x,y,z) = \Sigma_{nz} \int_0^\infty d\kappa_x\, A_{nz,\kappa x} \exp(-\kappa_x|x|)\exp(-\kappa_y|y|)\sin(k_z z)$$

so if we try our pillbox condition in the z direction,

$$\partial_z u_-(x,y,z) - \partial_z u_+(x,y,z) = 4\pi q\, \delta(x)\delta(y)$$

the LHS will be zero. What does this mean? It means exactly one thing: the Smythian form is **No Good**! A successful Smythian form must "support" Gauss's Law about the location of the point charge!

You might next think of making this u(x,y,z) be the potential only of the induced charge on the plates. But then the form is wrong because it is making u = 0 on the plates, which is no longer true.

Attempt B. Suppose we try a Smythian form that *does* have two oscillatory dimensions, such as

$$u(x,y,z) = \int_0^\infty dk_x \int_0^\infty dk_y \cos(k_x x)\cos(k_y y)\,[\, A_{kx,ky}\sinh(\kappa_z z) + B_{kx,ky}\cosh(\kappa_z z)\,]$$

where $\kappa_z = \sqrt{k_x^2 + k_y^2}$

This form is certainly less intuitive since we don't "see" the expected decay for large |x| and |y| values, and we don't "see" the potential vanishing on the plates. But let's just keep going. To get things vanishing on the plates we need

z=0:

$A_{kx,ky}\sinh(\kappa_z z) + B_{kx,ky}\cosh(\kappa_z z) = 0$

$A_{kx,ky}\sinh(0) + B_{kx,ky}\cosh(0) = 0$

$\Rightarrow B_{kx,ky} * 1 = 0$

$z=a_z$:

$A_{kx,ky}\sinh(\kappa_z z) = 0$

$A_{kx,ky}\sinh(\kappa_z a_z) = 0$

$\Rightarrow A_{kx,ky} = 0$

So this Smythian form is also **No Good!** [But see B1 below!]

Attempt B1 (the correct one!). Let's be a little more general that in B:

$$u(x,y,z) = \int_0^\infty dk_x \int_0^\infty dk_y \cos(k_x x)\cos(k_y y)\,[\, A_{kx,ky}\sinh(\kappa_z z) + B_{kx,ky}\cosh(\kappa_z z)\,] \quad z < b$$

$$u(x,y,z) = \int_0^\infty dk_x \int_0^\infty dk_y \cos(k_x x)\cos(k_y y)\,[\, A'_{kx,ky}\sinh(\kappa_z z) + B'_{kx,ky}\cosh(\kappa_z z)\,] \quad z > b$$

where $\kappa_z = \sqrt{k_x^2 + k_y^2}$

The Green's point charge is at x = y = 0 and z = b. I use cos functions in x and y because the solution should be even in x, and even in y. The requirement that the potential match at z = b says

$$A_{kx,ky}\sinh(\kappa_z b) + B_{kx,ky}\cosh(\kappa_z b) = A'_{kx,ky}\sinh(\kappa_z b) + B'_{kx,ky}\cosh(\kappa_z b)$$

which is just one condition on 4 coefficients. Two more conditions make the potential vanish on the plates:

$$A_{kx,ky}\sinh(\kappa_z 0) + B_{kx,ky}\cosh(\kappa_z 0) = 0 \qquad \Rightarrow \qquad B_{kx,ky} = 0$$

$A'_{kx,ky}\sinh(\kappa_z d) + B'_{kx,ky}\cosh(\kappa_z d) = 0$

So at this point we have three conditions on our 4 coefficients, and the remaining condition will come from the pillbox condition, and we have the required 2 oscillatory dimensions perp to our pillbox direction. I think one could solve the problem this way on short order, but I will not do that because I already solved it elsewhere by a slightly different method. In the nearby doc "question on Fourier..." I solve this problem and get this result:

$$u(x,y,z) = (1/\pi^2)\int_0^\infty \int_0^\infty dk_x dk_y \cos(k_x x)\cos(k_y y)\sinh[\kappa_z(d - z_>)]\sinh(\kappa_z z_<)/[\kappa_z \sinh(\kappa_z d)]$$

$z_> = \max(z,b)$ $z_< = \min(z,b)$ $0 < b < d$ (between the plates) $\kappa_z = \sqrt{k_x^2 + k_y^2}$

You see for example that when $z < b$, we have $\sinh(\kappa_z z)$ only, as the form above predicts with $B = 0$.

So this Attempt B1 is in fact a workable Smythian form! I did not realize it when I first did this writeup because, as shown in Attempt B above, I was assuming the same functional form in z on both sides of the Green's point charge plane. But we know that you must have a different form! Just a lapse.

Attempt C. As we shall see in the cylindricals solution of this problem below, it helps to start off with the point charge not on the z axis, then you can have some variation of the potential on the inside and outside of that point charge. So if we put our point charge let us say at $x = x_1$ ($y=0$) we could write (adding lots of new coefficients)

$$u_+(x,y,z) = \int_0^\infty dk_x \int_0^\infty dk_y [C\sin(k_x x) + D\cos(k_x x)]\cos(k_y y)[A\sinh(\kappa_z z) + B\cosh(\kappa_z z)] \quad x > x_1$$

$$u_i(x,y,z) = \int_0^\infty dk_x \int_0^\infty dk_y [E\sin(k_x x) + F\cos(k_x x)]\cos(k_y y)[A\sinh(\kappa_z z) + B\cosh(\kappa_z z)] \quad x < x_1$$

But this does not alleviate the problem just noted that we cannot make u vanish on the plates. So once again, this form is **No Good**. [But same logical error as in Attempt B, resolved in B1]

Attempt D. What happens then if we try this same idea with our previous form with one osc coordinate:

$$u_1(x,y,z) = \Sigma_{nz} \int_0^\infty d\kappa_x [A\sinh(\kappa_x x) + B\cosh(\kappa_x x)] \exp(-\kappa_y |y|)\sin(k_z z) \qquad -x_1 < x < x_1$$

$$u_2(x,y,z) = \Sigma_{nz} \int_0^\infty d\kappa_x \, C\exp(-\kappa_x x)\exp(-\kappa_y |y|)\sin(k_z z) \qquad x > x_1$$

$$u_3(x,y,z) = \Sigma_{nz} \int_0^\infty d\kappa_x \, D\exp(\kappa_x x)\exp(-\kappa_y |y|)\sin(k_z z) \qquad x < -x_1$$

OK, this vanishes on the plates, and it "relieves" our difficulty with the z direction pillbox condition because now you "cannot apply" that condition since the point charge is on the boundary between our two regions if you try to apply it! The form also has decay for large x and y in both directions.

Examining things at our two boundaries $x = x_1$ and $x = -x_1$, we require

$[\ Ash(\kappa_x x_1) \ + Bch(\kappa_x x_1)] = \ Cexp(-\kappa_x x_1)$

$[- Ash(\kappa_x x_1) + Bch(\kappa_x x_1)] \ = Dexp(-\kappa_x x_1)$

so we can then remove C and D and rewrite the above as

$$u_1(x,y,z) \ = \Sigma_{nz} \ \int_0^\infty d\kappa_x \ \ [Ash(\kappa_x x) + Bch(\kappa_x x)] \qquad\qquad \exp(-\kappa_y |y| \ ) \sin(k_z z)$$

$$u_2(x,y,z) \ = \Sigma_{nz} \ \int_0^\infty d\kappa_x \ [\ Ash(\kappa_x x_1) \ + Bch(\kappa_x x_1)]\exp(-\kappa_x(x-x_1)) \exp(-\kappa_y |y| \ )\sin(k_z z)$$

$$u_3(x,y,z) \ = \Sigma_{nz} \ \int_0^\infty d\kappa_x \ [- Ash(\kappa_x x_1) + Bch(\kappa_x x_1)] \exp(\kappa_x(x+x_1)) \exp(-\kappa_y |y| \ )\sin(k_z z)$$

But now we require slope continuity at $x = -x_1$,

$$\partial_x \ u_1(x,y,z) \ = \Sigma_{nz} \ \int_0^\infty d\kappa_x \ \ \kappa_x \ [Ach(\kappa_x x) + Bsh(\kappa_x x)] \qquad\qquad \exp(-\kappa_y |y| \ ) \sin(k_z z)$$

$$\partial_x \ u_2(x,y,z) \ = \Sigma_{nz} \ \int_0^\infty d\kappa_x \ (-\kappa_x)[\ Ash(\kappa_x x_1) \ + Bch(\kappa_x x_1)]\exp(-\kappa_x(x-x_1)) \exp(-\kappa_y |y| \ )\sin(k_z z)$$

$$\partial_x \ u_3(x,y,z) \ = \Sigma_{nz} \ \int_0^\infty d\kappa_x \ (+\kappa_x)[- Ash(\kappa_x x_1) + Bch(\kappa_x x_1)] \exp(\kappa_x(x+x_1)) \exp(-\kappa_y |y| \ )\sin(k_z z)$$

The 2-3 match where $x = -x_1$ tells us that,

$$(-\kappa_x)[\ Ash(\kappa_x x_1) \ + Bch(\kappa_x x_1)]\exp(-\kappa_x(-x_1-x_1)) \ = \ (+\kappa_x)[- Ash(\kappa_x x_1) + Bch(\kappa_x x_1)] \exp(\kappa_x(-x_1+x_1))$$

$$[\ Ash(\kappa_x x_1) \ + Bch(\kappa_x x_1)]\exp(+2\kappa_x x_1) \ = \ [\ Ash(\kappa_x x_1) - Bch(\kappa_x x_1)]$$

$$[\ Ash(\kappa_x x_1) \ + Bch(\kappa_x x_1)]\exp(+\kappa_x x_1) \ = \ [\ Ash(\kappa_x x_1) - Bch(\kappa_x x_1)] \exp(-\kappa_x x_1)$$

$$Ash(\kappa_x x_1) [\exp(+\kappa_x x_1) - \exp(-\kappa_x x_1)] = - Bch(\kappa_x x_1) [\exp(-\kappa_x x_1) + \exp(+\kappa_x x_1)]$$

$$Ash(\kappa_x x_1) \ 2 \ sh(\kappa_x x_1) = - Bch(\kappa_x x_1) \ 2 \ ch(\kappa_x x_1)$$

$$A \ sh^2(\kappa_x x_1) \ \ = - Bch^2(\kappa_x x_1))$$

$$B = - A \ \ th^2(\kappa_x x_1)$$

so we just keep in mind then that B can we replaced by a multiple of A and then everything is fine on the left side.

Now finally we come down to the right side pillbox condition at $x = x_1$

$$[\partial_x u_1(x,y,z) - \partial_x u_2(x,y,z)]^{x=x1} \ = 4\pi q \ \delta(y)\delta(z-b)$$

$$[\partial_x u_2(x,y,z)]^{x=x1} \ = \ \Sigma_{nz} \ \int_0^\infty d\kappa_x \ (-\kappa_x)[\ Ash(\kappa_x x_1) \ + Bch(\kappa_x x_1)] \exp(-\kappa_y |y| \ )\sin(k_z z)$$

$$[\partial_x u_1(x,y,z)]^{x=x1} = \Sigma_{nz} \int_0^\infty d\kappa_x \;\; \kappa_x \, [Ach(\kappa_x x_1) + Bsh(\kappa_x x_1)] \quad \exp(-\kappa_y |y|) \sin(k_z z)$$

so we get just a doubling of one of these terms, so our pillbox condition is then

$$2\,\Sigma_{nz} \int_0^\infty d\kappa_x \;\; \kappa_x \, [Ach(\kappa_x x_1) + Bsh(\kappa_x x_1)] \exp(-\kappa_y |y|) \sin(k_z z) = 4\pi q\, \delta(y)\delta(z-b)$$

where $\kappa_y = \sqrt{k_z^2 - \kappa_x^2}$ and where $B = - A \; th^2(\kappa_x x_1)$

which we rewrite as

$$2\,\Sigma_{nz} \int_0^\infty d\kappa_x \;\; \kappa_x \, A\, [ch(\kappa_x x_1) - th^2(\kappa_x x_1)] \exp(-\kappa_y |y|) \sin(k_z z) = 4\pi q\, \delta(y)\delta(z-b)$$

where $\kappa_y = \sqrt{k_z^2 - \kappa_x^2}$

I have probably made some errors getting to this point, but I think the general form is correct. Now we have an illustration of what happens when at least one of the perp coordinates is non-oscillatory! We want to solve for the A coefficient. We have orthog in z which removes the sum on the left and replaces it with a π and replaces $\delta(z-b)$ by a sine on the right,

$$2\,\pi \int_0^\infty d\kappa_x \;\; \kappa_x \, A\, [ch(\kappa_x x_1) - th^2(\kappa_x x_1)] \exp(-\kappa_y |y|) = 4\pi q\, \delta(y) \sin(k_z b)$$

$$\int_0^\infty d\kappa_x \;\; \kappa_x \, A(k_z, \kappa_x) \; [ch(\kappa_x x_1) - th^2(\kappa_x x_1)] \exp(-\sqrt{k_z^2 - \kappa_x^2} \; |y|) = 2q \sin(k_z b)\delta(y)$$

$$\int_0^\infty d\kappa_y \;\; \sqrt{k_z^2 - \kappa_y^2} \;\; a(k_z, \kappa_y) \; [ch(\sqrt{k_z^2 - \kappa_y^2}\, x_1) - th^2(\sqrt{k_z^2 - \kappa_y^2}\, x_1)] \exp(-\kappa_y |y|) = 2q \sin(k_z b)\delta(y)$$

where in the last line I have tried to replace κ_x by κ_y as the independent variable. Here is then our Big Problem writ large. What do we do next in our effort to find coefficient $a(k_z, \kappa_y)$? The function of y on the left, $\exp(-\kappa_y |y|)$, is not a complete-set basis function, so we cannot use y orthogonality.

You might wonder if some kind of Laplace Transform thinking might solve this problem, but I don't think so. The form in that sense is this: (let $t = \kappa_y$ and let $s = y$)

$$\int_0^\infty dt \; a(t) \, F(t) \, e^{-t|s|} = K\, \delta(|s|)$$

so what function a(s)F(s) is the Laplace transform of a delta function? None that I know of. If someone says the following and asks you to find f(t)

$$\int_0^\infty dt\, f(t)\, e^{-st} = \delta(s)$$

I don't think this is a well-posed situation. You might claim $f(t) = (2\pi i)^{-1}$ but then that doesn't reproduce the $\delta(s)$.

So basically we end up with some kind of ill-posed equation for our pillbox condition when we try to apply it in a situation where at least one of the perp dimensions is non-oscillatory! I guess this is what I wanted to see happen, and I then argue against attempting pillbox condition in this manner!

Dirichlet Problem. I think the Dirichlet problem would meet the same ending here. You specify a potential on the entire plane z = b, say. Try something like this

$$u(x,y,z) = \Sigma_{nz} \int_0^\infty d\kappa_x\, A_{nz,\kappa x} \exp(-\kappa_x |x|) \exp(-\kappa_y |y|) \sin(k_z z)$$

Then our Dirichlet condition becomes u(x,y,z=b) = f(x,y) so we have

$$\Sigma_{nz} \int_0^\infty d\kappa_x\, A_{nz,\kappa x} \exp(-\kappa_x |x|) \exp(-\kappa_y |y|) \sin(k_z b) = f(x,y)$$

We might hopefully write this as (ignoring details)

$$\sim \Sigma_{nz} \int_0^\infty d\kappa_x \int_0^\infty d\kappa_y \{A_{nz,\kappa x} \sin(k_z b)\, \delta(\kappa_x^2+\kappa_y^2 - k_z^2)\} \exp(-\kappa_x |x|) \exp(-\kappa_y |y|) = f(x,y)$$

The point is again this: if those exp functions were oscillatory members of a complete basis set, then we could perhaps invert this thing to find the A coefficient. But as it is, it is not clear what to do next. Again on could consider the LHS as a double Laplace Transform and maybe this would resolve the issue somehow. But it ain't purdy.

We can compare our Attempt D to the successful cylindrical approach below. Because it uses ρ, it does not have the problem we had above that we have to worry about x < 0 as well as x > 0. So the form could stay simpler. The idea of I and K functions below is similar to what we did with the x functions above, we need to have decay in the distance. But in the Cartesian attempt, we end up dead in the water with δ(y) sitting on the RHS and no way to deal with it.

There is probably some way to do the Green's and Dirichlet problems for parallel plates in Cartesian coordinates, I just don't know what it is!

3. Example 3: Infinite Parallel Plate Green or Dirichlet problems done in Cylindricals

Our atomic forms in general are these, in a form suitable for the parallel plates (sine in kz, I and K functions in Bessel)

$$[\sin(kz), \cos(kz)],\ [I_m(k\rho), K_m(k\rho)],\ [\sin(m\varphi), \cos(m\varphi)]$$

So let's put our point charge at $z = z_1$ and $\rho = \rho_1$ and $\varphi = 0$. Our plates cause k_z to quantize as in our Cartesian attempt, and we then have (A_m should really be written $A_{m,nz}$)

$u_i(z,\rho,\varphi) = \Sigma_{nz}\ \sin(k_z z)\ \Sigma_m\ A_m\ K_m(k_z\rho_1)\ I_m(k_z\rho)\ \cos(m\varphi)$ $\qquad k_z = n_z(\pi/a_z)$

$u_o(z,\rho,\varphi) = \Sigma_{nz}\ \sin(k_z z)\ \Sigma_m\ A_m\ K_m(k_z\rho)\ I_m(k_z\rho_1)\cos(m\varphi)$

where in true Smythian form we have built in continuity of u at $\rho = \rho_1$. Our general pillbox condition says

$\partial_{q1}V_i - \partial_{q1}V_o = (q/\varepsilon)\ (h_1/h_2h_3)\ \delta(q_2-q_2')\ \delta(q_3-q_3')$

or

$\partial_\rho u_i - \partial_\rho u_o = 4\pi q\ (1/\rho)\ \delta(z-z_1)\ \delta(\varphi')$

Computing the LHS, this last says

$\Sigma_{nz}\ \sin(k_z z)\ \Sigma_m\ k_z\ A_m\ [\ K_m(k_z\rho_1)\ I_m'(k_z\rho) - I_m(k_z\rho_1)K_m'(k_z\rho)]\ \cos(m\varphi) = 4\pi q\ (1/\rho)\ \delta(z-z_1)\ \delta(\varphi')$

and we get to set $\rho = \rho_1$ since that is the location of our pillbox, hence

$\Sigma_{nz}\ \sin(k_z z)\ \Sigma_m\ k_z\ A_m\ W[K_m(k_z\rho_1), I_m(k_z\rho_1)]\ \cos(m\varphi) = 4\pi q\ (1/\rho)\ \delta(z-z_1)\ \delta(\varphi')$

$\Sigma_{nz}\ \sin(k_z z)\ \Sigma_m\ k_z\ A_m\ (1/k_z\rho_1)\ \cos(m\varphi) = 4\pi q\ (1/\rho_1)\ \delta(z-z_1)\ \delta(\varphi')$ // Jackson p 86

$\Sigma_{nz}\ \sin(k_z z)\ \Sigma_m\ A_m\ \cos(m\varphi) = 4\pi q\ \delta(z-z_1)\ \delta(\varphi')$

Now, since we started with two oscillatory dimensions in our form, we can solve for the A_m. Let's do it, and this will serve as a little test for my transforms.doc. First, we quote and edit

$\int_0^\pi d\varphi\ \cos(m\varphi)\cos(m'\varphi) = (\pi/\varepsilon_m)\ \delta_{mm'}$ // orthogonality

so apply $\int_0^\pi d\varphi\ \cos(m'\varphi)$ to both sides and we get

$\Sigma_{nz}\ \sin(k_z z)\ \Sigma_m\ A_m\ \ (\pi/\varepsilon_m)\ \delta_{mm'} = 4\pi q\ \delta(z-z_1)$

$\Sigma_{nz}\ \sin(k_z z)A_{m'}(\pi/\varepsilon_{m'}) = 4\pi q\ \delta(z-z_1)$

$\Sigma_{nz}\ \sin(k_z z)A_m(\pi/\varepsilon_m) = 4\pi q\ \delta(z-z_1)$ (*)

Now assume $k_z = n_z(\pi/S)$ where S is the separation of the parallel plates and $n_z = 1,2,3....$ Then

$\sin(k_z z) = \sin([n_z(\pi/S)]z) = \sin(n_z\theta_z)$ $\qquad \theta_z \equiv \pi(z/S)$ which ranges 0 to π

Also, we write

$(S/\pi)\ \delta(z-z_1) = \delta([\pi/S]z-[\pi/S]z_1) = \delta(\theta_z-\theta_{z1})$ $\qquad \Rightarrow \qquad \delta(z-z_1) = (\pi/S)\ \delta(\theta_z-\theta_{z1})$

so (*) above is // for later: $n_z\,\theta_{z1} = n_z\,\pi(z_1/S) = k_z z_1$

$$\Sigma_{nz}\ \sin(n_z\theta_z)A_m(\pi/\varepsilon_m) = 4\pi q\ (\pi/S)\ \delta(\theta_z-\theta_{z1}) = (4q\pi^2/S)\ \delta(\theta_z-\theta_{z1}) \qquad (**)$$

Then we quote and edit

$$\int_0^{2\pi} d\theta_z \sin(n_z\theta_z)\sin(n_z'\theta_z) = \pi\,\delta_{nz,nz'}(1-\delta_{nz,0}) \qquad \text{// but we never get } n_z = 0$$

Aside: We know that the θ_z range in our problem is $(0,\pi)$ corresponding to (0,S) in the gap. But θ_z in the sine series transform has the range $(0,2\pi)$, or $(-\pi,\pi)$. So we have to "extend" our current problem's range to be $(0,2\pi)$ in order to use this transform. Since our potential vanishes everywhere outside the tube, this should not be a problem. If we don't extend the range and apply the transform, we get a result that is off by a power of 2 ! So it really does matter.

so apply $\int_0^{2\pi} d\theta_z \sin(n_z'\theta_z)$ to both sides of (**) to get

$$\Sigma_{nz}\ \pi\,\delta_{nz,nz'}(1-\delta_{nz,0})A_m(\pi/\varepsilon_m) = (4q\pi^2/S)\sin(n_z'\theta_{z1})$$

$$\pi\,(1-\delta_{nz',0})A_m(\pi/\varepsilon_m) = (4q\pi^2/S)\sin(n_z'\theta_{z1})$$

$$(1-\delta_{nz',0})A_m(1/\varepsilon_m) = (4q/S)\sin(n_z'\theta_{z1})$$

$$(1-\delta_{nz',0})A_{m,nz'} = (4q/S)\,\varepsilon_m \sin(n_z'\theta_{z1})$$

$$A_{m,nz} = (4q/S)\,\varepsilon_m \sin(n_z\theta_{z1})$$

Our solution to this problem is then

$$u_i(z,\rho,\varphi) = (4q/S)\,\Sigma_{nz}\ \sin(k_z z)\sin(k_z z_1)\Sigma_m\ \varepsilon_m\ K_m(k_z\rho_1)\ I_m(k_z\rho)\ \cos(m\varphi)$$
$$u_o(z,\rho,\varphi) = (4q/S)\,\Sigma_{nz}\ \sin(k_z z)\sin(k_z z_1)\Sigma_m\ \varepsilon_m\ K_m(k_z\rho)\ I_m(k_z\rho_1)\cos(m\varphi)$$

or

$$u(z,\rho,\varphi) = (4q/S)\,\Sigma_{nz}\ \sin(k_z z_>)\sin(k_z z_<)\,\Sigma_m\ \varepsilon_m\ K_m(k_z\rho_>)\ I_m(k_z\rho_<)\ \cos(m\varphi)$$

where $k_z = n_z(\pi/S)$ sums are: $\Sigma_{nz=1}^{\infty}$ and $\Sigma_{m=0}^{\infty}$

Now, finally, we can think about taking the limit $\rho_1 \to 0$ so our point charge is on the z axis, and then we expect to have only $u_o(z,\rho,\varphi)$ to think about. We know from Jackson p 75 $x \ll 1$ limits that

$$I_m(k_z\rho_1) \to 1/\Gamma(m+1) * (k_z\rho_1/2)^m \to 0 \text{ for } m = 1,2,3\ldots \text{ and} \to 1 \text{ for } m = 0.$$

so as expected we have only m = 0 and our result is

$$u_o(z,\rho,\varphi) = (4q/S)\,\Sigma_{nz=1}^{\infty}\ \sin(k_z z)\sin(k_z z_1)\,K_0(k_z\rho) \qquad \text{**** Result ****}$$

where $k_z = n_z(\pi/S)$

Our only discomfort now is that $K_0(k_z\rho) \rightarrow - \ln(k_z\rho/2) - .5772$ as $\rho \rightarrow 0$. This says that the solution to this Green's problem seems to diverge logarithmically at every point on the z axis between the plates.

Take limit of result as we go near the point charge.

Next, does this result do the right thing as we get near our point charge?

Aside: We know that

$$1/R = (2/\pi) \Sigma_{m=0}^{\infty} \varepsilon_m \int_0^{\infty} dk \cos[k(z-z')] I_m(k\rho_<) K_m(k\rho_>) \cos[m(\varphi-\varphi')]$$

so let's put the primed point at our point charge, so $\varphi' = 0$ and $z' = z_1$ and $\rho_< = \rho_1 = 0$, so

$$1/R = (2/\pi) \varepsilon_0 \int_0^{\infty} dk \cos[k(z-z_1)] I_0(k\rho_1) K_0(k\rho) = (2/\pi) \int_0^{\infty} dk \cos[k(z-z_1)] K_0(k\rho)$$

so that, very close to our point charge, we expect to have

$$u = q/R = (2q/\pi) \int_0^{\infty} dk \cos[k(z-z_1)] K_0(k\rho)$$

This is of course the famous integral Jackson p 86 and we get

$$u = q/R = q/\sqrt{\rho^2 + (z-z_1)^2}$$

and all we have done is verified our 1/R formula in this application. We have learned nothing really about our series solution above.

Resume. Just out of curiosity, do I have any series formulas that look similar to the solution above? GR7 page 933 has some "series of Bessel functions". The closest we see there is page 939 which is not quite right. What about our friends PBM? The closest we get is this:

$$4. \sum_{k=1}^{\infty} \cos kaK_0(kx) = \frac{\pi}{2\sqrt{x^2+a^2}} + \frac{1}{2}\left(C + \ln\frac{x}{4\pi}\right) + \\ + \frac{\pi}{2}\sum_{k=1}^{\infty}\left[\frac{1}{\sqrt{(2k\pi-a)^2+x^2}} - \frac{1}{2k\pi}\right] + \frac{\pi}{2}\sum_{k=1}^{\infty}\left[\frac{1}{\sqrt{(2k\pi+a)^2+x^2}} - \frac{1}{2k\pi}\right]$$
$$[x > 0, \ \operatorname{Im} a = 0].$$

Let's go back to our solution above and change some names to protect the innocent,

$u_o(z,\rho,\varphi) = (4q/S) \Sigma_{k=1}^{\infty} \sin(K_z z) \sin(K_z z_1) K_0(K_z \rho)$

where $K_z = k(\pi/S)$

$u_o(z,\rho,\varphi) = (4q/S) \Sigma_{k=1}^{\infty} \sin(k(\pi/S)z) \sin(k(\pi/S)z_1) K_0(k(\pi/S)\rho)$

Let $x = (\pi/S)\rho$ and let $a_1 = (\pi/S)z_1$ and $a = (\pi/S)z$, then

$u_o(z,\rho,\varphi) = (4q/S) \Sigma_{k=1}^{\infty} \sin(ka) \sin(ka_1) K_0(kx)$

Suppose then we set $z \approx z_1$ so this becomes (close to the point charge, keep $\rho > 0$ so not on it)

$u_o(z,\rho,\varphi) = (4q/S) \Sigma_{k=1}^{\infty} \sin^2(ka_1) K_0(kx)$

$= (1/2)(4q/S) \Sigma_{k=1}^{\infty} [\ 1 - \cos(k2a_1)] K_0(kx)$

We can apply the above formula then for each term. The first term says $a = 0$

$\Sigma_{k=1}^{\infty} K_0(kx) = (\pi/2)*(1/x) + (1/2) (C + \ln(x/4\pi)) + 0 + 0$

The second term says $a = 2a_1$ = not small

$\Sigma_{k=1}^{\infty} \cos(k2a_1)K_0(kx) = (\pi/2)*(1/\sqrt{x^2 + 4a_1^2}\ ) + (1/2) (C + \ln(x/4\pi))$

$+ \Sigma_{k=1}^{\infty} [(1/[2\pi k-2a_1] + (1/[2\pi k+2a_1])$ $\qquad x \approx 0$

In our difference the term $(1/2) (C + \ln(x/4\pi))$ will cancel, and I think the $(\pi/2)*(1/x)$ is the dominant term for small x. So we have roughly that

$u_o(z,\rho,\varphi) \approx (1/2)(4q/S) \Sigma_{k=1}^{\infty} [\ 1 - \cos(k2a_1)] K_0(kx)$

$= (1/2)(4q/S) (\pi/2)*(1/x) =$

$= (1/2)(4q/S) (\pi/2)*(1/\rho) (S/\pi) =$

$= q/\rho$

which is exactly what I wanted to see! We are on a little ring of radius ρ around our point charge in the $z = z_1$ plane, and this is just what we expect.

Actually the GR result would have worked as well:

1. $$\sum_{k=1}^{\infty} K_0(kx)\cos kxt = \frac{1}{2}\left(C + \ln\frac{x}{4\pi}\right) + \frac{\pi}{2x\sqrt{1+t^2}} + \frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2+(2l\pi - tx)^2}} - \frac{1}{2l\pi}\right\} + \frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2+(2l\pi + tx)^2}} - \frac{1}{2l\pi}\right\}$$

MO 61

So the fact that my result seems right very close to the point charge gives some confidence to the answer I got for this problem.

It is a little harder to look at $\rho = 0$ and $z\text{-}z_1 = \Delta z$. I will just let this ride.

Can I find someone else's solution of this elementary problem??

I just found on scribd a 214 page claim to be solutions for Jackson 3rd Ed which would be good to have, but scribd is super slow today.

Classical Electrodynamics 3rd Ed J.D. Jackson - Solutions - 214 Pg

For future reference. Here is something elsewhere

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8.4 Exercises
Exercise 1: Parallel plates and strips
Exercise 2: Parallel plates and a point charge
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But its an NB file, no thanks. Here is another:

Point charge between two parallel grounded planes

American Journal of Physics -- November 1976 -- Volume 44, Issue 11, pp. 1132-1134

Issue Date: November 1976

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But Marriott cannot get it. Some people have this as a problem with an infinite set of image charges. I did this in 2D in Stak. Alas, I cannot find anyone treating this simple problem. Well I searched new Jackson and found this:

3.17 The Dirichlet Green function for the unbounded space between the planes at $z = 0$ and $z = L$ allows discussion of a point charge or a distribution of charge between parallel conducting planes held at zero potential.

(a) Using cylindrical coordinates show that one form of the Green function is

$$G(\mathbf{x},\mathbf{x}') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right)$$

(b) Show that an alternative form of the Green function is

$$G(\mathbf{x}, \mathbf{x}') = 2 \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\ e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<) \sinh[k(L - z_>)]}{\sinh(kL)}$$

I can now compare his form (a) to my result above:

$$G(\mathbf{x}, \mathbf{x}') = \frac{4}{L} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right)$$

$u(z,\rho,\varphi) = (4q/S)\ \Sigma_{nz}\ \sin(k_z z_>) \sin(k_z z_<)\ \Sigma_m\ \varepsilon_m\ K_m(k_z\rho_>)\ I_m(k_z\rho_<)\ \cos(m\varphi)$ $\qquad k_z = n_z(\pi/S)$

I can of course just keep the cos(m(φ-φ')) in his result, fold this over in m, and they indeed I get exactly my result. So there is the verification I have been looking for!!

Then Jackson has more on this same subject!

3.20 **(a)** From the results of Problem 3.17 or from first principles show that the potential at a point charge q between two infinite parallel conducting planes held at zero potential can be written as

$$\Phi(z, \rho) = \frac{q}{\pi\epsilon_0 L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z_0}{L}\right) \sin\left(\frac{n\pi z}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right)$$

where the planes are at $z = 0$ and $z = L$ and the charge is on the z axis at the point $z = z_0$.

(b) Calculate the induced surface-charge densities $\sigma_0(\rho)$ and $\sigma_L(\rho)$ on the lower and upper plates. The result for $\sigma_L(\rho)$ is

$$\sigma_L(\rho) = \frac{q}{L^2} \sum_{n=1}^{\infty} (-1)^n n \sin\left(\frac{n\pi z_0}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right)$$

Discuss the connection of this expression with that of Problem 3.19b and 3.19c.

(c) From the answer in part b, calculate the total charge Q_L on the plate at $z = L$. By summing the Fourier series or by other means of comparison, check your answer against the known expression of Problem 1.13 [C. Y. Fong and C. Kittel, *Am. J. Phys.* **35**, 1091 (1967).]

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So now compare part (a) above to my result

$$\Phi(z, \rho) = \frac{q}{\pi\epsilon_0 L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z_0}{L}\right) \sin\left(\frac{n\pi z}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right)$$

$u_o(z,\rho,\varphi) = (4q/S)\ \Sigma_{nz=1}^{\infty}\ \sin(k_z z) \sin(k_z z_1)\ K_0(k_z\rho)$

and of course set $\varepsilon_0 = 1/4\pi$ in his and you get my result! $k_z = n_z(\pi/S)$. So there it is!!!

Summary of results of this section:

Imagine two grounded metal parallel plates separated by distance S. We place a cylindrical coordinate system origin on the lower plate so the plates are at z = 0 and z = S. We then position a point charge q at position (z_1, ρ_1, $\varphi_1 = 0$) in the gap between the plates.

- Here is the potential between the plates: (the Green's function if q = 1) $k_z = n_z(\pi/S)$

$$u(z,\rho,\varphi) = (4q/S) \sum_{nz=1}^{\infty} \sin(k_z z) \sin(k_z z_1) \sum_m \varepsilon_m K_m(k_z \rho_>) I_m(k_z \rho_<) \cos(m\varphi)$$

- If we move the point charge to the z axis of our coordinate system so $\rho_1 = 0$, the result becomes

$$u_o(z,\rho,\varphi) = (4q/S) \sum_{nz=1}^{\infty} \sin(k_z z) \sin(k_z z_1) K_0(k_z \rho)$$

We note that for small ρ, the K function has a logarithmic divergence.

- Both these results are confirmed in blue Jackson.

4. Comment on the need for two oscillatory dimensions.

When you choose two parts of the atomic form to be oscillatory, then essentially you expand your potential in two complete sets of functions in those two variables, a sort of double generalized Fourier expansion. You know that your expansion will be well-posed, so to speak, meaning it will be an expansion that means something as the two sums go off to infinity. In each expansion term, if you select as your "radial factor" the appropriate "third atom", then not only do you have a well-posed double-convergent power series, but each term in that series satisfies the Laplace equation! Perhaps you can think of that third atomic factor as the solution of the residual 1D Green's problem in "the radial direction".

In any event, if you try to set yourself up with only *one* oscillatory function, the pillbox method will fail because it needs two oscillatory coordinates perp to the radial direction in order to work. But I think there is a more fundamental problem in this kind of setup. You have only a single expansion in one of the three variables, and that leaves you with some Green's function problem in 2 dimensions that you don't know how to solve, because neither of those dimensions is oscillatory. If you go ahead and try to write a double expansion in function sets only one of which is a complete set, probably it is not a well-posed expansion and just not meaningful in the sense of infinite sums. Remember how the coefficients keep moving if you don't have a complete set! That basic idea comes back again and again.

In the above successful Cylindricals solution of the Green's function for parallel plates, we are oscillatory in φ and in z, and radial only in the ρ direction. We know that, in general, other problems will be radial in z, and oscillatory in ρ. But no one ever talks about being radial in two of these three!

Another example: in the Cartesian solution to the plates problem found nearby, we again have two coordinates oscillatory and one expo, but this time it is expo in z, whereas the cylindrical solution was oscillatory in z.