Question on Fourier; Parallel Plate Cartesian Green's PhL 7.17.10

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Preliminary Question: (overview starts down 1/2 page) How does a 3D Fourier expansion (part of the usual transform) relate to the subject of Cartesian atoms and Smythian forms? For example, consider this 3D Fourier expansion of 1/R which appears in MF

$$1/R = (1/2\pi^2)\int d^3k \exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,] / k^2$$

In the notes below, I show that the integrand here is in fact NOT a Cartesian atom, yet the integral still solves Laplace. The usual atoms reappear when the above is rewritten in its 2D "transverse form"

$$1/R = (1/\pi)\int d^2k_t \exp[\, \pm i\mathbf{k}_t\bullet(\mathbf{r}-\mathbf{r}_1) \exp(-k_t|z-z_1|)/(2k_t) \qquad \text{Appendix A A.55}$$

as discussed below using a contour integration. Meanwhile, the solution to the parallel plate problem in Cartesian coordinates comes out being: ($\mathbf{k}_t = (k_x,k_y)$ and $k_t = \kappa_z = \sqrt{k_x^2+k_y^2}$)

$$G(\mathbf{r};\mathbf{r}_1) = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2k_t \,(\exp[-i\,\mathbf{k}_t\bullet(\mathbf{r}-\mathbf{r}_1)] /k_t) \sinh[k_t(d-z_>)] \sinh(k_t z_<) /\sinh(k_t d)$$

$z_> = \max(z,z_1)$ $z_< = \min(z,z_1)$ $0 < z_1 < d$ (between the plates)

where again in this Smythian form you see our usual Cartesian atomic form (written in exponentials)

1' $[\exp(k_x x), \exp(-k_x x)], [\exp(k_y y), \exp(-k_y y)], [\exp(\kappa_z z), \exp(-\kappa_z z)\,]$ k_z = imaginary = $i\kappa_z$

where x and y are oscillatory and z is radial/expo.

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0. Overview (2 pages, added 9.8.10)

In **Section 1** I shows that if we assume that

$$1/R = (1/2\pi^2)\int d^3k \exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,] / k^2$$

we find that

$$\nabla^2 (1/R) = -\, 4\pi\, q\, \delta^3(\mathbf{r}-\mathbf{r}_1)$$

which is the correct Green's equation for a unit point charge (q=1) at $\mathbf{r} = \mathbf{r}_1$.

In **Section 2** I comment that the *integrand* $\exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,] / k^2$ is itself NOT a Cartesian atomic form, since $\nabla^2 \exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,]/k^2 = -\exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,] \neq 0$ and thus does not satisfy Laplace. Yet everywhere away from $\mathbf{r} = \mathbf{r}_1$ the *integral* of this integrand *does* satisfy Laplace.

In **Section 3** I quote the Rothwell/Cloud solution to our problem of the Green's Function for parallel plates treated in Cartesian coordinates, and I was lucky to find this solution in a web search. I first show that it is in fact a superposition of valid Cartesian atomic forms, but I don't at this point verify their solution which, by the way, is only that part of the solution due to the induced charge on the plates, which they call the "secondary" potential in a problem like this.

In **Section 4** I start with $-\nabla^2 u(\mathbf{r};\mathbf{r}_1) = 4\pi\delta(\mathbf{r}-\mathbf{r}_1)$ and do a 3D Fourier to find $U = \mathcal{F}_{3D}(u)$ is given by

$$U(\mathbf{k};\, \mathbf{r}_1) = 4\pi\, (2\pi)^{-3} e^{+i\mathbf{k}\bullet\mathbf{r}1}/k^2$$

and this then is just another way to confirm the 1/R expansion shown above.

Section 5 has several lettered parts, to wit:

In **Part (a)** I start again with $-\nabla^2 u(\mathbf{r};\mathbf{r}_1) = 4\pi\delta(\mathbf{r}-\mathbf{r}_1)$ but now I do only a 2D Fourier expansion which is essentially doing a Stakgold "partial eigenfunction expansion" to solve this parallel plates problem. You are then left with this 1D Green's function problem in coordinate z, where now $U = \mathcal{F}_{2D}(u)$,

$(k_t^2 - \partial_z^2)\, U(\mathbf{k}_t;\, z;\, \mathbf{r}_1) = 4\pi\, (2\pi)^{-2}\, \delta(z-z_1) \exp(i\mathbf{k}_t\bullet\mathbf{r}_1)$
$U(\mathbf{k}_t;\, z=0;\, \mathbf{r}_1) = 0 \qquad U(\mathbf{k}_t;\, z=d;\, \mathbf{r}_1) = 0$ // potential vanishes on the plates!

Then in **Part (b)** I solve this 1D Green's function problem (lots of detail) and I get

$$g(z;\, z_1) = -(a/k)\, \sinh[k(z_> - d)]\, \sinh(kz_<) / \sinh(kd) \qquad z_> = \max(z, z_1)$$

where $g(z;\, z_1)$ is shorthand for $U(\mathbf{k}_t;\, z;\, \mathbf{r}_1)$ and k is really k_t and $a = \exp(i\mathbf{k}_t\bullet\mathbf{r}_1)/\pi$.

In **Part (c)** I write this out as

$$U(\mathbf{k_t};\, z;\, \mathbf{r_1}) \;=\; -\; \exp(i\mathbf{k_t}\bullet\mathbf{r_1})\,(1/\pi k_t)\,\sinh[k_t(z_>-d)]\,\sinh(k_t z_<)/\sinh(k_t d)$$

and I insert this into the 2D Fourier expansion to get the final result

$$G(\mathbf{r};\mathbf{r_1}) = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2k_t\,(\exp[-i\,\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r_1})]/k_t)\,\sinh[k_t(d-z_>)]\,\sinh(k_t z_<)/\sinh(k_t d)$$

$$= (1/\pi^2)\int_0^{\infty}\int_0^{\infty} dk_x dk_y\,\cos[k_x(x-x_1)]\,\cos[k_y(y-y_1)]\,\sinh[\kappa_z(d-z_>)]\sinh(\kappa_z z_<)/[\,\kappa_z\sinh(\kappa_z d)]$$

$z_> = \max(z,z_1)$ $\quad z_< = \min(z,z_1)$ $\quad 0 < z_1 < d$ $\quad$ (between the plates) $\quad \kappa_z = \sqrt{k_x^2+k_y^2}$.

which is the Green's function for a unit positive point charge located at $\mathbf{r_1} = (x_1, y_1, z_1)$ between parallel plates located at z=0 and z=d, when observed at point $\mathbf{r} = (x,y,z)$, and where $\mathbf{k_t} = (k_{tx},\ k_{ty}, 0)$ and we are in green Jackson cgs units. Either form for G is just a linear combination of Cartesian atoms, and I note the manifest $\mathbf{r} \leftrightarrow \mathbf{r_1}$ symmetry.

In order to verify Rothwell and Cloud's claimed result for the potential due just to the charge induced on the plates, I need to subtract out the contribution of the point charge to the above $G(\mathbf{r};\mathbf{r_1})$ result. Since the above G involves a d^2k_t integration, it is very useful to have a similar integral for the point charge 1/R.

So in **Part (d)** I develop this 1/R expansion, and show that it agrees with Appendix A.55 of R&C,

$$1/R = (1/\pi)\int d^2k_t\,\exp[\,\pm i\mathbf{k_t}\bullet\mathbf{(r-r_1)}\,\exp(-k_t|z-z_1|)/(2k_t) \qquad \text{Appendix A A.55}$$

which requires playing with a contour integral, something familiar to me from "propagator theory".

Finally then in **Part (e)** I subtract out this 1/R contribution and I get this result :

$$G_{ind}(\mathbf{r};\mathbf{r_1}) = -(2\pi)^{-2}\int d^2k_t\{2\sinh[k_t(z_>-d)]\sinh(k_t z_<)/\sinh(k_t d) + \exp(-k_t|z-z_1|)\}\,\exp[i\,\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r_1})]/(2k_t)$$

$$G^s(\mathbf{r}|\mathbf{r}') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-e^{-k_\rho(d-z)}\frac{\sinh k_\rho z'}{\sinh k_\rho d} - e^{-k_\rho z}\frac{\sinh k_\rho(d-z')}{\sinh k_\rho d}\right]\frac{e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}}{2k_\rho}\,d^2k_\rho. \qquad (3.212)$$

where I now show a direct comparison with the R&C result. It is not obvious that the two results agree, but, using Maple to do the algebra, I show that they *do* in fact agree exactly, except R&C have an error in their exponential argument -- they should really have $\exp(-j\ \mathbf{k_\rho}\bullet[\mathbf{r}'-\mathbf{r}])$. Mr. Rothwell confirmed this erratum and I have copied the email comm at the end of this doc.

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1. Verification of Fourier 3D 1/R expansion. First of all, how would we verify this?

$$\nabla^2\,|\mathbf{r}-\mathbf{r_1}|^{-1} = \nabla^2\,(1/R) = \nabla_1^2\,\{(1/2\pi^2)\int d^3k\,\exp[\,i\mathbf{k}\bullet\mathbf{(r-r_1)}\,]/k^2\}$$

$$= (1/2\pi^2) \int d^3k\, k^{-2}\, \nabla^2 \exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,]$$

$$= (1/2\pi^2) \int d^3k\, k^{-2}\, i^2 k^2 \exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,]$$

$$= -(1/2\pi^2) \int d^3k \exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,]$$

$$= -4\pi\,(1/2\pi)^3\,(2\pi)^3\,\delta(\mathbf{r}-\mathbf{r}_1)$$

$$= -4\pi\, q\, \delta^3(\mathbf{r}-\mathbf{r}_1) \qquad // \; q = 1$$

The Poisson equation says

$$-\nabla^2 V = 4\pi\rho$$

For a point charge q located at position $\mathbf{r}_1$ this says

$$-\nabla^2 V = 4\pi q\, \delta^3(\mathbf{r}-\mathbf{r}_1)$$

But we know that the potential of a point charge is just $V = q\ |\mathbf{r}-\mathbf{r}_1|^{-1}$ and we know it is unique. Thus, the above expansion must be equal to 1/R. The 1/R expansion shown above is the solution of this Poisson equation which we can regard as just a 3D Green's Function problem with no BC's, the Stak "fundamental solution".

2. Do we have a new atomic form for Cartesians? Notice that 1/R is a solution of Laplace as long as you stay away from the point charge. Staying away, then, we *seem* to have a "separated" atomic form as follows, where all three terms are oscillatory,

$$\exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,] = \exp[ik_x(x-x_1)]\,\exp[ik_y(y-y_1)]\,\exp[ik_z(z-z_1)]$$

But this product of atoms does NOT satisfy Laplace, since each term makes the same sign contribution to the curvature,

$$\nabla^2 \exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,] = -k^2 \exp[\, i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,] \neq 0$$

However, we can construct *with* this atomic form a function which, in a very large region of space, IS a solution of Laplace, and that is the 1/R shown above. Rather than having the k_i^2 components add up to zero for each term, we have global phase cancellation at all points except at the point charge.

Since the atom itself does not satisfy Laplace, I don't think I would classify it as one of my "atomic forms".

3. Examining the Rothwell/Cloud parallel plate Green's Function Solution

I found the following solution in a book Electromagnetics by **Rothwell and Cloud (**I have it). Here is the picture and the solution: (this really is a 3D problem, by the way)

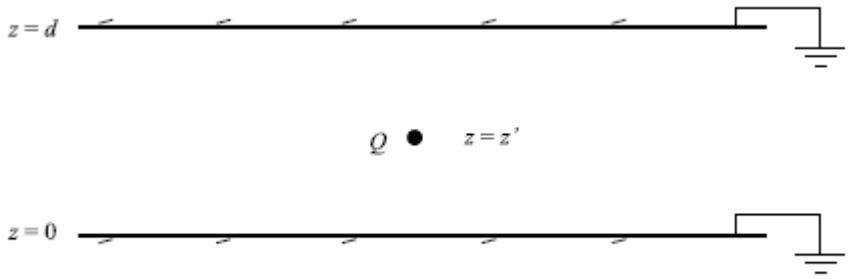

Figure 3.26: Geometry for computing Green's function for parallel plates.

3.7 Consider a point charge between two grounded conducting plates as shown in Figure 3.26. Write the Green's function as the sum of primary and secondary terms and apply the boundary conditions to show that the secondary Green's function is

$$G^s(\mathbf{r}|\mathbf{r}') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-e^{-k_\rho(d-z)}\frac{\sinh k_\rho z'}{\sinh k_\rho d} - e^{-k_\rho z}\frac{\sinh k_\rho(d-z')}{\sinh k_\rho d}\right]\frac{e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}}{2k_\rho}\,d^2k_\rho. \tag{3.212}$$

And here while we're at it is a related follow-on problem:

3.8 Use the expansion

$$\frac{1}{\sinh k_\rho d} = \operatorname{csch} k_\rho d = 2\sum_{n=0}^{\infty} e^{-(2n+1)k_\rho d}$$

to show that the secondary Green's function for parallel conducting plates (3.212) may be written as an infinite sequence of images of the primary point charge. Identify the geometrical meaning of each image term.

In this book, the authors use the word "secondary" to mean the potential due to induced charge, and "primary" is the exciting charge, here a point charge.

The notation used above is certainly hazy, but if we look back in the text we see this info:

$$G^p(\mathbf{r}|\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{-k_\rho|z-z'|}}{2k_\rho}e^{j\mathbf{k}_\rho\cdot(\mathbf{r}-\mathbf{r}')}\,d^2k_\rho \tag{3.77}$$

is the primary Green's function with $\mathbf{k}_\rho = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y$, $k_\rho = |\mathbf{k}_\rho|$, and $d^2k_\rho = dk_x\,dk_y$. We also find in (A.56) that a solution of Laplace's equation can be written as

$$\Phi^s(\mathbf{r}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[A(\mathbf{k}_\rho)e^{k_\rho z} + B(\mathbf{k}_\rho)e^{-k_\rho z}\right]e^{j\mathbf{k}_\rho\cdot\mathbf{r}}\,d^2k_\rho \tag{3.78}$$

where $A(\mathbf{k}_\rho)$ and $B(\mathbf{k}_\rho)$ must be found by the application of appropriate boundary conditions.

In the notes below, I derive the expansion 3.77 which is in fact correct. I also derive the parallel plate solution shown above, and I did find a significant erratum which I sent off to the author (namely, that the expo in 3.212 should be as it appears in 3.77). As for 3.78, it is a viable "atomic form" for this simple reason: If you apply ∇^2, you get the sum of two terms which are from the [...] and from the expo factor. The [...] contributes $+k_\rho^2$ and the expo contributes $- k_\rho^2$, so indeed you get $\nabla^2\Phi^2(\mathbf{r}) = 0$.

How would I enter this in my list of "atoms" ? One way is this:

$$[\ \exp(k_z z), \exp(-k_z z)\ ]\ [\cos(k_x x + k_y y)\ ]$$

which can be written as

$$[\ \exp(k_z z), \exp(-k_z z)\ ]\ [\cos(k_x x)\cos(k_y y) - \sin(k_x x)\sin(k_y y)\ ]$$

This is a *subset* of the more general atomic form

$$[\ \exp(k_z z), \exp(-k_z z)\ ]\ [\sin(k_x x), \cos(k_x x)\ ]\ [\sin(k_y y), \cos(k_y y)\ ]$$

so I don't think (3.78) is as fully general as my atomic form. In any event, you see that we are expo in z, and oscillatory in x and y, and we integrate over k_x and k_y, and $(0,\infty)$ would be general enough I think for the spectra. So I don't think I have anything new to add to my atomic forms for Cartesians.

My PDF **Rothwell and Cloud** book has no page numbers and is perhaps an early draft? I think it matches their 2001 First Edition however.

4. Rederive the 1/R expansion using the 3D Fourier Transform Notation

Nothing new relative to section 1, above, just doing it in a different notation. Suppose you start with a raw Green's equation [true Green's would be $G = u/(4\pi)$]

$$-\nabla^2 u(\mathbf{r};\mathbf{r}_1) = 4\pi\delta(\mathbf{r}-\mathbf{r}_1) \qquad \text{// Jackson convention with unit cgs charge}$$

You could Fourier-expand the δ and the u solution:

$$\delta(\mathbf{r}-\mathbf{r}_1) = (2\pi)^{-3} \int d^3k\ e^{-i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)}$$

$$u(\mathbf{r};\mathbf{r}_1) = \int d^3k\ e^{-i\mathbf{k}\bullet\mathbf{r}}\ \ U(\mathbf{k};\mathbf{r}_1)$$

$$-\nabla^2 u(\mathbf{r};\mathbf{r}_1) = \int d^3k\ e^{-i\mathbf{k}\bullet\mathbf{r}}\ k^2\ U(\mathbf{k};\mathbf{r}_1)$$

We then have the following k-space equation

$$k^2\ U(\mathbf{k};\mathbf{r}_1) \ = \ 4\pi\ (2\pi)^{-3} e^{+i\mathbf{k}\bullet\mathbf{r}_1}$$

and we have a solution to our problem in k-space, which is just this:

$$U(\mathbf{k};\mathbf{r_1}) = 4\pi\,(2\pi)^{-3}\;e^{+i\mathbf{k}\bullet\mathbf{r_1}}/k^2$$

We then want to transform this solution back to r-space,

$$u(\mathbf{r};\mathbf{r_1}) = \int d^3k\, e^{-i\mathbf{k}\bullet\mathbf{r}}\; U(\mathbf{k};\mathbf{r_1}) \;= \int d^3k\, e^{-i\mathbf{k}\bullet\mathbf{r}}\;\{4\pi\;(2\pi)^{-3}e^{+i\mathbf{k}\bullet\mathbf{r_1}}/k^2\;\}$$

$$= 4\pi\,(2\pi)^{-3}\;\int d^3k\, e^{-i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r_1})}/k^2 \;= (1/2\pi^2)\int d^3k\, e^{-i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r_1})}/k^2 \;= 1/R$$

where in the last step we use our result above. So this is another way to obtain that result, if you will.

5. How would you solve the parallel-plate problem using a "transverse Fourier Series"?

(a) do the Fourier Transform only in x and y, not z.

Think of our plates in z, and we decide to transform *only* the x and y directions. We start again with

$$-\nabla^2 u(\mathbf{r};\mathbf{r_1}) = 4\pi\delta^3(\mathbf{r}-\mathbf{r_1})$$

and we make these expansions:

$$\delta^3(\mathbf{r}-\mathbf{r_1}) = \delta(z-z_1)\,(2\pi)^{-2}\int d^2k_t\,\exp[-i\,\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r_1})]$$

$$u(\mathbf{r};\mathbf{r_1}) = \int d^2k_t\,\exp[-i\,\mathbf{k_t}\bullet\mathbf{r}]\;\; U(\mathbf{k_t};z;\mathbf{r_1})$$

where $\mathbf{k_t}$ is a 2-component transverse k vector, but can think as 3D vector with $k_{tz} = 0$. We next compute the effect of ∇^2 on u :

$$-\nabla^2\, u(\mathbf{r};\mathbf{r_1}) = -(\nabla_t^2 + \partial_z^2)\, u(\mathbf{r};\mathbf{r_1}) \;=$$

$$= \int d^2k_t\,(-\nabla_t^2\exp[-i\,\mathbf{k_t}\bullet\mathbf{r}])\;\; U(\mathbf{k_t};z;\mathbf{r_1}) \;+ \int d^2k_t\,\exp[-i\,\mathbf{k_t}\bullet\mathbf{r}]\;\; (-\partial_z^2\, U(\mathbf{k_t};z;\mathbf{r_1}))$$

$$= \int d^2k_t\,\exp[-i\,\mathbf{k_t}\bullet\mathbf{r}])\; k_t^2 U(\mathbf{k_t};z;\mathbf{r_1}) \;+ \int d^2k_t\,\exp[-i\,\mathbf{k_t}\bullet\mathbf{r}]\;\; (-\partial_z^2\, U(\mathbf{k_t};z;\mathbf{r_1}))$$

$$= \int d^2k_t\,\exp[-i\,\mathbf{k_t}\bullet\mathbf{r}])\;\{\,(k_t^2 - \partial_z^2)\, U(\mathbf{k_t};z;\mathbf{r_1})\,\}$$

Our Green's equation then reads:

$$\int d^2k_t \exp[-i\,\mathbf{k_t}\bullet\mathbf{r}])\ \{ (k_t^2 - \partial_z^2)\, U(\mathbf{k_t}; z; \mathbf{r_1}) \}\ = 4\pi \int d^2k_t \exp[-i\,\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r_1})]\ \{ \delta(z-z_1)\,(2\pi)^{-2}\}$$

and thus in k_t space we have this equation to solve:

$$(k_t^2 - \partial_z^2)\, U(\mathbf{k_t}; z; \mathbf{r_1}) = 4\pi\,(2\pi)^{-2}\,\delta(z-z_1)\exp(i\mathbf{k_t}\bullet\mathbf{r_1})$$

We can now regard this, for a particular value of $\mathbf{k_t}$, as **a 1D Helmholtz Green's Function problem**. But we now want to impose our boundary conditions

$$U(\mathbf{k_t}; z=0; \mathbf{r_1}) = 0 \qquad\qquad U(\mathbf{k_t}; z=d; \mathbf{r_1}) = 0$$

(b) Solve the 1D Helmholtz Green's Function problem in z which arises.

So let's simplify the notation and write $k = k_t$

$$(k^2 - \partial_z^2)\, g(z; z_1) = a\,\delta(z-z_1) \qquad\qquad a = 4\pi\,(2\pi)^{-2}\exp(i\mathbf{k_t}\bullet\mathbf{r_1}) = \exp(i\mathbf{k_t}\bullet\mathbf{r_1})/\pi$$

The homo solutions of this equation are sh(kz) and ch(kz) due to the relative minus sign. So we construct our Green's function like this:

$$\begin{aligned} g(z; z_1) &= Ash(kz) && 0 \le z \le z_1 \\ g(z; z_1) &= Bsh(kz) + Cch(kz) && z_1 \le z \le d \end{aligned}$$

The first line meets the BC at z = 0, and we need to work on the second one:

$$Bsh(kd) + Cch(kd) = 0 \Rightarrow \qquad C = -B\,th(kd)$$

So we now have it down to two coefficients:

$$\begin{aligned} g(z; z_1) &= Ash(kz) && 0 \le z \le z_1 \\ g(z; z_1) &= B[\, sh(kz) - th(kd)ch(kz)\,] && z_1 \le z \le d \end{aligned}$$

The function must be continuous at z_1 so we have

$$Ash(kz_1) = B[\, sh(kz_1) - th(kd)ch(kz_1)\,]$$

Then we have our jump condition to worry about. I will just rederive that right here rather than look it up. Start with

$$(k^2 - \partial_z^2)\, g(z; z_1) = a\,\delta(z-z_1)$$

Now integrate both sides around z_1 to get

$$\int_{z1-\varepsilon}^{z1+\varepsilon} (k^2 - \partial_z^2)\, g(z; z_1) = a$$

The k^2 term gives nothing, so we have

$$\int_{z1-\varepsilon}^{z1+\varepsilon} \partial_z^2\, g(z; z_1) = -a$$

The integrand is a perfect differential, so we have derived our jump condition,

$$\partial_z g(z; z_1)^+ - \partial_z g(z; z_1)^- = -a$$

But from above we have

$$\begin{aligned} g(z; z_1) &= A\,sh(kz) && 0 \le z \le z_1 \\ g(z; z_1) &= B[\, sh(kz) - th(kd)ch(kz) \,] && z_1 \le z \le d \end{aligned}$$

$$\begin{aligned} \partial_z\, g(z; z_1) &= A\,k\,ch(kz) && 0 \le z \le z_1 \\ \partial_z\, g(z; z_1) &= B\,k\,[\, ch(kz) - th(kd)sh(kz) \,] && z_1 \le z \le d \end{aligned}$$

So our jump condition is this:

$$\partial_z g(z; z_1)^+ - \partial_z g(z; z_1)^- = -a$$

$$B\,k\,[\, ch(kz_1) - th(kd)sh(kz_1) \,] - A\,k\,ch(kz_1) = -a$$

So let's now assemble our continuity and our jump conditions:

$$A\,sh(kz_1) = B[\, sh(kz_1) - th(kd)ch(kz_1) \,]$$

$$A\,ch(kz_1) = B[\, ch(kz_1) - th(kd)sh(kz_1) \,] + a/k$$

We want then to solve these equations for A and B, then we will have found our Green's 1D solution. This is just a Cramer's Rule problem which Maple is happy to solve for us:

```
e1 :=  A*sinh(k*z1) = B*(sinh(k*z1) - tanh(k*d)*cosh(k*z1));
```

$$e1 := A\sinh(k\,z1) = B\,(\sinh(k\,z1) - \tanh(k\,d)\cosh(k\,z1))$$

```
e2 := A*cosh(k*z1) = B*(cosh(k*z1) - tanh(k*d)*sinh(k*z1)) + a/k;
```

$$e2 := A\cosh(k\,z1) = B\,(\cosh(k\,z1) - \tanh(k\,d)\sinh(k\,z1)) + \frac{a}{k}$$

```
solve({e1,e2},{A,B}):simplify(%);
```

$$\left\{A = \frac{a\,(-1+\mathbf{e}^{(-2\,k\,(z1-d))})\,\mathbf{e}^{(k\,z1)}}{k\,(-1+\mathbf{e}^{(2\,k\,d)})},\ B = \frac{1}{2}\,\frac{(-\mathbf{e}^{(2\,k\,d)} - 1 + \mathbf{e}^{(-2\,k\,(z1-d))} + \mathbf{e}^{(-2\,k\,z1)})\,a\,\mathbf{e}^{(k\,z1)}}{k\,(-1+\mathbf{e}^{(2\,k\,d)})}\right\}$$

where it puts everything into expo notation. Manual fiddling then shows that

A = - (a/k) sinh[k(z_1-d)] / sinh(kd)

B = -(a/k) sinh(kz_1) coth(kd)

I then verified this with Maple by entering these things as AA and BB and then subtracting.

So, we now know our 1D Green's function which meets the parallel plates BC's:

g(z; z_1)	=	Asinh(kz)	$0 \le z \le z_1$
g(z; z_1)	=	B[sinh(kz) - tanh(kd)cosh(kz)]	$z_1 \le z \le d$

g(z; z_1)	=	- (a/k)sinh(kz) sinh[k(z_1-d)] / sinh(kd)	$0 \le z \le z_1$
g(z; z_1)	=	-(a/k) [sinh(kz) - tanh(kd)cosh(kz)] sinh(kz_1) coth(kd)	$z_1 \le z \le d$

We can simplify the last expression a little:

-(a/k) [sinh(kz) - tanh(kd)cosh(kz)] sinh(kz_1) coth(kd)

= -(a/k) [sinh(kz) - sinh(kd)cosh(kz)/cosh(kd)] sinh(kz_1) cosh(kd)/sinh(kd

= -(a/k) [sinh(kz)cosh(kd) - sinh(kd)cosh(kz)] sinh(kz_1) /sinh(kd)

= -(a/k) [sinh[k(z-d)] sinh(kz_1) /sinh(kd)

Then write

g(z; z_1)	=	-(a/k) sinh[k(z_1-d)] sinh(kz) / sinh(kd)	$0 \le z \le z_1$
g(z; z_1)	=	-(a/k) sinh[k(z-d)] sinh(kz_1) /sinh(kd)	$z_1 \le z \le d$

which is our familiar form

$$g(z; z_1) = -(a/k)\sinh[k(z_>-d)]\sinh(kz_<)/\sinh(kd) \qquad z_> = \max(z,z_1)$$

We can confirm that:

(1) both these g forms are linear combinations of sh and ch and solve our homo ODE
(2) the first g = 0 on the lower plate, the second g=0 on the upper plate

(c) Resumption of Main Flow; Statement of the Parallel Plate Cartesian Green's Function

Now let's go back to our original problem where $a = \exp(i\mathbf{k_t}\bullet\mathbf{r_1})/\pi$ and $k^2 = k_t^2$ and we have shown then that

$$U(\mathbf{k_t}; z; \mathbf{r_1}) = -\exp(i\mathbf{k_t}\bullet\mathbf{r_1})(1/\pi k_t)\sinh[k_t(z_>-d)]\sinh(k_t z_<)/\sinh(k_t d)$$

Now we can insert this back into our transverse Fourier transform to get

$$u(\mathbf{r};\mathbf{r_1}) = \int d^2k_t \exp[-i\,\mathbf{k_t}\bullet\mathbf{r}]\;\; U(\mathbf{k_t}; z; \mathbf{r_1})$$

$$= \int d^2k_t \exp[-i\,\mathbf{k_t}\bullet\mathbf{r}]\;\; \{\; -\exp(i\mathbf{k_t}\bullet\mathbf{r_1})(1/\pi k_t)\sinh[k_t(z_>-d)]\sinh(k_t z_<)/\sinh(k_t d)\;\}$$

$$= -(1/\pi)\int d^2k_t\,(\exp[-i\,\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r_1})]/k_t)\sinh[k_t(z_>-d)]\sinh(k_t z_<)/\sinh(k_t d)$$

This is the complete solution to this problem, it includes the effect of the point charge and of the induced charge. If we wanted a true Green's function, we divide by 4π and say

$$G(\mathbf{r};\mathbf{r_1}) = -(2\pi)^{-2}\int d^2k_t\,(\exp[-i\,\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r_1})]/k_t)\sinh[k_t(z_>-d)]\sinh(k_t z_<)/\sinh(k_t d)$$

where $z_> = \max(z,z_1)$. I think it is more logical to change the sinh argument so it is positive, then

$$G(\mathbf{r};\mathbf{r_1}) = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2k_t\,(\exp[-i\,\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r_1})]/k_t)\sinh[k_t(d-z_>)]\sinh(k_t z_<)/\sinh(k_t d)$$

$z_> = \max(z,z_1)$ $\quad z_< = \min(z,z_1)$ $\quad 0 < z_1 < d$ $\quad$ (between the plates) $\quad$ *********

which is the Green's function for a unit positive point charge located at $\mathbf{r_1} = (x_1, y_1, z_1)$ between parallel plates located at z=0 and z=d, when observed at point $\mathbf{r} = (x,y,z)$, and where $\mathbf{k_t} = (k_{tx}, k_{ty}, 0)$ and we are in green Jackson cgs units.

Note that the symmetry of G is manifest: swapping $\mathbf{r} \leftrightarrow \mathbf{r_1}$ causes $z \leftrightarrow z_1$ which switch has no effect on the $z_<$ and $z_>$ part of the solution (eg, $\max(z,z_1) = \max(z_1,z)$). It causes the phase to change sign which makes no difference. Note also that, as per the general Stak theorem, G is positive everywhere.

Let's try a rewrite of the above. We replace the expo with cos, so we have

$$\exp[-i\,\mathbf{k}_t\bullet(\mathbf{r}-\mathbf{r}_1)] \rightarrow \cos[\mathbf{k}_t\bullet(\mathbf{r}-\mathbf{r}_1)] = \cos[k_{tx}(x-x_1) + k_{ty}(y-y_1)]$$

$$= \cos[k_{tx}(x-x_1)]\cos[k_{ty}(y-y_1)] - \sin[k_{tx}(x-x_1)]\sin[k_{ty}(y-y_1)]$$

But the sin sin term vanishes because, for example, $\sin[k_{tx}(x-x_1)]$ is odd in k_{tx} but the integration range is even, and the rest of the integrand is even since $k_t = \sqrt{k_{tx}^2+k_{ty}^2}$. So we drop the sin term and get

$$G(\mathbf{r};\mathbf{r}_1) = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2k_t(\cos[k_{tx}(x-x_1)]\cos[k_{ty}(y-y_1)]/k_t)\sinh[k_t(d-z_>)]\sinh(k_t z_<)/\sinh(k_t d)$$

We can then fold those two integrals to get this final form:

$$G(\mathbf{r};\mathbf{r}_1) = (1/\pi^2) \int_0^{\infty} \int_0^{\infty} d^2k_t\ \cos[k_{tx}(x-x_1)]\cos[k_{ty}(y-y_1)]\sinh[k_t(d-z_>)]\sinh(k_t z_<)/[k_t\sinh(k_t d)]$$

$$G(\mathbf{r};\mathbf{r}_1) = (1/\pi^2) \int_0^{\infty} \int_0^{\infty} dk_x dk_y\ \cos[k_x(x-x_1)]\cos[k_y(y-y_1)]\sinh[\kappa_z(d-z_>)]\sinh(\kappa_z z_<)/[\kappa_z\sinh(\kappa_z d)]$$

$z_> = \max(z,z_1)$ $z_< = \min(z,z_1)$ $0 < z_1 < d$ (between the plates) *********

where $\kappa_z = \sqrt{k_x^2+k_y^2}$. Here we see one of our standard Cartesian atomic forms in action!

1 $[\sin(k_x x), \cos(k_x x)], [\sin(k_y y), \cos(k_y y)], [\exp(\kappa_z z), \exp(-\kappa_z z)]$ k_z = imaginary = $i\kappa_z$

so there is nothing new under the sun regarding atoms!

(d) Transverse Fourier Expansion for 1/R.

If we want, we can get the potential due just to the induced charge by subtracting out (from u)

$$1/R = (1/2\pi^2)\int d^3k\ \exp[\,i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_1)\,]/k^2$$

but we want to get this into a 2D Fourier form,

$$1/R = (1/2\pi^2)\int d^2k_t\, dk_z\ \exp[\,i\mathbf{k}_t\bullet(\mathbf{r}-\mathbf{r}_1)\,]\exp[\,ik_z(z-z_1)\,]/(k_t^2+k_z^2)$$

$$= (1/2\pi^2)\int d^2k_t\ \exp[\,i\mathbf{k}_t\bullet(\mathbf{r}-\mathbf{r}_1)\,] \int_{-\infty}^{\infty} dk_z\ \exp[\,ik_z(z-z_1)\,]/(k_t^2+k_z^2)$$

$$= (1/2\pi^2)\int d^2k_t\ \exp[\,i\mathbf{k}_t\bullet(\mathbf{r}-\mathbf{r}_1)\,] \int_{-\infty}^{\infty} dk_z\ \cos[\,k_z(z-z_1)\,]/(k_t^2+k_z^2)$$

$$= (1/\pi^2)\int d^2k_t \exp[\, i\mathbf{k_t}\bullet\mathbf{(r-r_1)} \quad \int_0^\infty dk_z \cos[\, k_z(z-z_1)\,] / (k_t^2 + k_z^2)$$

We have from (GR7 p 435)

$$2. \qquad \int_0^\infty \frac{\cos(ax)\,dx}{\sqrt{\beta^2+x^2}} = K_0(a\beta) \qquad [a>0, \quad \mathrm{Re}\,\beta>0]$$

WA 191(1), GW(333)(78a)

so let $\beta = k_t$ and $a = |z-z_1|$ and our integral is then

$$\int_0^\infty dk_z \cos[\, k_z(z-z_1)\,] / (k_t^2 + k_z^2) \;=\; K_o[k_t\,|z-z_1|] \qquad \text{// modified Bessel function}$$

So we then have

$$1/R \;= (1/\pi^2)\int d^2k_t \exp[\, i\mathbf{k_t}\bullet\mathbf{(r-r_1)}]\; K_o[k_t\,|z-z_1|]$$

This is no doubt correct, but I think *there is a simpler result available*. We had earlier that

$$= (1/2\pi^2)\int d^2k_t \exp[\, i\mathbf{k_t}\bullet\mathbf{(r-r_1)} \quad \int_{-\infty}^\infty dk_z \exp[\, ik_z(z-z_1)\,] / (k_t^2 + k_z^2)$$

Suppose we treat the rightmost integral as a *contour* integral, where we have poles at $\pm ik_t$ in the k_z plane. Maybe write it like this:

$$I = \int_{-\infty}^\infty dw \exp[\, iw(z-z_1)\,] / (k_t^2 + w^2)$$

$$= \int_{-\infty}^\infty dw \exp[\, iw(z-z_1)\,] / [(w-ik_t)\,(w+ik_t)]$$

If $z>z_1$, we close the contour up top since $\exp[\, i(i|w|(z-z_1)\,] = \exp[\, -|w|(z-z_1)\,] \to 0$. We then pick up the residue of the pole at $w = +ik_t$ and we get

$$I = (2\pi i)\exp[\, i(ik_t)(z-z_1)]/\,(2ik_t)$$

With the other sign choice, we close bottom and get (leading minus since wrong circulation sense)

$$I = -\,(2\pi i)\exp[\, i(-ik_t)(z-z_1)]/\,(-2ik_t) \;=\; (2\pi i)^{-1}\exp[\, i(ik_t)(z_1-z)]/\,(2ik_t)$$

We combine these results then to have

$$I = (2\pi i)\exp[\, i(ik_t)|z-z_1|]/\,(2ik_t) \;=\; (2\pi)\exp(-k_t|z-z_1|)/(2k_t)$$

Then we find that

$$1/R = (1/2\pi^2)\int d^2k_t \exp[\, i\mathbf{k_t}\bullet\mathbf{(r-r_1)} \int_{-\infty}^{\infty} dk_z \exp[\, ik_z(z-z_1)\,]\,/\,(k_t^2 + k_z^2)$$

$$= (1/2\pi^2)\int d^2k_t \exp[\, i\mathbf{k_t}\bullet\mathbf{(r-r_1)}\ \ I$$

$$= (1/2\pi^2)\int d^2k_t \exp[\, i\mathbf{k_t}\bullet\mathbf{(r-r_1)}\ \ \{(2\pi)\exp(-k_t|z-z_1|)/(2k_t)\}$$

$$= (1/\pi)\int d^2k_t \exp[\, \pm i\mathbf{k_t}\bullet\mathbf{(r-r_1)}\exp(-k_t|z-z_1|)/(2k_t)$$

and this result agrees with Rothwell and Cloud Appendix A A.55. I suspect I have gone down this same road many times in my prior life, perhaps with Kittel, perhaps with some wave stuff, but fine, here it is again. I show explicitly that either sign is OK in the phase exponent since 1/R is real.

(e) Subtracting off the point charge potential

So the a/R above is what we want to subtract off to get just the potential of the induced charge,

$$u_{ind}(\mathbf{r};\mathbf{r_1}) = -(1/\pi)\int d^2k_t\,(\exp[i\,\mathbf{k_t}\bullet\mathbf{(r-r_1)}/k_t)\sinh[k_t(z_>-d)]\sinh(k_tz_<)/\sinh(k_td)$$

$$- (1/\pi)\int d^2k_t \exp[\, i\mathbf{k_t}\bullet\mathbf{(r-r_1)}\exp(-k_t|z-z_1|)/(2k_t)$$

$$= -(1/\pi)\int d^2k_t\,(\exp[i\,\mathbf{k_t}\bullet\mathbf{(r-r_1)}]/(2k_t))\ \{\,2\sinh[k_t(z_>-d)]\sinh(k_tz_<)/\sinh(k_td) + \exp(-k_t|z-z_1|)\,\}$$

$$= -(1/\pi)\int d^2k_t\ \{\,2\sinh[k_t(z_>-d)]\sinh(k_tz_<)/\sinh(k_td) + \exp(-k_t|z-z_1|)\,\}\exp[i\,\mathbf{k_t}\bullet\mathbf{(r-r_1)}]/(2k_t)$$

Now recall the comment above that "true Green's would be G = u/(4π)" . So our true Green's function then is this (that is, G satisfies the ODE driven by a δ with no 4π)

$$G_{ind}(\mathbf{r};\mathbf{r_1}) = -(2\pi)^{-2}\int d^2k_t\{2\sinh[k_t(z_>-d)]\sinh(k_tz_<)/\sinh(k_td) + \exp(-k_t|z-z_1|)\,\}\exp[i\,\mathbf{k_t}\bullet\mathbf{(r-r_1)}]/(2k_t)$$

$$G^s(\mathbf{r}|\mathbf{r}') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-e^{-k_\rho(d-z)}\frac{\sinh k_\rho z'}{\sinh k_\rho d} - e^{-k_\rho z}\frac{\sinh k_\rho(d-z')}{\sinh k_\rho d}\right]\frac{e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}}{2k_\rho}\,d^2k_\rho. \qquad (3.212)$$

where I show *their* solution for comparison. I think they have a typo in the exponential where they don't show **r**. Let's evaluate the {..} factor in the two cases:

$$\{\,2\sinh[k_t(z_>-d)]\sinh(k_tz_<)/\sinh(k_td) + \exp(-k_t|z-z_1|)\,\} \qquad z > z_1$$

$= 2 \sinh[k_t(z-d)] \sinh(k_t z_1) / \sinh(k_t d) + \exp(-k_t(z-z_1))$

$= [\, 2 \sinh[k_t(z-d)] \sinh(k_t z_1) + \sinh(k_t d)\exp(-k_t(z-z_1)) \,] / \sinh(k_t d)$

So make this agree with their result, I would have to show that

$2 \sinh[k_t(z-d)] \sinh(k_t z_1) + \sinh(k_t d)\exp(-k_t(z-z_1))$

$= \exp(-k_t(d-z)) \sinh(k_t z_1) + \exp(-k_t z) \sinh(k_t(d-z_1))$

Maple confirms that this is in fact an equality!

Now, what happens with the other sign?

$\{ 2 \sinh[k_t(z_> - d)] \sinh(k_t z_<) / \sinh(k_t d) + \exp(-k_t|z-z_1|) \}$ $\qquad z < z_1$

$= 2 \sinh[k_t(z_1 - d)] \sinh(k_t z) / \sinh(k_t d) + \exp(-k_t(z_1 - z))$

$= [2 \sinh[k_t(z_1 - d)] \sinh(k_t z) + \sinh(k_t d)\exp(-k_t(z_1 - z))] / \sinh(k_t d)$

To make this agree with their result, I would have to show that

$2 \sinh[k_t(z_1 - d)] \sinh(k_t z) + \sinh(k_t d)\exp(-k_t(z_1 - z))$

$= \exp(-k_t(d-z)) \sinh(k_t z_1) + \exp(-k_t z) \sinh(k_t(d-z_1))$

and Maple verifies this as well! ("question on Fourier 2")

Therefore I have shown this fact:

- [...] = {....}

so our answers differ only inasmuch as they have omitted **r** in their expo!

(f) Errata Search. Google books shows the 2001 edition having the error. (this book of course has page numbers). I look at Edward Rothwell's website and see that a 2nd edition is out, October 28, 2008. I don't have any download on this new book so cannot check errata. I sent him an email about this error, I hope to get some kind of response, probably that it was fixed in 2nd Ed.

// I did in fact get an email response, and Mr. Rothwell confirmed the error, saying they found it when they wrote their book of solutions!

Hi Ed,

For a very obscure reason, I was looking at your first edition Electromagnetics book and noted a small erratum that you have no doubt fixed in your 2nd edition. Since I don't have access to that 2nd edition, I don't know if it has been fixed or not, so I report it anyway. It involves a certain problem in the book,

3.7 Consider a point charge between two grounded conducting plates as shown in Figure 3.26. Write the Green's function as the sum of primary and secondary terms and apply the boundary conditions to show that the secondary Green's function is

$$G^s(\mathbf{r}|\mathbf{r}') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-e^{-k_\rho(d-z)}\frac{\sinh k_\rho z'}{\sinh k_\rho d} - e^{-k_\rho z}\frac{\sinh k_\rho(d-z')}{\sinh k_\rho d}\right]\frac{e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}}{2k_\rho}d^2k_\rho. \tag{3.212}$$

This result is exactly correct (by my painful calculation) except the exponential factor should be

$$\exp[\, j\, \mathbf{k}_\rho \bullet (\mathbf{r} - \mathbf{r}')\,]$$

Without this correction, the Green's Function appears to be independent of r !

I looked a bit for errata on your two editions but could not find any online.

I have not read much in your book but it looks very good to me! I was a Jackson student circa 1971 (green Jackson), and have recently been reading my two old volumes of Ivar Stakgold on BV problems which enlarge the playing field somewhat.

Best regards,

-Phil Lucht
Salt Lake City

Hi Phil:

You're absolutely right. We didn't discover that typo until we wrote a solution manual for the 2nd ed. We cleaned up a lot of typos and minor errors, but I'm sure there are still a few lurking in there.

I took "physics" EM out of Jackson as well. My red copy has duct tape holding it together :-)

The biggest difference between the second and first edition of our book is that we added a chapter on integral equations and numerical solutions. We also added a lot more problems and, as I mentioned, tried to clear up the errata. If you're not into integral equations, there's probably no reason to get the second edition.

Thanks for letting me know about the error, and if you spot any more please pass them along.

Take care,

Ed
